# Lieb-Schultz-Mattis Meets Symmetry Protected Topological Phases

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In this brief term paper, following [6], I will discuss the interplay of conventional Lieb-Schultz-Mattis (LSM) theorem and bosonic symmetry protected topological (SPT) phases. The original LSM theorem prevents the ground state (GS) of a system from being boring. It will be shown that, with further crystalline symmetries and constraints on local Hilbert space, another interestingness appears in the generalized LSM theorem. To be precise, the GS in this case is guaranteed to be in a *non-trivial* SPT phase, which would be forbidden by the original LSM theorem.

## INTRODUCTION

Since the discovery of free fermion topological insulators [3, 10], symmetry protected topological (SPT) phases in general have been extensively studied. An important question in this field is to classify all possible SPT phases based on symmetries and dimensionality. Tremendous progress has been made in classifying free fermion SPTs [11], but the problem becomes more subtle when interactions are introduced [8, 14]. Different from fermionic SPTs, bosonic SPTs require interactions to stabilize, with famous examples being the spin-1 Haldane chain [2] and the bosonic integer quantum Hall insulator [12]. Group cohomology has been very successful in classifying bosonic SPTs [1]. However, there does exist bosonic SPTs that are beyond the group cohomology classification [15]. We will see another such phase in this paper.

Another powerful results in many-body physics, but seemingly irrelevant to SPTs, is the Lieb-Schultz-Mattis (LSM) theorem [4, 7, 9], which says a translation symmetric system with half integer spins per unit cell cannot be trivially gapped, i.e. the ground state (GS) is either gapless, or topologically ordered, or there must be spontaneous symmetry breaking. In another words, LSM theorem forbids SPT phases. However, as we will see in the following discussions, by adding restrictions on the symmetry action on the local Hilbert space, projective representations to be precise, then the resulting phase has to be a non-trivial SPT.

## A 1D EXAMPLE OF SPT-LSM SYSTEMS

### Symmetries and local Hilbert spaces

Consider a 1D chain, where the local Hilbert space on each site is given by $|\alpha,\beta\rangle$, with $\alpha,\beta = 0,1,2,3$, i.e. the local Hilbert space dimension is 16. The on-site symmetry of this system is taken to be $Z_4^g \times Z_4^h$, where $g,h$ label the generators. The schematic diagram of the chain is shown in Fig 1. Notice that each site contains two "spins" and each unit cell contains two sites. Therefore, the translation symmetry is given by $T_{2x}$. Also, there

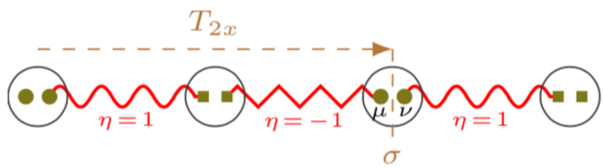


**FIG. 1:** Schematic diagram for the 1D chain. Each site contains two "spins" and each unit cell consists of two sites.

is a mirror symmetry $\sigma$ along each site center, which will turn out to be crucial in obtaining the SPT-LSM system. A crucial point is that we require the state on each site to facilitate a projective representation of the symmetry group $Z_4^g \times Z_4^h$. The projective representations are classified by the second cohomology group $H^2[Z_4^g \times Z_4^h, U(1)] = Z_4$. To be precise, using $W_{g,h}$ as representations, we have

$$W_g^4 = W_h^4 = 1, W_g W_h = i^\eta W_h W_g, \tag{0.1}$$

with $\eta = 0,1,2,3$. For our purposes we take $\eta = 2$, as a result of which, the symmetry still acts linearly on each unit cell (consisting of two sites). More concretely, by introducing the $4\times 4$ matrices $\mu,\nu$ acting as

$$\begin{aligned}\mu^z|\alpha,\beta\rangle = i^\alpha|\alpha,\beta\rangle, \mu^x|\alpha,\beta\rangle = |\alpha+1,\beta\rangle;\\ \nu^z|\alpha,\beta\rangle = i^\beta|\alpha,\beta\rangle, \nu^x|\alpha,\beta\rangle = |\alpha,\beta+1\rangle,\end{aligned} \tag{0.2}$$

the $W_{g,h}$ can be written as

$$\begin{aligned}W_g(j) &= \mu_j^x \otimes \nu_j^x;\\ W_h(j) &= \begin{cases}\mu_j^z \otimes \nu_j^z, j \text{ even}\\ (\mu_j^z)^3 \otimes (\nu_j^z)^3, j \text{ odd}\end{cases}\end{aligned} \tag{0.3}$$

which satisfy the relations in Eq. 0.1 for $\eta = 2$ as desired.

### An exactly solvable model Hamiltonian

Based on the above definitions, we can write down an AKLT style Hamiltonian that is exactly solvable:

$$H = \sum_j (1 - P^x_{j+1/2} P^z_{j+1/2}), \tag{0.4}$$

where

$$P^x_{j+1/2} = \frac{1}{4}(1 + \nu^x_j \mu^x_{j+1} + (\nu^x_j \mu^x_{j+1})^2 + (\nu^x_j \mu^x_{j+1})^3),$$
$$P^z_{j+1/2} = \frac{1}{4}(1 + \nu^z_j (\mu^z_{j+1})^3 + (\nu^z_j \mu^z_{j+1})^2 + (\nu^z_j)^3 \mu^z_{j+1}). \tag{0.5}$$

It can be easily shown that the unique zero energy GS is given by

$$|\Psi\rangle = \otimes_j \left[ \sum_{a=0}^{3} |\nu^z_j = a, \mu^z_{j+1} = a\rangle \right]. \tag{0.6}$$

For an open chain, the symmetry $Z^g_4 \times Z^h_4$ acts on the two edges as $W^L_{g,h} = \mu^{x,z}, W^R_{g,h} = \nu^{x,z}$, which corresponds to $\eta = -1$.

### Comparison with group cohomology classification

Based on the "crystalline equivalence principle" elucidated in [13], the 1D bosonic SPT phases are classified by the second group cohomology

$$H^2[Z^g_4 \times Z^h_4 \times Z^\sigma_2, U(1)_\sigma] = Z_2 \times Z_2, \tag{0.7}$$

where $Z^g_2$ is the mirror reflection and $U(1)_\sigma$ means the reflection acts non-trivially on the $U(1)$ phase. Here the translation symmetry $T_{2x}$ is not included because it's not essential for the non-triviality of the SPT phase. There are two root SPT phases based on this classification. The first one is the Haldane phase protected by $\sigma$, the other is the SPT phase protected by $Z^g_4 \times Z^h_4$ with $\eta = 2$. Notice that here the $\eta = 2$ projective representation is about the edge states of the SPT. However, in our model, the edge states have $\eta = \pm 1$ even though each bulk site has $\eta = 2$. Therefore, the non-trivial SPT phase obtained before goes beyond the group cohomology classification.

## SPT PHASES FROM DOMAIN WALL CONDENSATION

Domain wall condensation, or more generally defect condensation, can be used to construct possible SPT phases. The important thing here is that all symmetries have to be preserved when the defects are condensed. The defect operator in our 1D model can be constructed through a $Z_4$ version of Kramers-Wannier duality for the $Z^g_4$ part of the symmetry by defining

$$\tau^x_{j+1/2} = (\mu^z_j)^\dagger \mu^z_{j+1}, \tau^z_{j+1/2} = \prod_{j'<j} \mu^x_{j'} \nu^x_{j'}, \tag{0.8}$$

where $\tau^z_{j+1/2}$ is the $g$ domain wall creation operator. With $\eta = 2$ in Eq. 0.1, we can work out the symmetry actions on the domain wall:

$$\begin{aligned} g &: \tau^z_{j+1/2} \to \tau^z_{j+1/2}, \\ h &: \tau^z_{j+1/2} \to (-1)^j \tau^z_{j+1/2}, \\ \sigma &: \tau^z_{j+1/2} \to (\tau^z_{-j-1/2})^\dagger, \\ T_{2x} &: \tau^z_{j+1/2} \to \tau^z_{j+5/2}. \end{aligned} \tag{0.9}$$

From these symmetry actions, we immediately obtain the following relation

$$h\sigma \circ \tau^z = -\sigma h \circ \tau^z, \tag{0.10}$$

which says the condensation of $\tau^z$ cannot preserve $Z^h_4$ and $\sigma$ at the same time, i.e. by condensing the domain wall alone we are not able to obtain a symmetric phase. The trick here is to condense the bound state of the $g$ domain wall and the $h$-charge $\mu^x, \nu^x$. Defining this bound state as $\tilde{\tau}^z_{2j-1/2} \equiv \tau^z_{2j-1/2}(\nu^x_{2j})^\dagger$, we can have a similar set of symmetry transformation rules. It can be shown that $[h, \sigma] \circ \tilde{\tau}^z = 0$, which implies that the condensation of $\tilde{\tau}^z$ preserves all the symmetries.

## REAL-SPACE CONSTRUCTION: DECORATIONS

Decoration can be thought of as a generalization of layered constructions of SPTs [5]. More precisely, given a lattice of certain dimension, we can put certain SPTs on smaller regions (of varies dimensions) of the lattice in order for the whole system to be some specific SPT phase. Here we use this real-space construction to reproduce the 1D SPT phase discussed above.

Let's start with the bond $[0, 1]$, connecting site 0 and site 1, represented by the wiggly line shown in Fig 1. It's easy to see that the bond has global symmetry $Z^g_4 \times Z^h_4$, hence the possible SPT phases that can be decorated on the bond are classified by $\eta_{[0,1]} = H^2[Z^g_4 \times Z^h_4, U(1)] = Z_4$. Notice again that this $\eta_{[0,1]}$ here characterizes the edge, or the "spin" at the two ends of the bond, in this case. If the left edge has $\eta_{[0,1]}$, then the right edge has $-\eta_{[0,1]}$, giving a total of 0 for the whole bond. Then by translation and mirror reflection, we can obtain the decorations on all the other bonds. Based on the non-trivial action of $\sigma$, we have

$$\begin{aligned} \eta_{[-1,0]} &= \sigma \circ \eta_{[0,1]} = -\eta_{[0,1]} \bmod 4 \\ \eta_0 &= -\eta_{[-1,0]} + \eta_{[0,1]} = 2\eta_{[0,1]} \bmod 4, \end{aligned} \tag{0.11}$$

where the second relation is true because the site 0 consists of the right edge of $[-1, 0]$ and the left edge of $[0, 1]$. Recall that we have chosen $\eta_0 = 2$ for our model, therefore $\eta_{[0,1]} = \pm 1$, which specifies the SPT that should be decorated on the bond.

The punchline is, given the global symmetry and lattice structure, we can decorate the lattice by appropriately chosen SPTs to construct the desired phase.

## CONCLUSION AND REMARK ON HIGHER DIMENSIONS

In this paper we have discussed a generalized LSM theorem related to bosonic SPTs using a simple 1D model. The SPT-LSM systems can be constructed in real space through decoration or through condensation of symmetry defects. The approaches used in dealing with the 1D case are completely general and can be applied to arbitrary dimensions.

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