University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2022
Assignment 4 – Solutions

Due 11:59pm Monday, April 25, 2022

Thanks in advance for following the guidelines on HW01. Please ask me by email if you have any trouble.

---

1. **Galilean transformation of non-relativistic fields.**

Show that the action

$$S = \int dt d^d x \left(\Phi^\star \mathbf{i} \partial_t \Phi - \frac{1}{2m} \vec{\nabla} \Phi^\star \cdot \vec{\nabla} \Phi - V(|\Phi|) \right) \quad (1)$$

is invariant under Galilean boosts, in the form

$$\Phi(\vec{x}, t) \to \Phi'(\vec{x}', t') \quad \text{with} \quad \Phi(\vec{x}, t) = e^{-\frac{\mathbf{i}}{2} m v^2 t + \mathbf{i} m \vec{v} \cdot \vec{x}} \Phi'(\vec{x}', t') \quad (2)$$

with $t' = t, x_i' = x_i - v_i t$.

Note that this is also how the nonrelativistic single-particle wavefunction must transform in order to preserve the Schrödinger equation.

Don't forget that $\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu}$.

Let $\Phi = e^{\mathbf{i}\Theta} \Phi'$, so that

$$\vec{\nabla} \Phi = e^{\mathbf{i}\Theta} \left(\mathbf{i} m \vec{v} + \vec{\nabla}' \right) \Phi'$$

and

$$\partial_t \Phi = e^{\mathbf{i}\Theta} \left(\frac{1}{2} m v^2 + \mathbf{i} \partial_t \right) \Phi' = e^{\mathbf{i}\Theta} \left(\mathbf{i} \partial_{t'} - \mathbf{i} \vec{v} \cdot \vec{\nabla}' + \frac{1}{2} m v^2 \right) \Phi'.$$

In the last step of each line we used $\partial_t = \partial_{t'} - v^i \partial_i$ and $\vec{\nabla}_x = \vec{\nabla}_{x'}$. Therefore

$$\mathcal{L}(\Phi) = (\Phi')^\star e^{-\mathbf{i}\Theta} e^{\mathbf{i}\Theta} \left(\mathbf{i} \partial_{t'} - \mathbf{i} \vec{v} \cdot \vec{\nabla}' + \frac{1}{2} m v^2 \right) \Phi' - \frac{1}{2m} (\Phi')^\star (\overleftarrow{\nabla} - \mathbf{i} m \vec{v}) e^{-\mathbf{i}\Theta} \cdot e^{\mathbf{i}\Theta} (\vec{\nabla} + \mathbf{i} m \vec{v}) \Phi' \quad (3)$$

$$= \mathcal{L}(\Phi') + \frac{m v^2}{2} |\Phi'|^2 - \frac{m^2 v^2}{2m} |\Phi'|^2 - (\Phi')^\star \mathbf{i} \vec{v} \cdot \vec{\nabla}' \Phi' + \frac{\mathbf{i} m}{2m} \vec{v} \cdot (\Phi')^\star \vec{\nabla}' \Phi' \times 2 - \vec{\nabla}' \cdot \left(\frac{\mathbf{i}}{2} \vec{v} |\Phi'|^2 \right). \quad (4)$$

The last term is a total derivative, and everything else cancels but $\mathcal{L}(\Phi')$. Note that the measure transforms trivially $d^d x dt = d^d x' dt'$ because $\det \frac{\partial(x', t')}{\partial(x, t)} = 1$.

How does the boost act on the Goldstone mode in the symmetry-broken phase?

2. **Diagrammatic understanding of BCS instability of Fermi liquid theory.**

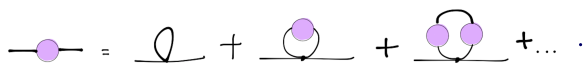
(a) Recall that only the four-fermion interactions with special kinematics are marginal. Keeping only these interactions, show that cactus diagrams (like this: ) dominate.

The diagrams which dominate are made of the marginal 4-fermion vertices, which have the momenta equal and opposite in pairs, *i.e.* $V(k_1, k_2, k_3, k_4) = V(k, -k, k', -k')$. This is automatic in cactus diagrams. The model which keeps only these terms is called the *Reduced BCS model*.

(b) To sum the cacti, we can make bubbles with a corrected propagator. Argue that this correction to the propagator is innocuous and can be ignored.

These diagrams do not depend on the external momenta. Therefore, they are merely a renormalization of the chemical potential. Fixing the propagator according to the correct particle density therefore removes all effects of these diagrams.

To resum their effects we use the self-energy with the pink blob which satisfies

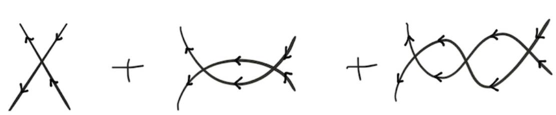


(c) Armed with these results, compute diagrammatically the Cooper-channel susceptibility (two-particle Green's function),

$$\chi(\omega_0) \equiv \left\langle \mathcal{T} \psi^\dagger_{\vec{k},\omega_3,\downarrow} \psi^\dagger_{-\vec{k},\omega_4,\uparrow} \psi_{\vec{p},\omega_1,\downarrow} \psi_{-\vec{p},\omega_2,\uparrow} \right\rangle$$

as a function of $\omega_0 \equiv \omega_1 + \omega_2$, the frequencies of the incoming particles. Think of χ as a two point function of the Cooper pair field $\Phi = \epsilon_{\alpha\beta}\psi_\alpha\psi_\beta$ at zero momentum.

Sum the geometric series in terms of a (one-loop) integral kernel.

$\chi(\omega_0) =$  $+ \cdots$ (5)

$$= -\mathbf{i}V + (-\mathbf{i}V)^2 \frac{1}{2} \int đ^d k d\epsilon G(\epsilon + \omega_0, \vec{k}) G(-\epsilon, -\vec{k}) + (-\mathbf{i}V)^3 \left(\frac{1}{2}\right)^2 \int GG \int GG + \cdots \quad (6)$$

$$\equiv -\mathbf{i}V \left(1 - \frac{\mathbf{i}}{2} V \int GG + (-\frac{\mathbf{i}}{2} V \int GG)^2 + \cdots \right) \quad (7)$$

$$= -\mathbf{i}V \left(1 - \mathcal{I} + \mathcal{I}^2 + \cdots\right) = \frac{-\mathbf{i}V}{1 + \mathcal{I}}. \quad (8)$$

The $\frac{1}{2}$ is a symmetry factor.

(d) Do the integrals. In the loops, restrict the range of momenta to $|\epsilon(k)| < E_D$, the Debye energy, since it is electrons with these energies that experience attractive interactions.

Consider for simplicity a round Fermi surface. For doing integrals of functions singular near a round Fermi surface, approximate the dispersion relation as $\epsilon(k) \simeq v_F(|k| - k_F)$, so that $d^dk \simeq k_F^{d-1}\frac{d\xi}{v_F}d\Omega_{d-1}$. I recommend doing to the frequency integral first (by residues).

Now we have to do the integral.

$$\mathcal{I} = \frac{\mathbf{i}}{2}V\int đ^dk d\epsilon G(\epsilon+\omega_0,\vec{k})G(-\epsilon,-\vec{k}) \tag{9}$$

$$= \frac{\mathbf{i}}{2}V\int đ^dk d\epsilon \frac{1}{(\epsilon+\omega_0)(1+\mathbf{i}\eta)-\xi(\vec{k})}\frac{1}{(-\epsilon)(1+\mathbf{i}\eta)-\xi(-\vec{k})} \tag{10}$$

$$= \frac{\mathbf{i}}{2}V\int đ^dk \frac{2\pi\mathbf{i}}{2\pi}(-1)^{\text{sign}(\xi(k))}\frac{1}{\omega_0 - 2\xi(k)} \tag{11}$$

$$= -\frac{V}{2}\int đ^dk(-1)^{\text{sign}(\xi(k))}\frac{1}{\omega_0 - 2\xi(k)} \tag{12}$$

In the third line we assumed parity $\xi(k) = \xi(-k)$, and did the frequency integral by residues, as recommended. The orientation of the contour depends on the sign of $\xi(k)$. Now we use the approximation $d^dk \simeq k_F^{d-1}\frac{d\xi}{v_F}d\Omega_{d-1}$ to write

$$\mathcal{I} = -V\underbrace{\frac{\int đ^{d-1}k}{2v_F}}_{\equiv N}\left(\int_0^{E_D}\frac{d\xi}{\omega_0-2\xi} - \int_{-E_D}^{0}\frac{d\xi}{\omega_0-2\xi}\right) \tag{13}$$

$$= -NV\left(\int_0^{E_D}\frac{d\xi}{\omega_0-2\xi} - \int_0^{E_D}\frac{d\xi}{\omega_0+2\xi}\right) \tag{14}$$

$$= -NV\left(-\frac{1}{2}\log\frac{\omega_0-2E_D}{\omega_0} - \frac{1}{2}\log\frac{\omega_0+2E_D}{\omega_0}\right) \tag{15}$$

$$\overset{\omega_0 \ll E_D}{\simeq} NV\left(\frac{1}{2}\log\frac{-2E_D}{\omega_0} + \frac{1}{2}\log\frac{+2E_D}{\omega_0}\right) \tag{16}$$

$$= NV\left(\log\frac{2E_D}{\omega_0} + \frac{\mathbf{i}\pi}{2}\right). \tag{17}$$

Note that bubbles in the t-channel would give zero in this approximation because both poles would be on the same side of the frequency contour.

(e) Show that when $V < 0$ is attractive, $\chi(\omega_0)$ has a pole. Does it represent a bound-state? Interpret this pole in the two-particle Green's function as indicating an instability of the Fermi liquid to superconductivity. Compare the location of the pole to the energy E_{BCS} where the Cooper-channel interaction becomes strong.

The pole occurs at

$$0 = 1 + \mathcal{I} = 1 + NV \left(\log \frac{2E_D}{\omega_0} + \frac{\mathbf{i}\pi}{2} \right)$$

which says

$$\omega_0 = 2\mathbf{i}E_D e^{-\frac{1}{NV}}.$$

Note the crucial factor of $\mathbf{i}$. This says that the pole is in the UHP of the ω_0 plane. The fact that the pole occurs in the UHP of the ω_0 plane means that the Fourier transform of this quantity grows exponentially in time (for short times at least). It is an instability of the Fermi liquid groundstate, not a boundstate.

(f) **Cooper problem.** [optional] We can compare this result to Cooper's influential analysis of the problem of two electrons interacting with each other in the presence of an inert Fermi sea. Consider a state with two electrons with antipodal momenta and opposite spin

$$|\psi\rangle = \sum_k a_k \psi^\dagger_{k,\uparrow} \psi^\dagger_{-k,\downarrow} |F\rangle$$

where $|F\rangle = \prod_{k<k_F} \psi^\dagger_{k,\uparrow}\psi^\dagger_{k,\downarrow} |0\rangle$ is a filled Fermi sea. Consider the Hamiltonian

$$H = \sum_k \epsilon_k \psi^\dagger_{k,\sigma}\psi_{k,\sigma} + \sum_{k,k'} V_{k,k'} \psi^\dagger_{k,\sigma}\psi_{k,\sigma}\psi^\dagger_{k',\sigma'}\psi_{k',\sigma'}.$$

Write the Schrödinger equation as

$$(\omega - 2\epsilon_k) a_k = \sum_{k'} V_{k,k'} a_{k'}.$$

Now assume (following Cooper) that the potential has the following form:

$$V_{k,k'} = V w^\star_{k'} w_k, \qquad w_k = \begin{cases} 1, & 0 < \epsilon_k < E_D \\ 0, & \text{else} \end{cases}.$$

Defining $C \equiv \sum_k \omega^\star_k a_k$, show that the Schrödinger equation requires

$$1 = V \sum_k \frac{|w_k|^2}{\omega - 2\epsilon_k}. \tag{18}$$

Assuming V is attractive, find a bound state. Compare (3) to the condition for a pole found from the bubble chains above.

This leads to a boundstate at ω such that

$$1 = VN \int_0^{E_D} \frac{d\xi}{\omega - 2\xi} = -\frac{VN}{2} \log\left(\frac{-2E_D}{\omega}\right)$$

which says

$$\omega = -2E_D e^{-\frac{2}{|V|N}}.$$

The Cooper bound-state equation (3) is just what we would get if we left out the contribution of the virtual electrons with $\xi < 0$ – the ones below the Fermi energy (which in fact I did when I was first writing this problem). This results in a factor of two in the exponent (so the Cooper pair binding energy is exponentially larger than the magnitude of the frequency found above). More importantly it results in a minus sign rather than a factor of $\mathbf{i}$ (a boundstate energy should be negative). Including (correctly) the effects of fluctuations below Fermi sea level changes the boundstate to an instability. I recommend the book by Schrieffer (called *Superconductivity*) for this subject.

3. **Fermion propagator in a metal.** [bonus problem]

Starting from

$$G(p,t) = -\frac{1}{2\pi\mathbf{i}} \langle \text{gs}| \mathcal{T} c_p(t) c_p^\dagger(0) |\text{gs}\rangle \tag{19}$$

and using the free fermion time evolution operator, and the fact that the groundstate has all levels filled up to the Fermi level:

$$\langle \text{gs}| c_p^\dagger c_p |gs\rangle = \begin{cases} 1, & \epsilon_p < 0 \\ 0, & \epsilon_p > 0 \end{cases} \tag{20}$$

show that the free fermion propagator can be written as

$$G(p,\omega) = \frac{a}{\omega - \epsilon_p - \mathbf{i}\eta b \text{sgn}(\epsilon_p)} \tag{21}$$

or

$$G(p,\omega) = \frac{a'}{\omega(1 + \mathbf{i}b'\eta) - \epsilon_p} \tag{22}$$

where $\eta = 0^+$ is an infinitesimal for some constants a, b, a', b' to be determined.