University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2022
Assignment 4

Due 11:59pm Monday, April 25, 2022

Thanks in advance for following the guidelines on HW01. Please ask me by email if you have any trouble.

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1. **Galilean transformation of non-relativistic fields.**

   Show that the action

   $$S = \int dt d^d x \left(\Phi^\star \mathbf{i} \partial_t \Phi - \frac{1}{2m} \vec{\nabla} \Phi^\star \cdot \vec{\nabla} \Phi - V(|\Phi|) \right) \quad (1)$$

   is invariant under Galilean boosts, in the form

   $$\Phi(\vec{x}, t) \to \Phi'(\vec{x}', t') \quad \text{with} \quad \Phi(\vec{x}, t) = e^{-\frac{\mathbf{i}}{2} m v^2 t + \mathbf{i} m \vec{v} \cdot \vec{x}} \Phi'(\vec{x}', t') \quad (2)$$

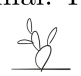
   with $t' = t, x'_i = x_i - v_i t$.

   Note that this is also how the nonrelativistic single-particle wavefunction must transform in order to preserve the Schrödinger equation.

   Don't forget that $\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu}$.

   How does the boost act on the Goldstone mode in the symmetry-broken phase?

2. **Diagrammatic understanding of BCS instability of Fermi liquid theory.**

   (a) Recall that only the four-fermion interactions with special kinematics are marginal. Keeping only these interactions, show that cactus diagrams (like this: ) dominate.

   (b) To sum the cacti, we can make bubbles with a corrected propagator. Argue that this correction to the propagator is innocuous and can be ignored.

   (c) Armed with these results, compute diagrammatically the Cooper-channel susceptibility (two-particle Green's function),

   $$\chi(\omega_0) \equiv \left\langle \mathcal{T} \psi^\dagger_{\vec{k},\omega_3,\downarrow} \psi^\dagger_{-\vec{k},\omega_4,\uparrow} \psi_{\vec{p},\omega_1,\downarrow} \psi_{-\vec{p},\omega_2,\uparrow} \right\rangle$$

   as a function of $\omega_0 \equiv \omega_1 + \omega_2$, the frequencies of the incoming particles. Think of χ as a two point function of the Cooper pair field $\Phi = \epsilon_{\alpha\beta} \psi_\alpha \psi_\beta$ at zero momentum.

   Sum the geometric series in terms of a (one-loop) integral kernel.

(d) Do the integrals. In the loops, restrict the range of momenta to $|\epsilon(k)| < E_D$, the Debye energy, since it is electrons with these energies that experience attractive interactions.

Consider for simplicity a round Fermi surface. For doing integrals of functions singular near a round Fermi surface, approximate the dispersion relation as $\epsilon(k) \simeq v_F(|k| - k_F)$, so that $d^dk \simeq k_F^{d-1} \frac{d\xi}{v_F} d\Omega_{d-1}$. I recommend doing to the frequency integral first (by residues).

(e) Show that when $V < 0$ is attractive, $\chi(\omega_0)$ has a pole. Does it represent a bound-state? Interpret this pole in the two-particle Green's function as indicating an instability of the Fermi liquid to superconductivity. Compare the location of the pole to the energy E_{BCS} where the Cooper-channel interaction becomes strong.

(f) **Cooper problem.** [optional] We can compare this result to Cooper's influential analysis of the problem of two electrons interacting with each other in the presence of an inert Fermi sea. Consider a state with two electrons with antipodal momenta and opposite spin

$$|\psi\rangle = \sum_k a_k \psi^\dagger_{k,\uparrow} \psi^\dagger_{-k,\downarrow} |F\rangle$$

where $|F\rangle = \prod_{k<k_F} \psi^\dagger_{k,\uparrow} \psi^\dagger_{k,\downarrow} |0\rangle$ is a filled Fermi sea. Consider the Hamiltonian

$$H = \sum_k \epsilon_k \psi^\dagger_{k,\sigma} \psi_{k,\sigma} + \sum_{k,k'} V_{k,k'} \psi^\dagger_{k,\sigma} \psi_{k,\sigma} \psi^\dagger_{k',\sigma'} \psi_{k',\sigma'}.$$

Write the Schrödinger equation as

$$(\omega - 2\epsilon_k) a_k = \sum_{k'} V_{k,k'} a_{k'}.$$

Now assume (following Cooper) that the potential has the following form:

$$V_{k,k'} = V w^\star_{k'} w_k, \qquad w_k = \begin{cases} 1, & 0 < \epsilon_k < E_D \\ 0, & \text{else} \end{cases}.$$

Defining $C \equiv \sum_k \omega^\star_k a_k$, show that the Schrödinger equation requires

$$1 = V \sum_k \frac{|w_k|^2}{\omega - 2\epsilon_k}. \tag{3}$$

Assuming V is attractive, find a bound state. Compare (3) to the condition for a pole found from the bubble chains above.

3. **Fermion propagator in a metal.** [bonus problem]

Starting from

$$G(p,t) = -\frac{1}{2\pi\mathbf{i}} \langle \mathrm{gs}| \mathcal{T} c_p(t) c_p^\dagger(0) |\mathrm{gs}\rangle \tag{4}$$

and using the free fermion time evolution operator, and the fact that the ground-state has all levels filled up to the Fermi level:

$$\langle \mathrm{gs}| c_p^\dagger c_p |gs\rangle = \begin{cases} 1, & \epsilon_p < 0 \\ 0, & \epsilon_p > 0 \end{cases} \tag{5}$$

show that the free fermion propagator can be written as

$$G(p,\omega) = \frac{a}{\omega - \epsilon_p - \mathbf{i}\eta b \mathrm{sgn}(\epsilon_p)} \tag{6}$$

or

$$G(p,\omega) = \frac{a'}{\omega(1 + \mathbf{i}b'\eta) - \epsilon_p} \tag{7}$$

where $\eta = 0^+$ is an infinitesimal for some constants a, b, a', b' to be determined.