Braiding Statistics and Classification of Extrinsic Defects in Topological States

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In this note, we review the twist defect and use a simple example to demonstrate the non-Abelian statistics among the twist defects. Then, the classification of the line defect is discussed. This classification relies on the folding process and the criterion of the gapped edge modes.

I. INTRODUCTION

Topological states form a novel set of gapped quantum states which are distinguished by topology instead of symmetry, which implies that adding local perturbation is unable to alter the physical properties of topological states. The robustness in topological state is especially useful in storing and processing quantum information. One special topological state, known as non-Abelian state, has been proposed to be utilized for intrinsically fault-tolerant quantum information storage. The quantum operation can be achieved by adabatically moving quasiparticles around. The braiding statistics between the quasiparticles is non-Abelian. Here, we review an interesting realization of the non-Abelian quasiparticles in the Abelian topological states. These quasiparticles are extrinsic defects. One example of the extrinsic defect is the lattice dislocation in the topological ordered state or the bilayer systems[1–3]. One important distinction between the extrinsic defects and the intrinsic quasiparticles is that the extrinsic defects are confined excitations. Therefore, the energy cost to pull the point defects apart is either logarithmically or linearly proportional to their distance. This suggests that the overall phase of the braiding statistics in the extrinsic defects is not well-defined. Consequently, these extrinsic defects form a projective representation of the braid group. In contrast, the intrinsic quasiparticles form a linear representation of the braid group. This opens a possibility of the novel behaviors in the extrinsic defects. In the rest of this note, we will first introduce a special extrinsic defect, the twist defect and investigate the braiding statistics between the twist defects. Then, we will turn to the classification of the extrinsic defects based on the gapped edge mode.

II. WHAT IS A TWIST DEFECT?

A topological ordered phase is characterized by a set of topologically nontrivial quasiparticles $\{\gamma_i\}$ for $i = 1, \cdots, N_{qp}$ where N_{qp} is the number of quasiparticles. Let's describe the topological property of these quasiparticles. The first is the fusion rule among two quasiparticles (γ_i, γ_j). The idea is to view the combination of these two quasiparticles as a linear combination of single quasiparticle states,

$$\gamma_i \times \gamma_j = \sum_k N_{ij}^k \gamma_k \tag{1}$$

The second is the gained phase $e^{i\theta_{ij}^k}$ when two quasiparticles $\{\gamma_i, \gamma_j\}$ wind around each other. θ_{ij}^k tells us the information about the braid statistics between these quasiparticles. For Abelian topological phase, N_{ij}^k is nonzero for only value of k (see example in Appendix A).

After defining the topological ordered phase in terms of the quasiparticles, we can further define the discrete symmetry g in the topological ordered phase. The operation of the discrete symmetry g is an automorphism which maps a quasiparticle γ_i to another quasiparticle $\gamma_{g(i)}$ as $\gamma_i \to \gamma_{g(i)}$ for $i = 1, \cdots, N_{qp}$. For example, the Z_N topological state (e.g. Z_N Toric code) has N^2 quasiparticles which can be classified into two categories, the electric and the magnetic particles. We can label these quasiparticles as (a, b) where $a, b = 0, \cdots, N-1$. The symmetry for this topological state is $Z_2 \times Z_2$. The first Z_2 represents the exchange of the electric and magnetic particles, $(a, b) \to (b, a)$. Another Z_2 stands for the exchange between the quasiparticles and their conjugate partners, $(a, b) \to (N-a, N-b)$.

Given a topological state with the discrete symmetry g, we are eventually ready to define the twist defect with the symmetry g. The twist defect is a point defect. For convenience, we usually connect two twist defects with a branch cut. The position of the branch cut is simply a gauge choice. When the quasiparticle γ_i winds around the twist defect, the quasiparticle would be mapped into another quasiparticle $\gamma_{g(i)}$. This twist defects can show up at the dislocations in Z_N topologically ordered models and topological nematic states. Note that the symmetry for the twist defect to carry depends on the starting topological state.

III. HOW DO THESE TWIST DEFECTS GIVE NON-ABELIAN BRAID STATISTICS?

A straightforward way to show the braid statistics of the twist defects is to move twist defects around and to investigate the braid statistics. This procedure can be done either in the bulk or from the edge theory [1]. Here, I would like to start with a simple example with twist defects and compute the quantum dimension of the twist defect. The nontrivial quantum dimension would suggest a non-Abelian braid statistic between these twist defects. Consider the case of bilayer decoupled $1/k$ Laughlin fractional quantum hall layers. The Lagrangian is

$$4\pi\mathcal{L} = K_{IJ} a_I da_J \tag{2}$$

where K_{IJ} is diagonal and the entries are both k. The symmetry in this topological state is the exchange of the layer indices. Since there are only two layers, this operation is Z_2. Therefore, the twist defect also belongs to Z_2.

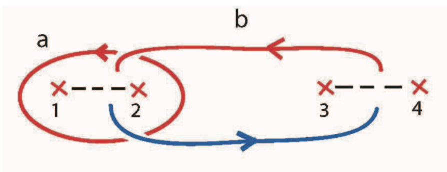

FIG. 1. The two distinct non-contractible loops when there are two pairs of the twist defects [1].

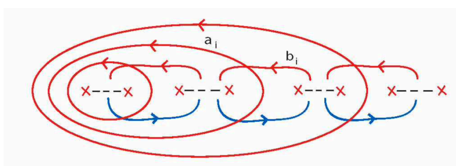

FIG. 2. The $2(n-1)$ distince non-contractible loops when there is n pairs of the twist defects. Here, $n = 4$[1].

We put two pairs of Z_2 twist defects in our system. These two twist defects induce two distinct non-contractible loops (shown in Fig. 1). The red/blue line represents the trajectory of the quasiparticles in the upper/lower layer. Note that these two loops only cross once since the quasiparticles at different layers commute with each other. The braid statistics between these two loops can be computed as

$$W(a)W(b) = W(b)W(a)e^{2\pi i m^T K^{-1} m} = W(b)W(a)e^{\frac{2\pi i}{k}} \tag{3}$$

where $m = (1,0)^T$ since this only involves the quasiparticle in the upper layer. The irreducible representation of this loop algebra is k-dimensional. Given n pairs of twist defects, there would be $2(n-1)$ non-contractible loops (shown in Fig. 2). Thus, we have $(n-1)$ copies of the loop algebra among quasiparticles (Eq. (3)). The full dimension becomes k^{n-1}. Suppose that the quantum dimension of each twist defect is d_t, the full dimension is d_t^{2n}. By taking large n limit and equating k^{n-1} with d_t^{2n}, we can get the quantum dimension as $d_t = \sqrt{k}$. When $k = 2$, the quantum dimension is $\sqrt{2}$. This suggests that a Majarona zero mode is localized at the twist defect. The corresponding braiding statistics is the braiding of Ising anyons.

IV. CLASSIFICATION BASED ON THE EDGE THEORY

For the second half of this note, we will discuss the classification of the general defects [3]. Here, the main question is whether we can understand the defects in the $2+1$ dimensional Abelian topological states. In $2+1$ dimension, there can be two kinds of defects, the line defect and the point defect. In this note, we will only focus on the line defect.

A general line defect is a one dimensional boundary between two topological states, A_1 and A_2. For the branch cut between the twist defects discussed in the previous sections, the two topological states are the same, $A = A_1 = A_2$. Generally, these two topological phases are distinct. For simplicity, we would fold the topological state A_2 onto A_1. The resultant phases are trivial phase and the phase of $A_1 \times \bar{A}_2$ where $\bar{A}_2$ is the parity-reversed of A_2. Note that the folding process flips one of the axis. Since there is no edge state on the trivial phase side, the edge state behavior in the phase of $A_1 \times \bar{A}_2$ can be used to classify the line defects.

A. Classification of Line Defect

We start with the classification of line defect. We only consider line defects that correspond to *gapped boundaries*. How the edges gain mass from the interaction is the main focus here. What is the form of the interaction? The interaction considered here is the backscattering term which stands for the interaction between left and right moving fermions on the edge (Ψ_L, Ψ_R). The fermion parity symmetry is preserved in the interaction. The interaction form is defined as

$$H_{\text{int}} = \sum_i g_i \cos(\Lambda_i^t K \phi) \tag{4}$$

where Λ_i is a integer vector. Suppose there is no further symmetry constraint, when can we gap out the edge mode? This question is answered in Ref. [4, 5]. The first criterion is that the number of left and right moving modes is equal. This criterion can be understood from the mass generation mechanism from backscattering. The backscattering term can only gaps out the left and right moving modes together. Note that the number of left/right moving modes is equal to that of the positive/negative eigenvalues in K matrix. The second criterion is a bit nontrivial. Suppose that there is $2N$ edge modes, the second criterion commands that there exists N linearly independent $\{\Lambda_i\}$ such that $\Lambda_i K \Lambda_j = 0$ for all i, j. In the language of backscattering interaction, we need N terms to gap those $2N$ modes. Appendix B shows two explicit examples ($\nu = 2/3$ and $\nu = 8/9$). The $\nu = 8/9$ edge can be gapped out if the charge conservation is broken. However, the $\nu = 2/3$ edge is protected in the sense that no single backscattering term can open the gap.

So far, we learn the criterions for the edges to be gapped out by the backscattering interaction. Could we understand these criterions in terms of the quasi-particles in the bulk topological state? Ref. [3, 4] answered these questions. The main connection is that each independent vector Λ_i corresponds to a quasi-particle m_i condensed on the boundary. The relation between them is

$$\Lambda_i = c_i K^{-1} m_i \tag{5}$$

where c_i is the minimal integer to make Λ_i become an integer vector. How do we know this m quasi-particle is condensed? This can be seen from the interaction term Eq. (4). The ground state condition pins down the phases to satisfy

$$\Lambda_i^T K \phi = 2\pi n_i \Rightarrow c_i m_i^T \phi = 2\pi n_i \Rightarrow m_i^T \phi = 2\pi \frac{n_i}{c_i} \tag{6}$$

for $n_i \in \mathbb{Z}$. The quasi-particle operator on the edge is given by

$$\chi_{m_i} = e^{i m_i^T \phi} \Rightarrow \langle \chi_{m_i} \rangle = e^{2\pi i \frac{n_i}{c_i}} \neq 0. \tag{7}$$

This suggests that this quasi-particle m_i is condensed on the boundary line. Here, we can see the equivalence between the gapped boundary and the condensed quasi-particles on the boundary.

These condensed quasiparticles form a "Lagrangian subgroup". To define the condensation consistently, this subgroup must satisfy the following conditions:

$$\begin{aligned} &1.\ e^{i\theta_{mm'}} = 1 \text{ for all } m, m' \in M \\ &2.\ e^{i\theta_{ml}} \neq 1 \text{ for at least one } m \in M, \text{ if } l \notin M. \end{aligned} \tag{8}$$

The first condition means that every two particles in M are mutually bosonic. Thus, they can be condensed simultaneously. The second condition commands that all other quasiparticles not in M are confined after the condensation of all quasiparticles in M.

Here are some warnings of the loopholes in the description mentioned above. The first is how we can guarantee that the relation between the quasiparticle m_i in the Lagrangian subgroup and the vector Λ_i (Eq. (5)) is satisfied. The second is whether the order of the Lagrangian subgroup is N since we have N linearly independent Λ_i. Ref. [3] proves a lemma which guaranteed the relation in Eq. (5) by constructing a topologically equivalent K' matrix. The K' matrix is built from enlarging the dimension of K matrix without adding additional topological quasiparticles. The new quasiparticle m_i' can be constructed based how we build the matrix K'. These new m_i' quasiparticles satisfy the relation in Eq. (5). This enlarging process also shows that the number of the left/right moving modes (this number is N in our setup) and the order in the Lagrangian subgroup can be different.

Different Lagrangian subgroups represent different kinds of gapped edge modes on the boundary. Since we only consider the line defect that correspond to the gapped edge, the Lagrangian subgroup can be utilized to classify the line defects.

V. SUMMARY

In this note, we briefly review the braiding statistics and the classification in extrinsic defects. For the braiding statistics in the twist defect, we utilize a simple model to exhibit the non-Abelian nature of these twist defects. The detailed calculations from the bulk or the edge mode picture can be found in [1]. The bulk picture utilizes a useful mapping from the topological state with n defects to a topological state without defect on a genus $g = n$ manifold. Effectively, we can think that the existence of the twist defects creates the genus. Therefore, the twist defect is also called genon. For the classification of line defect, the proof heavily relies on the criterion for gapped edge states. To classify the point defect, further braiding statistics between different Lagrangian subgroups is discussed in [3]. The effect of gauging these defects is also studied in [3].

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[1] M. Barkeshli, C.-M. Jian, and X.-L. Qi, Phys. Rev. B **87**, 045130 (2013).

[2] M. Barkeshli and X.-L. Qi, Phys. Rev. X **2**, 031013 (2012).

[3] M. Barkeshli, C.-M. Jian, and X.-L. Qi, Phys. Rev. B **88**, 235103 (2013).

[4] M. Levin, Phys. Rev. X **3**, 021009 (2013).

[5] F. D. M. Haldane, Phys. Rev. Lett. **74**, 2090 (1995).

[6] A. Kitaev and C. Laumann, "Topological phases and quantum computation," (2009), arXiv:0904.2771 [cond-mat.mes-hall].

Appendix A: Fusion rule for Abelian and non-Abelian quasiparticles in topological state

The Abelian example is the Z_2 toric code [6]. In the Z_2 toric code, there are three kinds of nontrivial particles. They are the electric particle (e), the magnetic particle (m) and the bound state of the electric and magnetic particles (ϵ). The fusion rules are

$$\begin{aligned} & e \times e \rightarrow 1, m \times m \rightarrow 1, e \times m \rightarrow \epsilon \\ & \epsilon \times \epsilon \rightarrow 1, e \times \epsilon \rightarrow m, m \times \epsilon \rightarrow e. \end{aligned} \tag{A1}$$

The non-Abelian example is the Ising anyons. There are two kinds of nontrivial particles, σ and ψ. The fusion rules are

$$\sigma \times \sigma \rightarrow 1 + \psi, \sigma \times \psi \rightarrow \sigma, \psi \times \psi \rightarrow 1. \tag{A2}$$

Appendix B: Examples of $\nu = 8/9$ and $\nu = 2/3$

For $\nu = 8/9$, the K matrix is

$$K = \begin{bmatrix} 1 & 0 \\ 0 & -9 \end{bmatrix}. \tag{B1}$$

Let's try to find the solution of Λ which satisfies $\Lambda^T K \Lambda = 0$. Setting $\Lambda = (a, b)$ gives $a^2 - 9b^2 = 0$. The solution is $\Lambda = (3, 1)$ or $(3, -1)$. and these solutions are valid since all the entries are integer. How would the interaction term look like? Let's plug these two solutions into the Eq. (4),

$$H_{\text{int}} = \cos(3\phi_L \pm 9\phi_R) \equiv \cos(\phi'_L \pm \phi'_R). \tag{B2}$$

The term with the positive sign corresponds to the charge conserving backscattering term such as $\Psi_L^\dagger \Psi_R + h.c.$. For the term with the negative sign, the interaction relates with the pairing in the superconductivity like $\Psi_L^\dagger \Psi_R^\dagger + h.c.$. Note that these two interactions both preserve the fermion parity.

For $\nu = 2/3$, the K matrix is

$$K = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}. \tag{B3}$$

One can show that the solution of Λ contains irrational number, which suggests the solution is not valid. Thus, the condition is not satisfied in $\nu = 2/3$.