Decorated Domain Wall Construction of Symmetry-Protected Phases

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We review the decorated domain wall construction that physically realize a variety of bosonic SPT phases. This explicitly verifies the mathematical classification and low-energy physics which most earlier studies are confined to. More recently, it has been used to describe anomalous SPTs, phases that can only be realized on the boundary of a higher dimensional bulk.

1 Introduction

Symmetry protected topological phases are short-ranged-entangled phases with a unique groud state, but cannot be adiabatically connected to a trivial ground state when the symmetry is enforced. The non-trivial property can be detected when the system has boudnary, where a symmetry-protected gapless state will arise. Fermionic SPTs were discovered and classified first, but later bosonic SPTs are studied, aspects include mathematical classification through group cohomology [1] and low energy physics.

To provide explicit physical systems that realize those bosonic SPT phases, the decorated domain wall construction was proposed [2], focusing on $Z_2 \times G$ SPT phases. Basically, higher dimensional SPTs are constructed by attaching lower-dimensional SPTs on the proliferated domain walls. It allows us to construct and understand higher dimensional SPTs in terms of lower dimensional ones, and provides concrete description of the edge dynamics.

This picture has been generalized to more complicated cases, where one can construct D-dimensional SPT phases that are anomalous, that is they only appear on the boundary of a $D+1$ bulk. Such cases can be detected through consistency requirements among those decorated domain walls are derived [3], and are mathematically described by spectral sequences.

In the following, we present the basic idea of decorating the domain walls in 2-d and 3-d, provide a concrete physical model, and then use it to check the edge dynamics, low-energy theory and group cohomology.

2 Construction of 2-d SPT phases

The physical idea for 2-d SPT phase is shown in Fig. 1. Normally proliferating domain walls create a symmetric ground state. However, by dressing these domain walls with Haldane chains, dangling spin 1/2 appear on the boundary, giving gapless edge states characteristic of SPT.

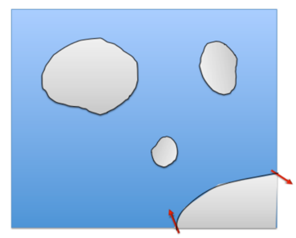

Figure 1: Decorating the domain walls with 1-d SPT (Haldane chains).

A concrete physical model with $Z_2 \times Z_2^T$ symmetry, where Z_2^T represents time reversal symmetry, is shown in Fig. 2. Each plaquette holds a Z_2 variable (the black dot) that flips between $|0\rangle$ and $|1\rangle$ by Z_2, and each vertex holds four spin 1/2 on which Z_2^T acts as $\mathcal{T} = i\sigma_y K$. The Hamiltonian contains a term V that effectively attaches a Haldane chain to each domain walls, by enforcing the two spin-1/2 of neighoring vertices on a Z_2 domain wall to form a singlet, as shown in Fig. 2(c). Finally we proliferate the domain walls to form a unique ground state that

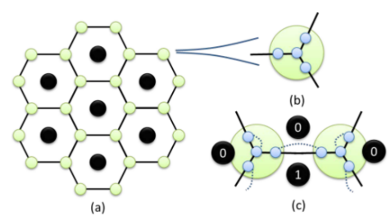

Figure 2: 2-d SPT model with $Z_2 \times Z_2^T$ symmetry.

observes the symmetry,

$$H = uV + \sum_i \tau_x^i \,, \tag{1}$$

where τ_x^i acts on the Z_2 variables.

This construction provides an explicit ground state wave function, which provides an explicit construction of the edge state, as well as a derivation of the effective field theory description and connection to group cohomology.

2.1 Edge state and its effective theory

Fig. 3 shows the edge if the system has a boundary,

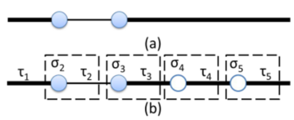

Figure 3: Edge state description.

where thick and thin lines represent Z_2 variables $|0\rangle$ and $|1\rangle$. The solid dots are the dangling spin 1/2 resulting from the boundary cutting through the domain wall and exist only there, which transform projectively. To treat the edge as a spin chain, it's easier to add dummy psuedospins (hollow dots in Fig. 3(b)), that transform differently under time reversal, $\mathcal{T} = \sigma_x K$, forming linear representation. After some manipulation (see Appendix A), the Hamiltonian can be reduced to the XY model,

$$H_e = \sum_i \tau_x^i \sigma_x^{i+1} + \tau_z^i \sigma_z^{i+1} + \sigma_x^i \tau_x^i + \sigma_z^i \tau_z^i \,. \tag{2}$$

The low energy effective theory of the XY model is

$$2\pi \mathcal{L}_{\mathrm{edge}} = \partial_t \phi_1 \partial_x \phi_2 - v \left[\left(\frac{\partial_x \phi_1}{2} \right)^2 + \left(\partial_x \phi_2 \right)^2 \right] \,, \tag{3}$$

and the domain wall creation operator can be shown to be (see Appendix B)

$$D(x) = e^{i\phi_2/2} \,. \tag{4}$$

and transforms as

$$D(x) \xrightarrow{\mathcal{T}^2} -D(x) \,, \tag{5}$$

so that the domain wall is indeed projective representation, as expected from an SPT phase.

2.2 Group cohomology

The boundary of an SPT phase can be classified by group cohomology from how the symmetry acts on it. This is well known, but the benefit of this construction is that we can explicitly check it using the degrees of freedom on the spin chain.

On the boundary, each of the degrees of freedom is $|\alpha_i\rangle$, α a group element and i the index of the site. The symmetry acts on the boundary as

$$O(\alpha) = \prod_i f^\alpha(\alpha_i, \alpha_{i+1}) \, |\alpha\alpha_1, \cdots, \alpha\alpha_N\rangle \,, \tag{6}$$

that is in addition to transforming $\alpha_i \to \alpha\alpha_i$ at each site, there's a phase factor for each pair α_i, α_{i+1} given by the non-trivial 3rd cocyle ω_3,

$$f^\alpha(\alpha_i, \alpha_{i+1}) = \omega_3\left(\alpha_i^{-1}\alpha_{i+1}, \alpha_{i+1}^{-1}\alpha^{-1}, \alpha\right) \,. \tag{7}$$

In the case here, Z_2 does not introduce any phase, whereas Z_2^T introduces a -1 factor if a domain wall is present (that is, a solid dot in Fig. 3 is involved), so

$$f^\alpha(\alpha_i, \alpha_{i+1}) = \begin{cases} \left(-\sigma_z^{i+1}\right)^{\frac{1-\tau_z^i \tau_z^{i+1}}{2}} & \mathcal{T} \text{ is involved} \\ 1 & \mathcal{T} \text{ not involved} \end{cases} \,, \tag{8}$$

which can be reorganized into

$$\omega_3\left(\alpha_i^{-1}\alpha_{i+1}, \alpha_{i+1}^{-1}\alpha^{-1}, \alpha\right) = f^\alpha(\alpha_i, \alpha_{i+1}) \,, \tag{9}$$

which can be checked to be a non-trivial third cocycle of group $Z_2 \times Z_2^T$.

3 Construction of 3d SPT phase

For 3d SPT phase, we again present the basic idea and then provide a concrete model. Consider the symmetry group to be $Z_2 \times \tilde{Z}_2$, where the tilde in the second Z_2 is there only for distinguish it from the first one. This group has 4 elements

$$\{1, g_1, g_2, g_3\} \,, \tag{10}$$

and we consider domain walls of two of those (say g_1 and g_2), which are closed 2-manifolds. The two kinds of manifolds may interset at a 1-d loop, as shown in Fig. 4. The wave function is then defined as

$$\psi_{3\mathrm{d}}(\mathcal{C}) = (-1)^{N_C^{g_1 g_2}} \,, \tag{11}$$

where $N_C^{g_1 g_2}$ is the number of such loops from intersecting g_1 and g_2 domain walls. This amounts to

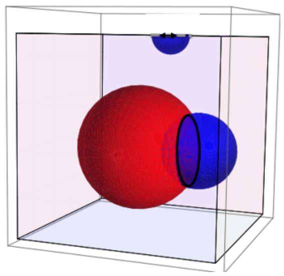

Figure 4: SPT in $d=3$.

putting 2d SPTs on domain walls, since 2d SPTs are characterized by

$$\psi_{2\mathrm{d}}\left(\mathcal{C}\right)=(-1)^{N_{\mathcal{C}}} \ , \tag{12}$$

where $N_{\mathcal{C}}$ is the number of domain walls (now they are 1-d domain walls for 2d SPTs).

The physical model is shown in Fig. 5, where the

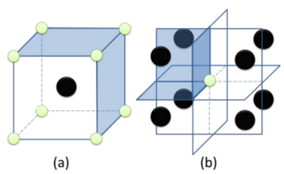

Figure 5: SPT in $d=3$, physical model.

black dots in the cubes are Z_2 variable, vertices are $\tilde{Z}_2$ variables. Just like in the 2-d case, the vertex configuration depends on the Z_2 variables. We first put $+\sigma_x$ for every vertex. Then for vertices living on Z_2 domain walls, an extra factor

$$(-)^{n_{\mathrm{dwp}}} \tag{13}$$

is given, where n_{dwp} is the number of $\tilde{Z}_2$ domain wall pairs long the loop of all nearest neighbor $\tilde{Z}_2$ spins on the same Z_2 domain wall as the original $\tilde{Z}_2$ spin. This operation effectively puts a $\tilde{Z}_2$ SPTs on Z_2 domain walls.

Similar to 2-d cases, we can also explicitly check the edge dynamics, low energy effective theory, and the group cohomology classification.

4 Obstructions and Anomalous SPTs

Recently, more elaborate domain wall decorations have been investigated, which include junctions of domain walls [3], such as in Fig. 6. Each domain wall segments and junctions are characterized by group

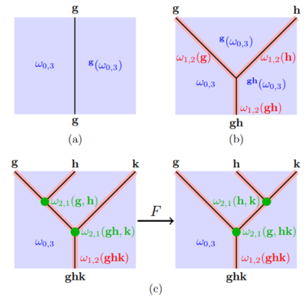

Figure 6: Domain wall obstructions.

cohomology, and consistency relations can be derived that those domain walls must satisfy, otherwise they cannot be realized in a strictly d-dimensional space but must appear on the boundary of a higher dimensional bulk.

5 Conclusions

We reviewed an explicit way to construct of 2d and 3d SPT wave functions, where concrete computation can be made, such as the edge dynamics, low energy effective theory, and group cohomology classification. We also point to an interesting recent direction where such construction provides a way to classify anomalous SPTs that can only apppear on the boundary of a higher-dimensional bulk.

Acknowledgement

We thank Professor McGreevy for the intriguing lectures and this fascinating topic. It's very unfortunate that we could not delve deeper into and do justice to it, but it's certainly a nice direction to read more about.

References

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[2] Xie Chen, Yuan-Ming Lu, and Ashvin Vishwanath. Symmetry-protected topological phases from decorated domain walls. *Nature communications*, 5(1):1–11, 2014.

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A The edge state Hamiltonian

To construct the Hamiltonian for the edge dynamics, we first see how the symmetries act on the edge,

$$\begin{aligned} Z_2 &: \prod_i \tau_x^i \\ Z_2^T &: \prod_{i,i+1} \left(\frac{\mathbb{I} + \tau_z^i \tau_z^{i+1}}{2} \sigma_x^{i+1} + \frac{\mathbb{I} - \tau_z^i \tau_z^{i+1}}{2} i\sigma_y^{i+1} \right) , \end{aligned} \tag{14}$$

since Z_2 only acts on Z_2 variables, while $\mathcal{T}$ is either σ_x or $i\sigma_y$ depending on whether its an actual dangling spin 1/2 or a pseudospin. The Hamiltonian must preserve the symmetry. Allowed terms are $\tau_x^i + \sigma_z^i \tau_x^i \sigma_z^{i+1}$ and $\sigma_x^i + \tau_z^{i-1} \sigma_x^i \tau_z^i$, so a possible Hamiltonian is

$$H_e = \sum_i \tau_x^i + \sigma_z^i \tau_x^i \sigma_z^{i+1} + \sigma_x^i + \tau_z^{i-1} \sigma_x^i \tau_z^i \,, \tag{15}$$

which reduces to the XY model after a unitary transformation

$$U = \left[\prod_n i \frac{\tau_z^n + \tau_x^n}{\sqrt{2}} \right] \cdot \left[\prod_n (-1)^{(1-\tau_z^n)(1-\sigma_z^n)/4} \right] . \tag{16}$$

B Domain wall creation operator for the effective theory

The low energy effective theory of the XY model is

$$2\pi \mathcal{L}_{\text{edge}} = \partial_t \phi_1 \partial_x \phi_2 - v \left[\left(\frac{\partial_x \phi_1}{2} \right)^2 + (\partial_x \phi_2)^2 \right] , \tag{17}$$

with

$$\left[\phi_1 (x) , \frac{\partial_{x'} \phi_2 (x')}{2\pi} \right] = i\delta (x - x') \ , \tag{18}$$

and the symmetry acts on the field as

$$\begin{aligned} Z_2 : &\phi_1 \to \phi_1 + \pi \\ &\phi_2 \to \phi_2 \\ Z_2^T : &\phi_1 \to -\phi_1 \\ &\phi_2 \to \phi_2 + \pi \,. \end{aligned} \tag{19}$$

The operator that creates a domain wall should thus rotate ϕ_1 by π everywhere to the right of the domain wall, and the one that does this is

$$D(x) = e^{i\phi_2/2} \,. \tag{20}$$

To check whether it is a projective representation, we see that

$$D(x) \xrightarrow{\mathcal{T}} e^{-i(\phi_2+\pi)/2} = -iD^*(x) \ , \tag{21}$$

so that

$$D(x) \xrightarrow{\mathcal{T}^2} -D(x) \ , \tag{22}$$