Introduction to Topological Non-linear σ-model

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This paper is a simple introduction to topological non-linear σ-model, which can be used to describe bosonic topological ordered phases. We will first introduce the construction of the topological non-linear σ-model, then we will present the 1-gauge theory as an example of σ-model. Finally, as an application, we will use this formalism to describe gravitational anomaly in $2+1D$ bosonic topological order.

I. INTRODUCTION

Topological ordered phase is an active aspect of research of topological phases of matter. It's very intriguing to find a universal topological invariant quantity that can characterize topological ordered phases. For bosonic topological orders, it was proposed that such a topological character might be topological partition function of non-linear σ-model.

In this paper, we will briefly introduce non-linear σ-model in topological partition function formalism, and then present 1-gauge theory as an example. Finally we will discuss the perturbative gravitational anomaly in terms of the topological partition function formalism.

II. TOPOLOGICAL NON-LINEAR σ-MODEL

Consider a bosonic system defined in $D=d+1$ spacetime manifld M^D, which is described by the partition function in terms of non-linear σ-model:

$$Z=\int \mathcal{D}\phi(x)e^{-\int_{M_D}\mathcal{L}}, \tag{1}$$

where $\mathcal{L}=\mathcal{L}(\phi,\partial\phi)$ and ϕ is the map from spacetime manifold to target space: $\phi: M_D\to K$. The topological partition function can be constructed by factoring out the volume term, which isn't topological[1,2]:

$$Z=e^{-\int\epsilon}\cdot Z^{\text{top}}, \tag{2}$$

where $\epsilon(x)$ is the energy density.

There is still another important ingredient in the definition of the partition function that need to be defined carefully, namely, the measure $\int\mathcal{D}\phi$, which, roughly speaking, can be understood as $\sum_{\phi(x)}$. In order to define it formally, we need to first decompose the M^{d+1} spacetime manifold to a set of $n-$simplex $M_n\subset M_D, n=0,1,2,\cdots$, together with the face map $d_i\subset\mathcal{D}$, which describes how the $n-1$-simplices form the n-simplex:

$$(\mathcal{M}_D,\mathcal{D}_M): M_0 \xleftarrow{d_0,d_1} M_1 \xleftarrow{d_0,d_1,d_2} \cdots \xleftarrow{d_0,d_1,\cdots,d_D} M_D \tag{3}$$

We can do the similar thing for the target space $K\to(\mathcal{K},\mathcal{D}_K)$. Then ϕ is now $\phi:\mathcal{M}_D\to\mathcal{K}$ is a set of maps $\phi_0: M_0\to K_0$, $\phi_1: M_1\to K_1,\cdots$, that also preserve the corresponding face maps. Therefore ϕ can be considered as a set of homomorphism $\phi:(\mathcal{M}_D,\mathcal{D}_M)\to(\mathcal{K},\mathcal{D}_K)$. Then integration measure $\int\mathcal{D}\phi(x)$ can be understand as summation over the set of homomorphism

$$\int\mathcal{D}\phi(x)=\sum_\phi. \tag{4}$$

To be more precise, denote the simplex in K as $v_i, l_{ij}, t_{ijk},\cdots$ for points, links, triangles, etc. Then

$$\phi_0: i\in\to v_i, \phi_1:(ij)\to l_{ij}, \phi_2:(ijk)\to t_{ijk},\cdots, \tag{5}$$

where $i\in M_0,(ij)\in M_1,(ijk)\in M_2,\cdots$. Therefore, the measure can be written out explicitly

$$\int\mathcal{D}\phi=\sum_\phi=\sum_{v_i,l_{ij},t_{ijk},\cdots}. \tag{6}$$

Finally, the partition function Eq.(1) is formally defined.

With the formally defined partition function, we can define the non-linear σ-model as

$$Z(\mathcal{M}_D;\mathcal{K},\tilde{\omega}_D)=\sum_\phi e^{2\pi i\int\phi^*\tilde{\omega}_D}, \tag{7}$$

where $\tilde{\omega}_D$ is a real-valued D-cochain in $\mathcal{K}$, and ϕ^* is the pull-back, then $\phi^*\tilde{\omega}_D=\omega_D$ is a real-valued D-cochain on $\mathcal{M}_D$.

A comment: Actually, different triangulations $\mathcal{T}$ of M_D might give different phases. Therefore, we need to choose a special $(\bar{\omega}_D,\bar{\mathcal{K}})$, such that different $\mathcal{T}$ will give the same phases under the fine triangulation limit. However, it still might be problematic, as we will discuss in section IV.

III. 1-GAUGE THEORY

In this section, we will introduce an example of topological non-linear σ-model, namely, 1-gauge theory (Dijkgraaf-Witten gauge theory).

Consider a non-linear σmodel of G with target space K, that satisfies $\pi_1(K)=G,\pi_{k>1}(K)=0$. This is a 1-gauge theory. Denote $K(G)=BG$, as a simplical set. In order to define a topological partition function, we need to first triangulate $BG\to\mathcal{B}G$. In this case, we can choose $(\mathcal{B}G)_0=\{p_0\}$, where p_0 is the base point in BG. Then the links are loops through p_0, and $a_{ij}=$

$g \in G = \pi_1(BG) = (\mathcal{B}G)_1$. The higher set of simplices are $(\mathcal{B}G)_n = G^n$. Finally, we can obtain the partition function

$$Z = \sum_{\phi} \left[\prod_{n=0}^{d+1} (\omega_n)^{N_n} \right] e^{i2\pi \int \phi^* \bar{\omega}_D}, \qquad (8)$$

where N_n is the number of simplices in $\mathcal{M}_D$, and $\bar{\omega}_D \in H^D(\mathcal{B}G; \mathbb{R}/\mathbb{Z})$ is the cocycle in $\mathcal{B}G$.

Notice that exponent is invariant under re-triangulation, but the "measure"

$$\sum_{\phi} \prod_{n=0}^{d+1} (\omega_n)^{N_n} \qquad (9)$$

doesn't. We need to do something similar to "gauge fixing". The "gauge symmetry" here refers to the invariance under a homomorphism $\Phi : I \times \mathcal{M}_D \to \mathcal{B}G$, $\phi_1 = \Phi(t_1, \cdot), \phi_2 = \Phi(t_2, \cdot)$:

$$e^{2\pi i \int \phi_1^* \bar{\omega}_D} = e^{2\pi i \int \phi_2^* \bar{\omega}_D}, \quad \partial \mathcal{M}_D = 0. \qquad (10)$$

Such a gauge symmetry means the exponent only depends on homotopic class $[\phi]$. After some manipulation, we can re-write the partition function in terms of the homotopic class

$$\begin{aligned} Z &= \sum_{[\phi]} \left[\prod_{n=0}^{d+1} (\omega_n)^{N_n} \right] N([\phi], \mathcal{M}_D, \mathcal{B}G) e^{i2\pi \int \phi^* \bar{\omega}_D} \\ &= \sum_{[\phi]} W([\phi, M_D, \mathcal{B}G]) e^{i2\pi \int \phi^* \bar{\omega}_D}, \end{aligned} \qquad (11)$$

where $N([\phi], \mathcal{M}_D, \mathcal{B}G)$ is the number of homomorphism $\phi : \mathcal{M}_D \to \mathcal{B}G$ in the homotopic class $[\phi]$. This partition function is invariant under the re-triangulation.

In the end, we obtain a non-linear σ-model, which is classified by $\omega_D \in H^D(\mathcal{B}G, \mathbb{R}/\mathbb{Z})$. If ω_D is trivial, then it corresponds to a G-gauge theory, if not, then it gives a Dijkgraaf-Witten gauge theory.

IV. ANOMALIES IN TOPOLOGICAL ORDER

n the previous section, we only consider the closed spacetime manifold. If the M_D is not closed, then the topological order system might have anomaly in the boundary. The topological partition function enables us to study such anomalies in topological order.

A. Analogy in Symmetry Protected Topological Phases

Actually there are lots of similar things in SPT phase. Before we consider topological order, we can first briefly review the anomaly in SPT in terms of non-linear σ-model. A SPT state in $D = d+1$ dimensional spacetime M_D bulk manifold also has a non-linear σ-model description with target space G[3]

$$S = \int_{M_D} \left[\frac{\partial g}{\lambda} + iW(g) \right], \quad g \in G. \qquad (12)$$

This defines a lattice theory and W is the topological term which corresponds to the elements in $H^D(G, \mathbb{R}/\mathbb{Z})$ for bosonic SPT.

We can try to gauge the theory by integrating out $g(x)$:

$$S = \int \left[\frac{tr(F^2)}{\lambda} + i\tilde{W}(A) \right], \qquad (13)$$

where $\tilde{W}(g^{-1}dg) = W(g) \in H^D(G, \mathbb{R}/\mathbb{Z})$.

Then we can consider the θ term

$$\theta[g] = \int \tilde{W}(g^{-1}dg), \qquad (14)$$

which may not be 0 mod 2π if $\partial M_D \neq 0$, which indicates anomaly in the boundary.

B. Gravitational Anomalies in Topological Order

Now back to topological order. Under the triangulation limit, the triangulation dependence of the topological partition function will turn into metric dependence:

$$Z(M_D, \mathcal{T}) \to Z(M_D, g_{\mu\nu}). \qquad (15)$$

If the manifold has boundary $\partial M_D \neq 0$, then under the diffeomorphism of $g_{\mu\nu}$, the topological partition function might not be invariant but gain an extra phase[2]

$$\mathcal{A} = e^{i2\pi \oint_I \alpha} = e^{i2\pi \oint_{I \times M_D} \Omega}, \qquad (16)$$

where I is a segment in the moduli space $\mathcal{M}_{M_D} = \{g_{\mu\nu}\}/\text{Diff}$, and α is a 1-form on $\mathcal{M}_{M_D}$ and Ω is a closed $D+1$ form constructed from the curvature on $M_D \times I$.

For example, consider $D = 2+1$ topological order with boundary, $\Omega = \frac{c_-}{24} p_1$, where p_1 is the first Pontryagin class, the topological partition function is

$$Z^{\text{top}} = e^{i2\pi \frac{c_-}{24} \oint_{M_D} \omega_3}, \quad d\omega_3 = p_1, \qquad (17)$$

which corresponds to gravitational Chern-Simons term, and c_- is the chiral central charge, which indicates the perturbative gravitational anomaly[2,4].

V. SUMMARY

In this paper, we first carefully define the topological partition function. In order to make it well-defined, we need to carefully construct the definition of the measure, and also do "gauge fixing" to make it triangulation (metric) independent. We present the 1-gauge theory as an example of construction of a non-linear σ-model. In addition, if the spacetime manifold has boundary, usually the metric dependence of the topological partition function cannot be fully cancelled, then we obtain perturbative gravitational anomaly, which can be used to characterize the bulk topological order.

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