Anomalies

Zhengdi Sun[*,i]

[*]*Department of Physics, University of California, San Diego, La Jolla, CA 92093 USA*
[†]*Department of Physic*

In this note, we give a brief introduction on the anomalies in the space of coupling constants. Part of the note contains my own opinions, which should be read with caution. The author thanks John McGreevy and Dachuan Lu for helpful discussion.

June 2021

[i] z5sun@ucsd.edu

In this note, we will give a brief introduction on the anomalies in the space of coupling constants following [1]. I should probably give the warning that this note contains some of my own understanding which I have yet to discuss with other people. Therefore, you should read it with caution. Feedback and discussions are extremely welcome.

Many Lagrangian theories have the properties that transforming the coupling constant in a certain way leaves the theory invariant. For instance, the θ angle in $3+1$-dim Maxwell theory is invariant under $\theta \to \theta + 2\pi$ when placing the theory on a spin manifold. Those transformations leave the theory invariant naturally forms a group G_0, and therefore the actual space of coupling constants should be the naive space of coupling constants X_{naive} modulo the action of group G_0: $X_{\text{actual}} = X_{\text{naive}}/G_0$. In the example of the θ-angle, $X_{\text{naive}} = \mathbb{R}$, $G_0 = \mathbb{Z}$, $X_{\text{actual}} = \mathbb{R}/\mathbb{Z} = S^1$.

Having seen so many examples of anomalies, it is natural to ask if such identification under G action can fail. Indeed, in the above case of θ angle, if we couple the theory to some background 1-form $\mathbb{Z}_2$ gauge field for the $\mathbb{Z}_2 \subset U(1)$ 1-form symmetry, then the theory can have fractional instanton number, thus the identification of $\theta \to \theta + 2\pi$ will fail. This resembles the phenomenon of the mixed 't Hooft anomalies, i.e., the identifications on the space of coupling constants fail when coupling the theory to background gauge fields.

The similarity with mixed 't Hooft anomalies motivate another way to see those anomalies, that is, we can promote the coupling constant to be space-time dependent and perhaps with some defects or singularities. This amounts to change the presentation of mixed 't Hooft anomalies, and as a result, we will find out the theory is not invariant under the gauge transformation of the background gauge field for the involved global symmetries. This would also imply the singularities may trap some non-trivial theory (or zero modes) so soak up the anomaly.

Just as 't Hooft anomalies impose constraints on the low energy theories; the anomalies in the space of coupling constants will have dynamical consequences which will help us understand the low energy theory.

From the effective theory point of view, there are infinitely amount of couplings one can add to the theory. Then, a natural question to ask is how do we know we've already found all the anomalous coupling constants. As far as the known example shows, there are three classes of examples.

1. Symmetry breaking couplings. In this case, the coupling we add explicitly breaks some symmetry G_0 in the theory and we should view the coupling constant as the spurion of G_0. Then, the coupling constant anomaly is just a result of the mixed 't Hooft anomaly between G_0 and some other global symmetry G_1. Thus, to find all anomalous couplings in this class we simply need to find all mixed 't Hooft anomalies of the global symmetries.

2. (-1)-form symmetry. In this case, there exists a top form W with integral period. This implies the theory has topological sector labelled by $\int W \in \mathbb{Z}$. Hence, we can include $e^{i\theta \int W}$ in the path integral. θ is periodic in 2π and should be view as $U(1)$ rotation parameter

for the (-1)-form global symmetry. The identification of $\theta \to \theta + 2\pi$ can clash with the background fields of other global symmetries. Because we've already specified the field contents in the EFT approach, it should not be hard to enumerate all the possibilities (unless someone is considering some super complicated quiver gauge theory?).

3. Anomalies involving local Weyl transformation and local renormalization group flow. When coupling constants have anomalies, we can either see the anomaly through turning on background fields for global symmetries; or we can see the anomaly by promoting coupling constants to be space-time dependent. However, there seems to be an exception. We can study the anomaly between Weyl symmetry and coupling constants [2]. In this case, it's not obvious to me how to see the anomaly without promoting the marginal coupling to be space-time dependent. More broadly, this is an example of local renormalization group flow [3]–we promote the coupling constants to be spacetime dependent and study the variation of the partition function under the infinitesimal Weyl transformation. This case seems to be more intricate and I don't have a good understanding on this.

1. Couplings from spurion

For this class, we start with a simple example of Weyl fermion ψ in $3+1$-dim. As we all known, the fermion has an anomalous chiral $U(1)$ symmetry, which has a mixed 't Hooft anomaly with gravity. The anomaly polynomial is given by

$$I_6 = \frac{1}{24}\int c_1(A) \wedge p_1(X),$$

which implies under the chiral $U(1)$ rotation $e^{i\phi}$:

$$Z[g_{\mu\nu}] \to e^{i\phi \int_X \frac{p_1(X)}{24}} Z[g_{\mu\nu}].$$

We can add a complex mass m to the theory: $m\psi\psi + c.c.$. Notice that $\psi\psi$ has charge 2 under the chiral rotation, therefore the mass deformation breaks the $U(1)$ chiral symmetry. However, we can view m as a spurion of charge -2, hence the theory with $me^{i\phi}$ is the same as m, as we can do a $U(1)$ chiral rotation $e^{-i\phi/2}$ to soak up the $e^{i\phi}$. However, this is not quite true if we place the theory on a curved manifold–in this case, the partition function will pick up a phase under the chiral rotation due to the mixed 't Hooft anomaly:

$$Z[me^{i\phi}, g_{\mu\nu}] = e^{-i\phi \int_X \frac{p_1(X)}{48}} Z[m, g_{\mu\nu}].$$

This is indeed the anomaly in the space of coupling constant discovered in [1].

We can take a step further and interpret the mass m as the vev of some charge (-2) boson field ϕ, which couples to Weyl fermion via $\phi\psi\psi + c.c.$. Then, in this case, the $U(1)$ chiral symmetry is restored but spontaneously broken. As we learned in the lecture, the theory can contain string defect of boson winding around vacuum manifold in spacetime; and there will be fermion zero mode trapped in the 2d string, which can be understood from the anomaly inflow of the above mixed 't Hooft anomaly. This is exactly the second dynamical consequence derived in [1].

Generalizing the lesson where we should couple the theory to the background field whenever we see a global symmetry, here we learn that whenever we see a coupling, we should think it as a vev of some bosons.

Another example given in [1] is 3d real fermion. The real mass m has a coupling anomaly as the partition function pick up a gravitational CS term as $m \to -m$. This is the just manifestation of mixed 't Hooft anomaly between time reversal symmetry and the gravity.

2. (-1)-form symmetries

Again let's start with a simple, well-known example: particle on a ring. For reader's convenience, we write down the action:

$$S = \frac{1}{2}\int d\tau \dot{q}^2 - \frac{i}{2\pi}\int d\tau \theta \dot{q},$$

where q is a compact scalar $q \simeq q + 2\pi$. Since $\int dq$ is quantized, θ is then a periodic variable $\theta \simeq \theta + 2\pi$. The theory has a coupling constant anomaly. We can couple the theory to a background gauge field A_τ of the $U(1)$ rotation on the target space S^1. The generic action is then given by

$$S = \frac{1}{2}\int d\tau (\dot{q} - A_\tau)^2 - \frac{i}{2\pi}\int d\tau \theta(\dot{q} - A_\tau) + k\int d\tau A_\tau, \quad k \in \mathbb{Z}.$$

The last term is the 1-dim CS term and k is the quantized level. Shifting θ by 2π no longer leave the action invariant, instead, it shifts the CS level k by 1.

Why there is a (-1)-form symmetry in the theory? $dq = d\tau\dot{q}$ as a 1-form, hence $d*(*dq) = ddq = 0$ and we should interpret $*dq$ as a conserved current. In general, $(n+1)$-form currents correspond to n-form symmetries; so in this case, since $*dq$ is a 0-form, we should interpret it corresponds to some (-1)-form symmetries. Since $\int d\tau\dot{q}$ is quantized, the theory should have topological sectors labelled by integers $n = \int d\tau\dot{q}$. So the (-1)-form symmetry should be $U(1)$ and the effect of θ-term is to weight the topological sectors by $e^{in\theta}$.

There is a similar story in 4d gauge theory. As we known, the theory has a θ-term $e^{i\frac{\theta}{4\pi^2}\int \mathrm{tr}F\wedge F}$. The instanton number $\frac{1}{4\pi^2}\int \mathrm{Tr}F\wedge F$ is quantized, hence θ is periodic with period 2π. If the gauge group G has a non-trivial center C, then it has a 1-form symmetry C and we can consider couple it to background gauge field. This amounts to sum over G/C bundle with a specified 2nd Stiefel Whitney class. This leads to fractional instanton number, hence θ fails to be 2π-periodic.

In this case, we can view $\frac{1}{4\pi^2} * \mathrm{Tr}F\wedge F$ as the current for the (-1)-form symmetry.

Unfortunately I am tempted by the idea unifying this class with the spurion case. In particular, since shifting θ by 2π is the famous T-duality, which forms a group $\mathbb{Z}$. It is tempting to view θ as a spurion under the T-duality group $\mathbb{Z}$, which transform non-linearly as $\theta \mapsto \theta + 2n\pi$, $n \in \mathbb{Z}$.

The T-duality transformation is generated by a unitary operator U. In the particle on a ring, $U = e^{iq}$ and in the 4d gauge theory case, $U = e^{i\int CS[A]}$. As one can show,

$$U^\dagger H_\theta U = H_{\theta+2}$$

in both cases (see [4] [5]). And the signature of the anomaly is U transform non-trivially under the global symmetry.

The problem is, however, the duality group is really not a symmetry since none of them really commute with Hamiltonian.

References

[1] C. Córdova, D. S. Freed, H. T. Lam & N. Seiberg, *"Anomalies in the Space of Coupling Constants and Their Dynamical Applications I"*, SciPost Phys. **8**, 001 (2020), `arXiv:1905.09315 [hep-th]`.

[2] J. Gomis, P.-S. Hsin, Z. Komargodski, A. Schwimmer, N. Seiberg & S. Theisen, *"Anomalies, Conformal Manifolds, and Spheres"*, JHEP **1603**, 022 (2016), `arXiv:1509.08511 [hep-th]`.

[3] H. Osborn, *"Local Renormalisation Group Equations in Quantum Field Theory"* .

[4] D. Gaiotto, A. Kapustin, Z. Komargodski & N. Seiberg, *"Theta, Time Reversal, and Temperature"*, JHEP **1705**, 091 (2017), `arXiv:1703.00501 [hep-th]`.

[5] D. Tong, *"Gauge theory"*, Lecture notes **1705**, D. Tong (2018).