Characterization of crystalline topological phases

Arghadip Koner[1]

[1]*Department of Chemistry and Biochemistry, University of California at San Diego, La Jolla, CA 92093*

Crystallographic point groups are ubiquitous in study of solid state systems. However, they also play an important role in understanding topological phases. SPTs can be classified based on what patterns of quantum entanglement are protected by crystalline symmetries. Here we present two frameworks for classification of crystalline SPTs. The first one is based on the geometrical structure of the SPT, whereas the second one is based on response of the SPTs to elastic deformations. This is set up on the usual way of topological terms in the effective action.

INTRODUCTION

A topological phase of matter is a gapped phase of matter characterized by - 1. degeneracy in the ground state, 2. fractional statistics, and 3. long range entanglement in the ground state. These phases are characterized by the pattern of this ground state entanglement and not by broken symmetry. In these systems, there is an interplay between topological features and microscopic symmetries of the systems. States which are connected by smooth deformations in absence of symmetry, may now be distinct based on the action of the symmetry group on the topological degrees of freedom. Based on a lattice model, the symmetries can be divided into two classes – 1. **internal symmetries** which include charge conservation, time reversal and spin rotation which act locally at the crystal sites, and 2. **spatial symmetries** which include spatial translation and rotations which moves lattice sites in space. Topological phases respecting spatial symmetries is called a **crystalline topological phase**.
There have been competing approaches for classification of these crystalline topological phases (cSPTs). Two of these include

1. The cSPTs characterized by point groups [1] and invertible topological phases which refers to those with no non trivial topological excitations, can be understood in the framework similar to the coupled layer construction[2]. It is conjectured that all SPTs in dimension d protected only by crystalline symmetries can be built from lower dimensional (k-dimensional, $0 < k < d$) blocks of invertible topological phases arranged in some spatial configuration in a d dimensional space.

2. For non-invertible cSPTs *i.e.* those with non-trivial topological excitations, the approach involves smooth states – the states which vary in space slowly (in a length scale longer than the lattice spacing and correlation length) [3].

The framework using 'defect networks' as a generalization of the block state picture for invertible SPTs to the non-invertible ones [4] unifies the two frameworks and provide a more robust way for the classification. In the next section we shall try to understand these defect networks in more detail.

CLASSIFICATION USING DEFECT NETWORKS

Introduction to defect networks

A defect network consists of a G-symmetric network of defects in a G_{int}-symmetric topological phase (where $G_{int} \leq G$ is a subgroup of the internal symmetries [Fig. 1(a)].
A cSPT exists on a d-dimensional manifold X which represents the physical space in which the system is embedded. Let X be acted upon by a symmetry group G. The elements of G that act trivially on X represent the internal symmetries. Here, we have $X = \mathbb{R}^d$ and the group G acts by the isometries of Euclidean space.
The geometrical picture of the crystalline topological phase is referred to as the defect network. The defect network is defined in terms of a cell decomposition of X [Fig-1(b)]. We choose the cell decomposition of X such that the image of a cell under the group action of G is also a cell. For each cell Σ, let G_Σ be the kernal of the group action on Σ. In this framework, the cells are chosen to be very large as compared to the lattice spacing and correlation length. To classify crystalline phases we have to classify the phase on the defect networks. The idea is the following: classify the phase on the $d-$dimensional cells $\rightarrow$ classify the $d-2$ dimensional defects in that phase $\rightarrow$ for any configuration of the $d-1$ dimensional defects on $d-1$ cells, classify the possible junctions on $d-2$ cells $\rightarrow \ldots$ and so on.
Next we talk about another ingrediant of the recipe - smooth states.

Smooth states in a nutshell

A smooth states represent a certain kind of physical states. The idea is this – in scales much less than a radius of variation R, there is an approximate translation symmetry. On scales small as compared to R, the state varies very slowly with space. We define a smooth state

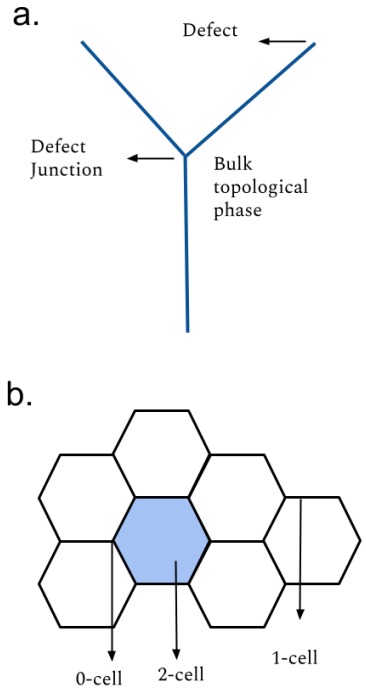

FIG. 1: (a) A defect network in 2-D consists of a bulk topological phase, 1-D defects and 0-D defect junctions. In the case where the 1-D defects are trivial, the 0-D defects are just point defects. In higher dimensions, we can have higher order junctions. (b) A cell decomposition of a manifold X. The 2-cells carry a 2-D topological phase. The 1-D ones carry defects and the 0-cells carry defect junctions.

on a spatial manifold X to be a map

$$f : X \to \Theta_d \tag{1}$$

for some space Θ_d which is an abstraction of the $d-$dimensional states in the neighborhood of a given point. One nice interpretation of Θ_d is the following: A point in the space Θ_d is a topological QFT. A continuous path in Θ_d is an isomorphism between the QFTs, a deformation between the paths is an equivalence between the isomorphisms, and so on. Now based on these concepts we can use the symmetries on a smooth state as a general classification of the crystalline phases.

Classification of cSPTs based on symmetries in a smooth state

The classification of topological phases in $d-$ dimensions with internal symmetry G is given by homotopy class of maps

$$f : BG \to \Theta_d \tag{2}$$

where, BG is the "classifying space" of a group G. $BG = EG/G$ upto homotopy equivalence, where EG is any contractible space with a free action G. The classification of topological phases in $d-$dimension with spatial symmetry G that acts on the manifold X is given by the homotopy class of maps

$$f : X//G \to \Theta_d \tag{3}$$

where $X//G = (X \times EG)/G$ is the homotopy quotient of X by the action of G. The action of G is diagonal on the product space $(X \times EG)$. In the case when $X = \mathbb{R}^d$, $X//G$ and BG are homotopy equivalent. This draws an equivalence between the action of internal symmetries and the action of spatial symmetries on X which gives the crystalline equivalence principle. According to the crystalline equivalence principle[3], classification of phases with spatial symmetry G has a one-to-one correspondence with classification of phases with the same symmetry G acting internally. One example of this is a spatial symmetry such as reflection which is unitary but orientation-reversing maps to an anti-unitary internal symmetry such as time reversal.

Finally lets look at an example to end this discussion. Let us consider the case where the only symmetries are translation $G = \mathbb{Z} \times \mathbb{Z}$. We can classify cSPTs using the crystalline equivalence principle and then use internal symmetries. In this case, the phases is classified by a group homomorphism $\rho = \mathbb{Z} \times \mathbb{Z} \to Aut(\mathcal{C})$, where $Aut(\mathcal{C})$ is a group of permutations of the anyon labels that leaves the braiding statistics unchanged, and by a symmetry fractionalization class $[\omega] = \mathcal{H}^2_\rho(\mathbb{Z} \times \mathbb{Z}, A) \cong A/A_\rho$, where A is a Abelian group of Abelian anyons, A_ρ is the subgroup generated by $\{(g \cdot a - a) : a \in A, g \in G\}$ and G acts on A according to its image by ρ. This thus provide a way for classification of $SPTs$.
Another approach for classification of cSPTs is by using elasticity theory which admits quantized topological terms. These terms have been shown to correspond to distinct phases of matter. This has been explored in the next section .

CLASSIFICATION USING ELASTICITY THEORY

A general framework for studying SPTs in general is by their response to background gauge fields. The idea here to, instead, characterize SPTs using their response to elastic deformations *i.e.* to phonons. This can even be generalised to quasi-crystalline SPTs [5].

An invertible topological phase in a d-dimensional lattice with d spatial translations and a $U(1)$ symmetry owing to charge conservation has an integer topological invariant ν (charge per unit cell). For a gapped state, with no other spatial symmetry other than translation, and the only internal symmetry $U(1)$, LSMOH theorem dictates an integer valued invariant ν which is the average charge per unit cell. For a system with invariant ν, the response to a elastic deformations and to a background gauge field A_μ is characterized by a topological term in the effective action. In addition to the kinetic terms, the Lagrangian contains

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \frac{\nu}{2\pi}\epsilon^{\mu\nu}\epsilon_I A_\mu \partial_\nu \theta^I dxdt \quad (d=1)\\
\mathcal{L} &= \mathcal{L}_0 + \frac{\nu}{8\pi^2}\epsilon^{\mu\nu\lambda}\epsilon_{IJ} A_\mu \partial_\nu \theta^I \partial_\lambda \theta^J d^2x dt \quad (d=2)\\
\mathcal{L} &= \mathcal{L}_0 + \frac{\nu}{48\pi^2}\epsilon^{\mu\nu\lambda\sigma}\epsilon_{IJK} A_\mu \partial_\nu \theta^I \partial_\lambda \theta^J \partial_\sigma \theta^K d^3x dt \quad (d=3)
\end{aligned}$$

as the topological term. Here μ, ν, ρ runs from $0, 1, \ldots, d$. The phases $\theta^1(\mathbf{x}, t)$ and $\theta^2(\mathbf{x}, t)$ are related to the phonon modes of the crystal. We see ν is the coefficient of the quantized topological term in effective action describing the dynamics of the phonons under the long wavelength elastic modes. These modes can be described by a slowly-varying displacement field $\mathbf{u}(\mathbf{x}, t)$ corresponding to displacement of atoms from their equilibrium positions. The displacement field is related to the phase angles $\theta^I (I = 1, \ldots, d)$ as

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{2\pi} a_I \theta^I(\mathbf{x},t) - \mathbf{x} \tag{4}$$

where $a^I (I = 1, \ldots, d)$ are a set of primitive lattice vectors. For the action to be invariant under large gauge transformations of A on a spacetime manifold, ν must be an integer. For the topological term, various properties can be derived. For $d = 2$ the charge density is

$$\rho = \frac{\delta S}{\delta A_0} = \frac{\nu}{8\pi^2}\epsilon^{ij}\epsilon_{IJ}\partial_i \theta^I \partial_j \theta^J, \tag{5}$$

Here the i, j index runs from $1, 2, \ldots, d$. In equilibrium (setting $\mathbf{u} = 0$ in Eq.4 we get $\theta^I = K^I_j x^i$, where K^I_i is the inverse of the matrix $\frac{1}{2\pi} a^i_I$ with the columns as vectors $\frac{1}{2\pi}\mathbf{a}^I$. Plugging this in, we get

$$\rho = \frac{\nu}{4\pi^2}\det K = \frac{\nu}{V_{unit}} \tag{6}$$

where V_{unit} =det(a) represents the volume of a unit cell. As the topological term is a part of the effective theory that describes the system on a length scale larger as compared to the unit cell size, ρ represents average charge density over a large length scale. This integer valued average charge density is what characterizes the cSPTs on the basis of a elasticity theory. The microscopic charge density can vary in the scale of unit cell and is not captured in this effective theory.

CONCLUSION

Here we have presented two separate ways for classification of SPTs. The first approach is based on the crystallographic equivalence principle which comes out of the defect network picture. This approach unifies two separately known ones basd on point groups and response to gauging of the symmetry. The second approach is much more physical and in based on the response of the cSPTs to elastic deformations. The advantage of this method is that it also generalises to quasicrystalline SPTs.

[1] H. Song, S.-J. Huang, L. Fu, and M. Hermele, Physical Review X **7** (2017), ISSN 2160-3308, URL `http://dx.doi.org/10.1103/PhysRevX.7.011020`.

[2] S.-J. Huang, H. Song, Y.-P. Huang, and M. Hermele, Physical Review B **96** (2017), ISSN 2469-9969, URL `http://dx.doi.org/10.1103/PhysRevB.96.205106`.

[3] R. Thorngren and D. V. Else, Physical Review X **8** (2018), ISSN 2160-3308, URL `http://dx.doi.org/10.1103/PhysRevX.8.011040`.

[4] D. V. Else and R. Thorngren, Physical Review B **99** (2019), ISSN 2469-9969, URL `http://dx.doi.org/10.1103/PhysRevB.99.115116`.

[5] D. V. Else, S.-J. Huang, A. Prem, and A. Gromov, *Quantum many-body topology of quasicrystals* (2021), 2103.13393.