# Non-Abelian Statistics and CFT

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## INTRODUCTION

We first briefly review some basic facts about non-abelian statistics and show that it share some common feature with rational conformal field theory. Then we will use fractional quantum hall system as an example to show the relation between nonabelian statistics and conformal field theory.

## NON-ABELIAN STATISTICS

The notion of particle statistics in quantum mechanics usually refers to the action of permutation group $S_n$ on the wavefunction for a collection of n identical indistinguishable particles: the wavefunction is taken to transform as a definite representation of this group. The usual examples are Bose and Fermi statistics. A more modern approach prefers to exchange particles along some definite paths in space and omits intersecting exchange paths. It turns out that for spatial dimension d $>$ 2 that the group of exchanges still reduces to $S_n$ and for d=2 the topologically distinct exchanges is the braid group $B_n$.

Braid group $B_n$ is known to have nonabelian representations. This implies that even when the positions and quantum numbers have been specified, the wavefunction is not unique but is a member of a vector space. It has been analized in[1] that in order for nonabelian statistics to be consistent with locality we need to introduce the notion of fusion. Fusion of two or more particles into a composite produces either annihilation of the particle or else a particle of some new type with its own statistics properties. The matrix elements for these processes denoted F, describe the fusion rules and will have to satisfy consistent relations involving B's. For example, taking a third particle around a pair before or after they fuse should give the same result, since the third particle is far from the fusing pair and so cannot distinguish the close pair from their composite. There are known action that can produce the nonabelian properties. These are the Chern-Simons terms for nonabelian gauge fields.

## SOME FACTS ABOUT CFT

Exactly at the critical point the correlation function of a collection of fields $\phi_{i_r}(x_r)$ can be split in the form

$$< \Pi_{r=1}^{n} \phi_{i_r}(x_r) >= \sum_p |F_{p;i_1...i_n}(z_1, ...z_n)|^2 \qquad (1)$$

We have assumed that the spectrum is diagonal which is equivalent to assuming that the operators do not carry spin. To further simplify the problem we can focus on a subset called rational CFT and the sum over p is finite in this case. As the z's are varied so as to exchange some $\phi_i's$ the function $F_p$ are analytically continued to different sheets but can be expressed as z independent linear combinations of the original functions through some braiding matrices B.[1] In this way the correlation function can be single valued.

Another operation that can be performed on the correlation functions is the operator product expansion. As the arguments $z_1$, $z_2$ of two fields approach one another, the operators merge into a linear combinations of single operators:

$$\phi_i(z)\phi_j(w) \sim C_{ij}^k(z-w)\phi_k(w) \qquad (2)$$

as z $\to$ w, where $\phi_k$ is some new field of type k and $C_{ij}^k$ is a singular coefficient function. This operation can be used to define some new matrices F, the fusion matrix that describes which fields k appear in the product of i and j. In[2] the consistency conditions that must be satisfied were analyzed. In fact these are also the consistent conditions for nonablian statistics. This indicate that there might be close relations between systems with nonabelian excitations and CFT. In the next section we will use fractional quantum hall system as an example.

## EXAMPLE: FRACTIONAL QUANTUM HALL SYSTEM

The fractional quantum hall system is an example of bulk-boundary correspondence. The edge modes on the boundary can be described by a conformal field theory. The wavefunction in the bulk can be constructed as correlation functions on the boundary.

Let's consider a fractional quantum hall system on a disc. Then the edge modes live on the cylinder with coordinate $(\sigma, t)$.

$$S = \frac{m}{4\pi}\int_{R\times S^1} dtd\sigma\partial_t\phi\partial_\sigma\phi - (\partial_\sigma\phi)^2 \qquad (3)$$

We can Wick rotate to Euclidean space and define the complex variable

$$w = \frac{2\pi}{L}\sigma + it \tag{4}$$

The next step is to map the cylinder to the plane by working with the single-valued complex coordinates

$$z = e^{-iw} \tag{5}$$

The chiral boson living on the boundary can be described by a holomorphic operator with propagator

$$< \phi(z)\phi(w) = -\frac{1}{m} log(z-w) > \tag{6}$$

The basic idea is to look at correlation functions involving insertions of electrons of the form

$$\psi =: e^{im\phi} :$$

Now consider the correlation function

$$G(z_i, \bar{z}_i) =< \psi(z_1)...\psi(z_N) exp(-\rho \int_\gamma d^2 z^{'} \phi(z^{'})) > \tag{7}$$

where$\rho_0 = 1/2\pi l_B^2$ and $\gamma$ is a disc-shaped region of radius R, large enough to encompass all point $z_i$. The correlation function is nonzero only if

$$mN = \rho_0 \int_\gamma d^2 z^{'} = \pi R^2 \rho_0$$

Using $\rho_0$=$1/\pi l_B^2$, we can see that we should take R=$\sqrt{2mN} l_B$ which is just the radius of the droplet described by the quantum hall wavefunction. Using (6) and Wick theorem, correlation function (7) can be written as

$$G(z_i, \bar{z}_i) \sim \prod_{i<j} (z_i - z_j)^m exp(-\rho_0 \sum_{i=1}^{N} \int_\gamma d^2 z^{'} log(z_i - z^{'})) \tag{8}$$

The imaginary part of the integral is ill-defined because of the branch cut in the logarithm. However it can be undone by a singular gauge transformation. Omitting overall constant and terms that are suppressed by $|z_i|/R$. the final result for the correlation function is

$$G(z_i, \bar{z}_i) \sim \prod_{i<j} (z_i - z_j)^m e^{-\sum_i |z_i|^2/4l_B^2} \tag{9}$$

This is the well-known Laughlin wavefunction.

We can extend this to wavefunction that involve quasi-holes. We can insert some number of quai-hole operators

$$\psi_{qh} =: e^{i\phi} :$$

into the correlation function

$$\begin{aligned} &\tilde{G}(z_i, \bar{z}_i; \eta_a, \bar{\eta_a}) \\ &=< \psi_{qh}(\eta_1)...\psi_{qh}(\eta_p)\psi(z_1)...\psi(z_N) exp(-\rho \int_\gamma d^2 z^{'} \phi(z^{'})) > \\ &= \prod_{a<b} (\eta_a - \eta_)^{1/m} \prod_{a,i} (z_i - \eta_a) \prod_{k<l} (z_k - z_l)^m \\ &\quad e^{-\sum_i |z_i|^2/4l_B^2 - \sum_a |\eta_a|^2/4ml_B^2} \end{aligned} \tag{10}$$

This is the laughlin wavefunction for the quasi-hole excitations. This bulk-boundary correspondence is reminiscent of what happens in dS/CFT correspondence. In spacetimes which areasymptotically de Sitter, the bulk Hartle-Hawking wavefunction at spacelike infinity is captured by a boundary Euclidean conformal field theory.

## CONCLUSION

It is conjectured that this correspondence between bulk wavefunction and boundary correlation functions extend to all quantum hall states. This means we don't need to guess quantum hall wavefunctions anymore. Instead we just guess a boundary CFT and compute its correlation functions. And it turns out that the CFT framework is most useful for studying the properties of quantum hall states, especially those with nonabelian anyons. Also the braiding properties of anyons are related to well-studied properties of CFTs.

## REFERENCE

[1]J.Frohlich,F.Gabbiani and P.Marchetti, Braid Statistics in Three-dimensional Local Quantum Theory.
[2]G.Moore and N. Seiberg, Classical and Quantum Conformal Field Theory.
[3]D. Tong, Lectures on Quantum Hall Effect.
[4]N.Read and G.Moore, Fractional Quantum Hall Effect and Nonabelian Statistics.

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[1] The braiding matrices can be understood from another point of view. Correlation functions in rational CFT obeys BPZ equations. For 4-point functions these become 2nd order ODEs with isolated singularities on the complex plane. As z goes around these singularities the solutions will transform into each other. The coefficient will be given by the B matrices mentioned above.