Anomaly matching in (continous) symmetry breaking phases

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This note reviews the consequences of 't Hooft anomaly and the anomaly matching in the continuous symmetry breaking phases. The anomaly in the ultraviolet theory is matched by Goldstone bosons with Wess-Zumino-Witten term and other (anomalous) low energy degrees of freedom. The anomaly matching in the discrete symmetry breaking phases leaves for future investigation.

INTRODUCTION

The 't Hooft anomaly for global symmetry G of a quantum field theory constrains the infrared phases to be

1. gapless with G symmetry
2. spontaneously symmetry breaking
3. topological order

but never a trivial gapped phase. The theory with 't Hooft anomaly is dubbed as "anomalous theory $\mathcal{T}$". When evaluating the partition function of such theory on a closed d-dimension manifold X, it yields an element in 1-dimension Hilbert space but not a complex number,

$$\mathcal{Z}_{\mathcal{T}}(X) \in \mathcal{H}(X)^*, \notin \mathbb{C}. \tag{0.1}$$

while nonanomalous theory on X takes value in $\mathbb{C}$. In other words, when performing a gauge transformation on the background gauge field which is associated to the global symmetry G, the partition function of the anomalous theory on X becomes,

$$\mathcal{Z}_{\mathcal{T}}[A + \delta_\theta A] \to \mathcal{Z}_{\mathcal{T}}[A] e^{\mathbf{i} S[\theta, A]}. \tag{0.2}$$

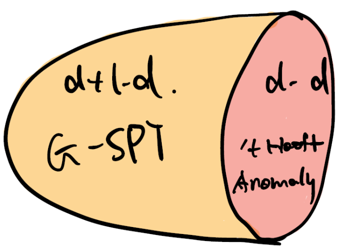
Namely, the phase is not canonically determined, thus we cannot regard it in $\mathbb{C}$ though it is 1-dimensional. On the other hand, the partition function of an invertible field theory $\mathcal{I}$ on $d+1$-manifold Y with boundary $\partial Y = X$ yields a vector in $\mathcal{H}(X)$,

$$\mathcal{Z}_{\mathcal{I}}(Y) \in \mathcal{H}(X). \tag{0.3}$$

When gluing the anomalous theory $\mathcal{T}$ in d and the invertible theory $\mathcal{I}$ in $d+1$, we can define a $\mathbb{C}$-valued partition function,

$$\mathcal{Z}_{\mathcal{T}}(X) \cdot \mathcal{Z}_{\mathcal{I}}(Y) \in \mathcal{H}(X)^* \otimes \mathcal{H}(X) = \mathbb{C} \tag{0.4}$$

and the pictoral description is,



The 't Hooft anomaly is a property of the Hilbert space, therefore, it should be matched in the ultraviolet and the infrared theory. From UV to IR, a common scenario is the spontaneously symmetry breaking. Suppose the UV theory $\mathcal{T}_{UV}$ has the global symmetry G, and it is spontaneously broken down to H in the IR, the IR theory contains [the $\mathcal{T}_{IR}$ with the unbroken H symmetry] and [Goldstone bosons U lives in the coset G/H],

$$\begin{array}{ll} \mathcal{T}_{UV}\ d \text{ anomalous theory} & \sim \mathcal{I} \text{ in } d+1 \\ \downarrow \text{ spontaneously symmetry breaking,} & \\ \mathcal{T}_{IR} + U & \end{array}$$

The $\mathcal{T}_{IR}$ together with U should match the anomaly in $\mathcal{T}_{UV}$.

The chiral anomaly is an example that the local (perturbative) anomaly which can be seen from the triangle diagram is known to be matched by the Goldstone boson with Wess-Zumino-Witten term [1, 2]. However, the global (non-perturbative) anomaly is more subtle and need proper definition for the WZW term, .

GENERAL WZW TERMS AND THE ANOMALY MATCHING

Assuming the d-dimensional UV theory has an anomaly described by $d+1$-dimension invertible theory $\mathcal{I}$, and the symmetry G is spontaneously broken to H in the IR, Ref. [3] defines the general WZW term associated to $\mathcal{I}$ so that the anomalies of UV and IR are matched.

The symmetry G could contain both the internal symmetry and the spacetime symmetry. When spontaneously breaking down to H, the Goldstone boson lives in the coset G/H, and locally, the Goldstone boson takes value in G and has a **gauge** symmetry $\hat{H}$. The $\hat{H}$ can be identical or cover of H. The Goldstone boson transforms as,

$$\hat{H}: \ U \to U\hat{h}, \quad G: \ U \to g^{-1}U \tag{0.5}$$

where $\hat{h} \in \hat{H}, g \in G$. Consider the connection A on the principal G-bundle,

$$A \to g^{-1}Ag + g^{-1}dg, \tag{0.6}$$

and define

$$A_U = U^{-1}AU + U^{-1}dU \tag{0.7}$$

which transforms invariant under G, but under $\hat{H}$,

$$A_U \to \hat{h}^{-1} A_U \hat{h} + \hat{h}^{-1} d\hat{h}. \tag{0.8}$$

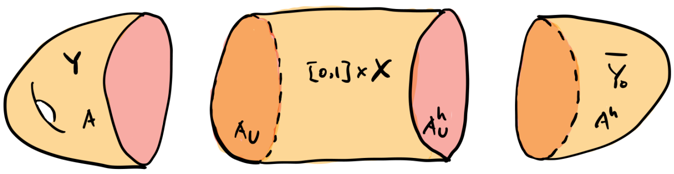
Therefore, A_U is the connection of the principal $\hat{H}$-bundle. We can decompose the gauge connection as,

$$A_U = A_U^{\mathfrak{h}} + (A_U - A_U^{\mathfrak{h}}) \equiv A_U^{\mathfrak{h}} + A_U^{\mathfrak{f}} \tag{0.9}$$

where $A_U^{\mathfrak{h}}$ is the connection of the principal $\hat{H}$-bundle and $A_U^{\mathfrak{f}}$ transforms homogeneously as $A_U^{\mathfrak{f}} \to \hat{h}^{-1} A_U^{\mathfrak{f}} \hat{h}$. It makes mathematical sense to consider a one-parameter family of gauge fields A_U^t,

$$A_U^t = A_U^{\mathfrak{h}} + (1-t) A_U^{\mathfrak{f}} = \begin{cases} A_U & t = 0 \\ A_U^{\mathfrak{h}} & t = 1 \end{cases} \tag{0.10}$$

The WZW term is constructed as,



1. The background gauge field A is extended to $Y, \partial Y = X$. Y may not exist.

2. Consider the manifold $[0,1] \times X$ with interpolation of the gauge field A_U^t, such that $A_U^{t=0} = A_U, A_U^{t=1} = A_U^{\mathfrak{h}}$. It can be shown that the construction does not depend on the choice of interpolation.

3. Take $Y_0, \partial Y_0 = X$ and extend $A_U^{\mathfrak{h}}$ to gauge field of $\hat{H}$ on Y_0. Y_0 may not exist.

4. Glue the three parts togother and form the closed manifold

$$Y_{total} = Y \cup ([0,1] \times X) \cup \bar{Y}_0, \tag{0.11}$$

and U is the transition function between Y and X, $A \to A_U$.

Therefore, the WZW term is defined as,

$$\Phi_{WZW} \equiv \mathcal{Z}_{\mathcal{I}}(Y_{total}) \in \mathsf{U}(1) \tag{0.12}$$

It is possible that the IR theory contains 't Hooft anomaly in H which is characterized by the invertible theory with only $\hat{H}$-bundle, the IR partition is defined with the use of Y_0,

$$\mathcal{Z}_{IR} \equiv \mathcal{Z}_{\mathcal{T}_{IR}}(X) \mathcal{Z}_{\mathcal{I}}(Y_0) \in \mathbb{C} \tag{0.13}$$

where $\mathcal{Z}_{\mathcal{T}_{IR}}(X) \in \mathcal{H}(X)^*$ is the partition function of $\mathcal{T}_{IR}$ which is coupled to $A_U^{\mathfrak{h}}$.

The properties of the WZW term is

1. Although Φ_{WZW} and $\mathcal{Z}_{IR}$ may depend on Y_0, their product $\Phi_{WZW}\mathcal{Z}_{IR}$ is independent of the choice of Y_0. By definition, the partition function of Y_0 is cancelled when taking the product.

   $$\begin{aligned} &\Phi_{WZW}\mathcal{Z}_{IR} \\ =&\mathcal{Z}_{\mathcal{I}}(Y) \cdot \mathcal{Z}_{\mathcal{I}}([0,1] \times X) \cdot \mathcal{Z}_{\mathcal{I}}(\bar{Y_0}) \mathcal{Z}_{\mathcal{T}_{IR}}(X) \mathcal{Z}_{\mathcal{I}}(Y_0) \\ =&...|\mathcal{Z}_{\mathcal{I}}(Y_0)|^2 = ... \times 1 \end{aligned}$$

   If IR theory is free from the anomaly, the WZW term is independent of Y_0.

2. Since U is only defined on X, the WZW term depends only on d-dimensional configuration of U.

3. WZW term depends on Y and the background gauge field A on it.

The last point coincides with the understanding of the symmetry protected topological phases, the definition of the WZW term depends on the manifold Y, but the dependence is only through the background gauge fields and this dependence is the anomaly of the corresponding symmetries.

The **local (perturbative) anomaly** represented by anomaly polynomial in 2 higher $d+2$ dimension can be reproduced by usual WZW term in non-linear sigma model [1, 4],

$$\Phi_{WZW}(U, A) = \exp\left(2\pi \mathbf{i} \int_Y \hat{A}(R) I(A_U)\right) \tag{0.14}$$

where $\hat{A}(R)$ is the Dirac genus, R is the Riemann curvature, and $I(A)$ is defined as,

$$I(A) = \int_0^1 dt \ \mathsf{tr}\left(\frac{\mathbf{i}}{2\pi} A \exp\left(\frac{\mathbf{i}}{2\pi}(tdA + t^2 A^2)\right)\right), \tag{0.15}$$

and when the background gauge field is 0, the WZW term reduces to the usual one [2],

$$\Phi_{WZW}(U, A) = \exp\left(2\pi \mathbf{i} \int_Y \hat{A}(R) I(U^{-1} dU)\right). \tag{0.16}$$

The WZW term reproduces the anomaly polynomial $\alpha(A)$ (omit gravitional part) by,

$$d\Phi_{WZW}(U, A) = d\mathsf{CS}(A) = \alpha(A) \tag{0.17}$$

However, the dependence on Y is more fundamental when we consider **global (non-perturbative) anomalies**.

TOPOLOGICAL θ ANGLE

The construction described in the previous section has several topological issues.

1. Whether the G-bundle can be reduced to $\hat{H}$-bundle. **Claim:** The partition function of the Goldstone bosons will be zero or very small when G-bundle cannot reduced to $\hat{H}$-bundle.

Suppose in the UV $\mathsf{U}(N_f)_L$ and $\mathsf{U}(N_f)_R$ bundle are topologically different. They are coupled to scalar S^i_j, then $\det S$ must vanish somewhere, otherwise S gives a bundle isomorphism which contradicts with the two bundles are different. The scalar S will spontaneously break $\mathsf{U}(N_f)_L \times \mathsf{U}(N_f)_R \to \mathsf{U}(N_f)_{diag}$. Then the point S vanishes will have larger energy thus the partition function will be small.

Those "topological solitons" in which some fields of the UV theory (such as the S above) go outside the effective field theory of Goldstone bosons have interesting anomalies.

2. Whether the G-bundle and $\hat{H}$-bundle on X can be extended to Y and Y_0. The absence of such extension implies the possibility of topological θ-angle of the groups. The general understanding coincides with the classification of the symmetry protected topological phases.

For the symmetry and structure we are interested in, one can compute the bordism group Ω^G_d, which characterizes the distinct d-manifold with symmetry G and other structures that can be realized as the boundary of $d+1$-dimensional manifold. The bordism group $\Omega^G_d = \mathbb{Z} \oplus ... \oplus \mathbb{Z}_n \oplus ...$, contains the free part $(\mathbb{Z}, \mathbb{R})$ and the torsion part $(\mathbb{Z}_n)$, the free parts correspond to various continuous θ-angles (instanton number), while the discrete parts correspond to the discrete θ-angle ($\mathsf{SU}(2)$ anomaly [5]).

3. The situation when $\hat{H}$-bundle is nontrivial while G-bundle is trivial. In other word, Y exists while Y_0 does not. One example is SO QCD_4, where the global fermion flavor symmetry $\mathsf{SU}(N_f)$ acting on N_f Weyl fermions will spontaneously breaks down to $\mathsf{SO}(N_f)$ for some N_f and N_c (the gauge color). The low enengy degrees of freedom are Goldstone bosons in G/H. The torsion part of the bordism group of the $SO(N_f)$ bundle is non-trivial and corresponds to discrete θ-angle relevant to the sign of Φ_{WZW}. The generalized θ-angle of the $\hat{H}$-bundle can be regarded as a topological term of the Goldstone bosons.

FINAL REMARK

The previous understanding of Wess-Zumino-Witten term is that the configuration of the Goldstone boson at ∞ should be the same, therefore, one can one-point compactify the spacetime manifold, yields $S^d \to G/H$, the WZW term exists when $\pi_{d+1}(G/H) = \mathbb{Z}, \pi_d(G/H) =$.

The understanding of the WZW term is updated by the study of symmetry protected topological phases. The WZW term can now be defined on more general manifold and match subtler anomaly, interesting examples are shown in Ref. [6–8].

I cannot resist to mention the 2+1d $\mathbb{C}P^1$ model with Hopf term. Based on the homotopy argument $\pi_3(\mathbb{C}P^1) = \mathbb{Z}$, the Hopf term seems to be $\mathbb{Z}$-valued,

$$\mathbf{i}\theta\mathrm{Hopf}[n],\ n \in \mathbb{C}P^1 \cong S^2 \tag{0.18}$$

however, the more sophisticated bordism group calculation shows [7, 8],

- $\mathsf{U}(1)$ factors in $\mathsf{Hom}(\Omega_3^{spin}(\mathbb{C}P^1), \mathsf{U}(1)) = 0$ classifies the ordinary theta angle,
- $\mathsf{Ext}(\Omega_3^{spin}(\mathbb{C}P^1), \mathbb{Z}) \cong \mathsf{Tors}(\Omega_3^{spin}(\mathbb{C}P^1)) = \mathbb{Z}_2$ classifies the discrete theta angles,
- $\mathsf{Hom}(\Omega_4^{spin}(\mathbb{C}P^1), \mathbb{Z}) \cong \mathsf{Free}(\Omega_4^{spin}(\mathbb{C}P^1)) = \mathbb{Z}$ classifies the Chern-Simons or Wess-Zumino terms.

The free part of $\Omega_4^{spin}(\mathbb{C}P^1)$ is the gravitational anomaly, the Hopf term corresponds to the Torsion part of $\Omega_3^{spin}(\mathbb{C}P^1)$ and that should be $\mathbb{Z}_2$-valued. Namely, only $\theta = 0, \pi$ is consistent as local and unitary QFTs and $k = \theta/\pi$ behaves as the level-k Chern-Simons theory.

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