University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2022
Assignment 2 – Solutions

Due 11:59pm Monday, April 11, 2022

Thanks in advance for following the guidelines on HW01. Please ask me by email if you have any trouble.

---

1. **Emergence of the Dirac equation.** Consider a chain of free fermions with

$$H = -t \sum_n c_n^\dagger c_{n+1} + h.c.$$

Show that the low-energy excitations at a generic value of the filling are described the the massless Dirac lagrangian in 1+1 dimensions. Find an explicit choice of $1+1$-d gamma matrices which matches the answer from the lattice model. Show that the right-movers are right-handed $\gamma^5 \equiv \gamma^0\gamma^1 = 1$ and the left-movers are left-handed.

This system has a conserved charge $N \equiv \sum_n c_n^\dagger c_n$ counting the number of fermions, which we get to pick. The easiest way to do this is to add a chemical potential $H \to H - \mu N$ and choose μ to get the desired number of particles on average. (This is the same as fixing the number of particles in the thermodynamic limit.) In that case we have

$$H = -t \sum_n c_n^\dagger c_{n+1} + h.c. - \mu \sum_n c_n^\dagger c_n = \oint_{\text{BZ}} đk c_k^\dagger c_k \epsilon_k$$

with $\epsilon_k = -2t \cos ka - \mu$, and the integral is over the Brillouin zone. $a = 1$ is the lattice spacing. By 'generic filling' I mean choose the number of particles per site to be between 0 and 1. The former and latter correspond to choosing $\mu = \pm 2t$ at the bottom or top of the band, where the dispersion is quadratic, rather than linear.

We can focus on the physics at the two Fermi points $k = \pm k_F$ (where k_F solves $\epsilon_{k_F} = 0$) by plugging in

$$\psi(x) \simeq \int_R đk e^{(k_F+k)x}\psi_R + \int_R đk e^{(-k_F+k)x}\psi_L$$

where R is a small-enough region in momentum space that the two domains don't overlap. This gives

$$H = \int_R đk \left(v_F k \psi_R^\dagger \psi_R - v_F k \psi_L^\dagger \psi_L \right)$$

where $v_F \equiv \partial_k \epsilon_k|_{k=k_F}$. Translating into an action, setting $v_F = 1$, and pretending R goes on forever (this is how we can fool ourselves that the chiral current is conserved), this is

$$S = \int dxdt \left(\psi_R^\dagger \left(\partial_t - \partial_x\right) \psi_R + \psi_L^\dagger \left(\partial_t + \partial_x\right) \psi_L \right) = \int d^2x \left(\bar{\Psi} \gamma^\mu \partial_\mu \Psi \right)$$

with

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

and

$$\gamma^0 = \sigma^1, \gamma^1 = \mathbf{i}\sigma^2, \gamma^5 \equiv \gamma^0\gamma^1 = -\sigma^3.$$

This gives

$$\gamma^5 \Psi = -\sigma^3 \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} -\psi_L \\ \psi_R \end{pmatrix}$$

so indeed the left-moving particle has left-handed chirality.

2. **Polyacetylene returns.**

   On HW01, you may have wondered what is the connection between the field theory we were studying (a scalar coupled to fermions in $D = 2$) and polyacetylene. I'd like to explain that connection a bit.

   Consider an extension of the model above to include also *phonon* modes, *i.e.* degrees of freedom encoding the positions of the ions in the solid. (Again we ignore the spins of the electrons for simplicity.)

$$H = -t \sum_n (1 + u_n) c_n^\dagger c_{n+1} + h.c. + \sum_n K (u_n - u_{n+1})^2 \equiv H_F + H_E.$$

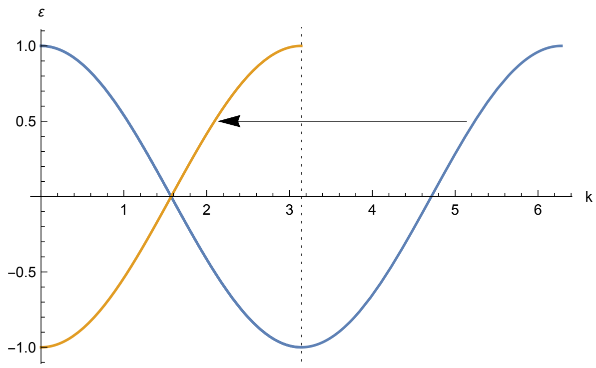
   Here u_n is the deviation of the nth ion from its equilibrium position (in the $+x$ direction), so the second term represents an elastic energy. Assume periodic boundary conditions and an even number of sites.

   (a) Consider a configuration

$$u_n = \phi(-1)^n \tag{1}$$

   where the ions move closer in pairs. Compute the electronic spectrum. (Hint: this enlarges the unit cell. Define $c_{2n} \equiv a_n, c_{2n+1} \equiv b_n$, and solve in Fourier space, $a_n \equiv \oint đk e^{2\mathbf{i}kn} a_k$ *etc.*) You should find that when $\phi \neq 0$ there is a gap in the electron spectrum (unlike $\phi = 0$). Expand the spectrum near the minimum gap and include the effects of the field ϕ in the continuum theory.

When doubling the unit cell, we halve the Brillouin zone. So even when $\phi = 0$, the spectrum gets folded on itself, like this:

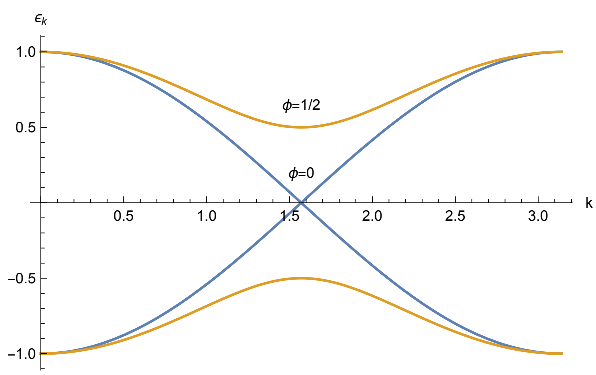


This means that at half-filling, with $\phi = 0$, it looks like there is a Dirac point at $k = \pi/2$.

Now, including ϕ, it allows the two branches of the Dirac point to mix with each other and produces a gap:

$$\epsilon(k) = \pm\sqrt{\cos^2 k + \phi^2 \sin^2 k}$$

which looks like this:



Near the minimum gap at $k = \pi/2$, we can expand to find

$$\epsilon(k = \frac{\pi}{2} + \delta k) = \pm\sqrt{\cos^2 k(1-\phi^2) + \phi^2} = \pm\sqrt{\delta k^2(1-\phi^2) + \phi^2}. \qquad (2)$$

Comparing to the spectrum of a Dirac fermion with action

$$S[\psi, \phi] = \int d^2x \left(\bar{\psi}\mathbf{i}\partial\!\!\!/\psi - \phi\bar{\psi}\psi\right)$$

which has

$$H = \gamma^0(\mathbf{i}\gamma^1\partial_x - \phi) = \begin{pmatrix} \phi & k \\ k & -\phi \end{pmatrix}$$

and therefore

$$\epsilon_k = \pm\sqrt{k^2 + \phi^2}$$

which agrees with (2) at small k (which is really the deviation from $k = \pi/2$) and small ϕ.

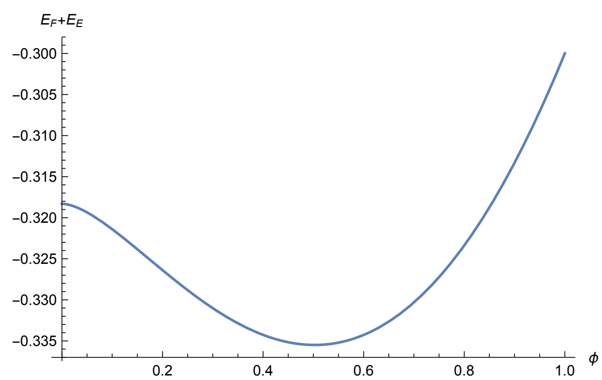
(b) **Peierls' instability.** Compute the groundstate energy of the electrons H_F in the configuration (1), at half-filling (*i.e.* the number of electrons is half the number of available states). Check that you recover the previous answer when $\phi = 0$. Interpret the answer when $\phi = 1$.

Compute H_E in this configuration, and plot the sum of the two as a function of ϕ. Choosing the parameters so the minimum is in the small-ϕ region, find the minimum.

At half-filling, in the groundstate the lower band is filled. The energy is

$$E_F(\phi) = -\oint dk\sqrt{\cos^2 k + \phi^2 \sin^2 k} = -\frac{1}{\pi}\text{EllipticE}(1 - \phi^2).$$

For $8K^2 = .2$ the total energy looks like this:



There is a minimum at $\phi^2 \neq 0$, *i.e.* two minima at $\phi = \pm\phi_0$. Increasing ϕ lowers the total energy because it lowers the energy of the filled states.

For small ϕ,

$$\text{EllipticE}(1 - \phi^2) = 1 + \frac{1}{4}\phi^2\left(\log\frac{\phi^2}{16} - 1\right) + \ldots \qquad (3)$$

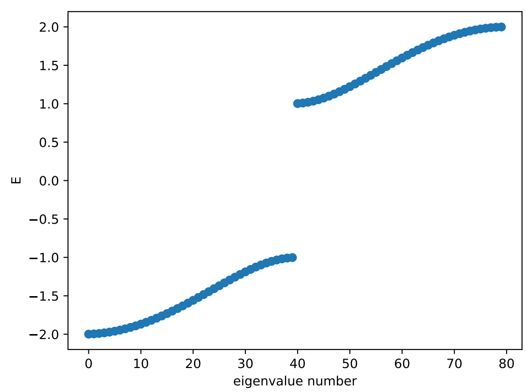
This functional form can be understood from the continuum.

(c) You should find that the energy is independent of the *sign* of ϕ. This means that there are two groundstates. We can consider a domain wall between a region of $+$ and a region of $-$. Show that this domain wall carries a fermion mode whose energy lies in the bandgap and has fermion number $\pm\frac{1}{2}$.

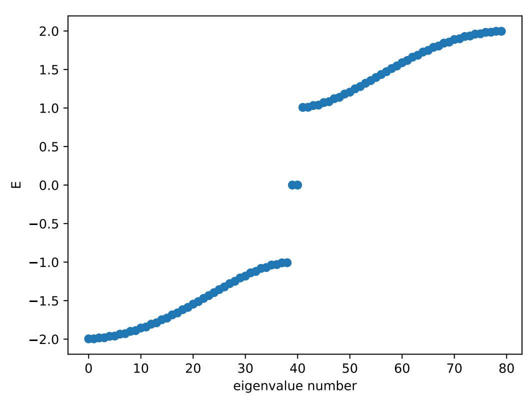
The basic idea is that ϕ must go through zero in between. We showed on HW01 that this is the case using the field theory we derived in the earlier parts of the problem. In particular, the two states (zero-mode occupied and zero-mode unoccupied) must have a fermion number which differ by 1, but they are related to each other by particle-hole symmetry, so they must have fermion number $\pm\frac{1}{2}$ (as we found on HW01).

(d) [bonus] Verify the result of the previous part by diagonalizing the relevant tight-binding matrix.

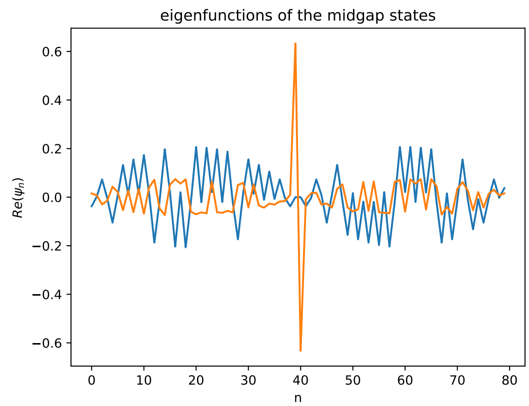
Here is the spectrum of a chain (of length 40) with $\phi = +0.5$ everywhere:



And here is the result when ϕ switches to -0.5 in the middle:



The wavefunctions of the states in the middle look like



(e) [bonus] Time-reversal played an important role here. If we allow complex hopping amplitudes, we can make a domain wall without midgap modes. Explain this from field theory. Bonus: explain this from the lattice hamiltonian.

If the mass is allowed to be complex, then we can interpolate between $-m$ and $+m$ without going through $m = 0$.

3. **Anomaly cancellation in the Standard Model.** If we try to gauge a chiral symmetry (such as hypercharge in the Standard Model (SM)), it is important that it is actually a symmetry, *i.e.* is not anomalous. In $D = 3 + 1$, a possible anomaly is associated with a choice of three currents, out of which to make a triangle diagram. We'll call a "$\mathsf{G}_1\mathsf{G}_2\mathsf{G}_3$ anomaly" the diagram with insertions of currents for $\mathsf{G}_1, \mathsf{G}_2$ and G_3. Generalizing a little, we showed that the divergence of the current for G_1 is

$$\partial_\mu j_1^{A\mu} = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} F^{2B}_{\mu\nu} F^{3C}_{\rho\sigma} \sum_f (-1)^f \mathrm{tr}_{R(f)}\{T_1^A, T_2^B\} T_3^C .$$

The sum is over each Weyl fermion, $R(f)$ is its representation under the combined group $\mathsf{G}_1 \times \mathsf{G}_2 \times \mathsf{G}_3$, and T_1^A are a basis of generators of the Lie algebra of G_1 etc. in the representation of the field f. By $(-1)^f$ I mean $\pm$ for left- and right-handed fermions respectively.

We consider the possibilities in turn.

Schwartz §30.4 does most of this pretty explicitly.

(a) Convince yourself that the divergence of the $\mathsf{U}(1)_Y$ hypercharge current gets a contribution of the form

$$\partial_\mu J^\mu_Y = \left(\sum_{\text{left}} Y_l^3 - \sum_{\text{right}} Y_r^3 \right) \frac{g'^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} B_{\rho\sigma}$$

from the triangle with three insertions of the current itself (here B is the hypercharge gauge field strength). The sum on the RHS is over all left- and right-handed Weyl spinors weighted by the cube of their hypercharge. Check that this sum evaluates to zero in the SM.

(b) Show that any anomaly of the form $\mathsf{SU}(N)\mathsf{U}(1)^2$ or $\mathsf{SU}(N)G_1G_2$ is zero.

(c) (Easy) Convince yourself that there is no $\mathsf{SU}(3)^3$ anomaly for QCD.

The charges of the fields under $\mathsf{SU}(3)$ are symmetric under $L \leftrightarrow R$ – *i.e.* QCD is non-chiral – so there is a cancellation between the contributions of left- and right-handed fields.

(d) Check that there is never an $\mathsf{SU}(2)^3$ anomaly. (Hint: the generators satisfy $\{\tau^a, \tau^b\} = 2\delta^{ab}$.)

(e) Show that the $\mathsf{SU}(3)^2\mathsf{U}(1)_Y$ anomaly demands that $2Y_Q - Y_u - Y_d = 0$. Check that this is true in the SM.

(f) Show that a necessary condition for hypercharge to not have an anomaly with the Electroweak gauge bosons on the RHS is $Y_L + 3Y_Q = 0$, where Y_L and Y_Q are the hypercharges of the left-handed leptons and quarks. Check that this works out in the SM.

It gets contributions only from left-handed fields (those charged under $\mathsf{SU}(2)_{EW}$): $\text{tr}\{\tau^a, \tau^b\}Y = \delta^{ab} \sum_{\text{left}} Y_l = Y_L + 3Y_Q$ because the quarks carry 3 colors.

(g) There is another kind of anomaly called a *gravitational anomaly*. This is a violation of current conservation in response to coupling to curved space. An example is of the form

$$\partial_\mu j^\mu_Y = a \text{tr} \mathcal{R} \wedge \mathcal{R}$$

where $\mathcal{R}$ is a two-form related to the curvature of spacetime (analogous to the field strength F). The coefficient a is proportional to $\sum_{\text{left}} \text{tr} Y_l - \sum_{\text{right}} \text{tr} Y_r$. Check that this too vanishes for hypercharge in the Standard Model.

These conditions, plus the assumption that the right-handed neutrino is neutral, actually determine all the hypercharge assignments.

(h) [bonus] Show that the previous statement is true.

There are various points of view from which the anomalies determine the charge assignments.

One is: Given the $\mathsf{SU}(3) \times \mathsf{SU}(2)_L$ representations, the actual hypercharges are the only way to satisfy all the anomaly constraints that is chiral. From this point of view, the fact that the hypercharges are all integer multiples of $1/6$ (so that $\mathsf{U}(1)_Y$ is compact) is an outcome of anomaly cancellation.

Another is: Assuming that the hypercharges are quantized (in some units), the choice in the SM is the only chiral choice, even without using the gravitational chiral anomaly constraint. This is a consequence of Fermat's Last Theorem.

(i) [bonus] Show that $\mathsf{U}(1)_B$ and $\mathsf{U}(1)_L$ are anomalous, but have all opposite anomalies, so that $\mathsf{U}(1)_{B-L}$ is non-anomalous. Here all quarks (antiquarks) have charge $1/3$ $(-1/3)$ under $\mathsf{U}(1)_B$, and all leptons (antileptons) have charge 1 (-1) under $\mathsf{U}(1)_L$.