University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2022
Assignment 1 – Solutions

Due 11:59pm Monday, April 4, 2022

- Please hand in your homework electronically via Canvas. The preferred option is to typeset your homework. It is easy to do and you need to learn to do it anyway as a practicing scientist. A LaTeX template file with some relevant examples is provided here. If you need help getting set up or have any other questions please email me. I am happy to give TeX advice.
- To hand in your homework, please submit a pdf file through the course's Canvas website, under the assignment labelled hw 01.

Thanks in advance for following these guidelines. Please ask me by email if you have any trouble.

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1. **Brain-warmer.** [optional] Find the constant a in

$$a \prod_{\mu=0}^{D-1} \gamma^\mu = \gamma^5$$

so that $(\gamma^5)^2 = 1$.

Do $D = 2$ and then use induction: in $D + 2$ dimensions, the chirality operator is $\gamma_\chi^{D+2} = \frac{a_{D+2}}{a_D} \gamma_\chi^D \gamma_{D-2} \gamma_{D-1}$. This costs an extra -1 in $(\gamma_\chi^{D+2})^2$.

2. **Chiral anomaly in two dimensions.**

Consider a massive relativistic Dirac fermion in 1+1 dimensions, with

$$S = \int \mathrm{d}x\mathrm{d}t \bar{\psi} \left(\mathbf{i}\gamma^\mu \left(\partial_\mu + eA_\mu \right) - m \right) \psi.$$

By heat-kernel regularization of its expectation value, show that the divergence of the axial current $j_\mu^5 \equiv \mathbf{i}\bar{\psi}\gamma_\mu\gamma^5\psi$ is

$$\partial_\mu j_\mu^5 = 2\mathbf{i}m\bar{\psi}\gamma^5\psi + \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}.$$

The calculation follows very closely the one in the lecture notes. There are two new ingredients: the mass, and the change to $D = 2$. We know that classically the mass contributes to a violation of the axial current (in any even dimension). From the derivation in terms of the path integral measure we can see that the effects of this and the anomaly contribute additively to the divergence of j^5_μ.

Again we expand the exponent using $(\mathbf{i}\not{D})^2 = D^2 + \frac{1}{2}i\Sigma^{\mu\nu}F_{\mu\nu}$. In $D = 2$, we use the fact that $\mathrm{tr}\gamma^5\Sigma^{\mu\nu} = 2\epsilon^{\mu\nu}$ (check by picking a basis or using the Clifford algebra directly) to find that the contribution from the anomaly is:

$$\partial^\mu \left\langle j^5_\mu \right\rangle = s\mathrm{tr}\gamma^5\Sigma^{\mu\nu}F_{\mu\nu}\underbrace{\int đ^2 k e^{-sk^2}}_{=\frac{1}{4\pi s}} + \mathcal{O}(s) \overset{s\to 0}{=} \frac{1}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}.$$

3. **Chiral anomaly in six dimensions.** [optional] Find the divergence of the axial current in QED in $D = 5 + 1$.

4. **An application of the anomaly to a theory without gauge fields.**

   Consider a 1+1d theory of Dirac fermions coupled to a background scalar field θ as follows:

$$\mathcal{L} = \bar{\Psi}\left(\mathbf{i}\not{\partial} + me^{\mathbf{i}\theta\gamma^5}\right)\Psi.$$

   We wish to ask: if we subject the fermion to various configurations of $\theta(x)$ (such as a domain wall where $\theta(x > 0) = \pi + \theta(x < 0)$) what does the fermion number do in the groundstate?

   (a) Convince yourself that when θ is constant

$$\langle j^\mu \rangle = 0$$

   where $j^\mu = \bar{\Psi}\gamma^\mu\Psi$ is the fermion number current.

   Lorentz invariance forbids an expectation value for a vector quantity.

   (b) Minimally couple the fermion to a *background* gauge field A_μ. Let $e^{\mathbf{i}\Gamma[A,\theta]} = \int [d\Psi]e^{\mathbf{i}S}$. Convince yourself that the term linear in A in $\Gamma[A,\theta] = \text{const} + \int A_\mu J^\mu + \mathcal{O}(A^2)$ is the vacuum expectation value of the current $\langle j^\mu \rangle = J^\mu$ (at $A = 0$).

   A is a source for j in the path integral:

$$\frac{1}{\mathbf{i}}\frac{\delta}{\delta A_\mu(x)}\log Z[A] = Z^{-1}\int D\psi j^\mu(x)e^{\mathbf{i}S} = \langle j^\mu(x) \rangle .$$

   (c) Show that by a local chiral transformation $\Psi \to e^{-\mathbf{i}\theta(x)\gamma^5/2}\Psi$ we can remove the dependence on θ from the mass term.

(d) Where does the theta-dependence go? Use the 2d chiral anomaly to relate $\langle j^\mu \rangle$ to $\partial\theta$. Notice that the result is independent of m. [This relation was found by Goldstone and Wilczek. The associated physics is realized in Polyacetylene.]

$$\mathbf{i}\Gamma[A,\theta] = \operatorname{tr}\log\left(\mathbf{i}\not{D} + me^{\mathbf{i}\theta}\right) = \mathbf{i}\int A_\mu J^\mu + \mathcal{O}(A^2).$$

It looks challenging to evaluate this determinant. But we've already done the necessary work in studying the chiral anomaly. The variation of the effective action under a (can be local!) chiral rotation by angle $\theta(x)$ is

$$\delta\Gamma = \int d^2x\theta(x)\frac{F_{\mu\nu}\epsilon^{\mu\nu}}{2\pi}.$$

Since we showed $\Gamma[\theta = \text{constant}] = 0$, the anomaly is the whole thing:

$$\Gamma[A,\theta] = \int d^2x\theta(x)\frac{F_{\mu\nu}\epsilon^{\mu\nu}}{2\pi} \overset{\text{IBP}}{=} -\int d^2x\frac{\partial_\mu\theta\epsilon^{\mu\nu}}{2\pi}A_\nu$$

Therefore

$$\langle j^\mu \rangle = -\frac{\partial_\mu\theta\epsilon^{\mu\nu}}{2\pi}.$$

(e) Show that a domain wall where θ jumps from 0 to π localizes *fractional* fermion number.

The charge on the domain wall is

$$Q = \int_{-\epsilon}^{\epsilon} dx j^0 = \int_{-\epsilon}^{\epsilon} dx\frac{\partial_x\theta}{2\pi} = \frac{1}{2\pi}\left(\theta(+\epsilon) - \theta(-\epsilon)\right) = \frac{1}{2}.$$

(f) Consider the Dirac hamiltonian in the presence of such a domain wall. Show that there is an exponentially-localized mode of zero energy.

In the basis for the gamma matrices where

$$\gamma^0 = \sigma^1, \gamma^1 = \mathbf{i}\sigma^2, \gamma^5 \equiv \gamma^0\gamma^1 = -\sigma^3,$$

the $D = 2$ Dirac Hamiltonian is

$$H = \gamma^0(\mathbf{i}\vec{\gamma}\cdot\vec{\nabla} + m(x)) = -\sigma^3\mathbf{i}\partial_x + \sigma^1 m(x), \quad m(x) = me^{\mathbf{i}\theta(x)\gamma^5} = \cos\theta - \mathbf{i}\sigma^3\sin\theta$$

so

$$H = -\sigma^3\mathbf{i}\partial_x + m(\sigma^1\cos\theta(x) - \sigma^2\sin\theta(x)).$$

The condition for a zero-mode is

$$0 = H\psi = \begin{pmatrix} -\mathbf{i}\partial_x & e^{\mathbf{i}\theta} \\ e^{-\mathbf{i}\theta} & \mathbf{i}\partial_x \end{pmatrix} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix},$$

with ψ normalizable. If $\theta(x) = \begin{cases} 0, x < 0 \\ \pi, x > 0 \end{cases}$, then $m(x) = \begin{cases} m, x < 0 \\ -m, x > 0 \end{cases}$.
Let's expand ψ in eigenstates of σ^2 ($\psi_\uparrow = \pm\mathbf{i}\psi_\downarrow$); since $\{H, \sigma^2\} = 0$, the terms don't mix, and we find

$$0 = -\mathbf{i}\partial_x (\pm\mathbf{i}\psi_\downarrow) + m(x)\psi_\downarrow = (\pm\partial_x + m(x))\, \psi_\downarrow \implies \psi_\downarrow(x) = \psi_\downarrow(0) e^{\mp \int_0^x dx' m(x')}.$$

Only one of the two choices (the lower sign) is normalizable far from the domain wall (on both sides), which gives the localized zero-energy mode

$$\psi(x) = \psi_\downarrow(0) e^{-m|x|} \begin{pmatrix} -\mathbf{i} \\ 1 \end{pmatrix}.$$