University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2022
Assignment 6 – Solutions

Due 11:59pm Monday, May 9, 2022

Thanks in advance for following the guidelines on HW01. Please ask me by email if you have any trouble.

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1. **Brain-warmers on spin coherent states.**

   (a) Show that

   $$\vec{n} = z^\dagger \vec{\boldsymbol{\sigma}} z$$

   where $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is a unit vector, and z_α are the components of the (normalized) spin coherent state $|\vec{n}\rangle$ ($\boldsymbol{\sigma}\cdot\vec{n}\,|\vec{n}\rangle = |\vec{n}\rangle$) in the Z-eigenbasis.

   One nice way to do this is to use the eigenvalue equation:

   $$(\vec{\sigma}\cdot\vec{n})_{\alpha\beta} z_\beta = z_\alpha .$$

   Multiply on the left by $z^\dagger_\alpha$ to get

   $$\vec{n}\cdot(z^\dagger\vec{\sigma}z) = z^\dagger(\vec{\sigma}\cdot\vec{n})z = 1.$$

   Since both $\vec{n}$ and $z^\dagger\vec{\sigma}z$ are unit vectors, this equation implies that they must be the *same* unit vector. To see that $z^\dagger\vec{\sigma}z$ is a unit vector, use $z^\dagger z = 1$ and the identity

   $$\sum_{a=1}^{3} \sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}.$$

   (b) In the same notation, check that

   $$\langle\vec{n}_1|\vec{n}_2\rangle = z_1^\dagger z_2.$$

   (c) Check that

   $$\mathbb{1}_{2\times2} = \int \frac{d^2n}{2\pi}|\vec{n}\rangle\langle\vec{n}|.$$

   (d) Check that

   $$\int dt\,\mathbf{i}z^\dagger\dot{z} = \int dt\,\mathbf{i}\frac{1}{2}\left(\cos\theta\dot{\phi} + \dot{\psi}\right) = 2\pi W_0[\hat{n}].$$

(e) Show that

$$\langle \check{n} | \vec{h} \cdot \vec{\mathbf{S}} | \check{n} \rangle = s\vec{h} \cdot \check{n}$$

where $|\check{n}\rangle = \mathcal{R} |s, s\rangle$ is a coherent state of spin s (where $|s, s\rangle$ is the eigenvector of $\mathbf{S}^z$ with maximal eigenvalue, and $\mathcal{R}$ is the rotation operator which takes $\check{z}$ to $\check{n}$).

(f) Show that for several spins and $i \neq j$

$$\langle \check{n} | \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j | \check{n} \rangle = s^2 \check{n}_i \cdot \check{n}_j,$$

where now $|\check{n}\rangle \equiv \otimes_j (\mathcal{R}_i |s_i\rangle)$ is a product of coherent states of each of the spins individually.

2. **Brain-warmer on Schwinger bosons.**

Recall the Schwinger-boson representation of the $\mathsf{SU}(2)$ algebra:

$$\mathbf{S}^+ = a^\dagger b,\ \mathbf{S}^- = b^\dagger a,\ \mathbf{S}^z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right), \tag{1}$$

where the modes a, b satisfy $[a, a^\dagger] = 1 = [b, b^\dagger], [a, b] = [a, b^\dagger] = 0$. This is the algebra of a simple harmonic oscillator in two dimensions,

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right).$$

Is the $\mathsf{SU}(2)$ a symmetry of this Hamiltonian? How does it act on the oscillator coordinates? Check that the oscillator algebra does indeed imply that $\vec{\mathbf{S}}$ defined this way satisfy the $\mathsf{SU}(2)$ algebra.

It is useful to write (1) in the same way we wrote $\vec{n} = z^\dagger \vec{\sigma} z$:

$$\mathbf{S}^i = \frac{1}{2} a^\dagger_\alpha \sigma^i_{\alpha\beta} a_\beta, \quad a_\alpha = (a, b)_\alpha.$$

Then

$$[\mathbf{S}^i, \mathbf{S}^j] = \frac{1}{4}\sigma^i_{\alpha\beta}\sigma^j_{\gamma\delta} \underbrace{[a^\dagger_\alpha a_\beta, a^\dagger_\gamma a_\delta]}_{=\delta_{\beta\gamma}a^\dagger_\alpha a_\delta + \delta_{\alpha\delta}a^\dagger_\beta a_\gamma} \tag{2}$$

$$= \frac{1}{4} a^\dagger_\alpha [\sigma^i, \sigma^j]_{\alpha\beta} a_\beta = \mathbf{i}\epsilon^{ijk}\mathbf{S}^k. \tag{3}$$

The $\mathsf{SU}(2)$ acts on the oscillators by $(a, b) = (x + \mathbf{i}p_x, y + \mathbf{i}p_y)$ is a doublet. We see that it is a symmetry of

$$H = a^\dagger a + b^\dagger b + 1 = n_a + n_b + 1$$

since $S^\pm$ preserve $n_a + n_b$. More precisely,

$$\mathbf{S}^2 \equiv \vec{\mathbf{S}} \cdot \vec{\mathbf{S}} = (n_a + n_b)(n_a + n_b + 1),$$

so the spin is $s = n_a + n_b$.

3. **Geometric Quantization of the 2-torus.**

   Redo the analysis that we did in lecture for the two-sphere for the case of the two-torus, $S^1 \times S^1$. The coordinates on the torus are $(x, y) \simeq (x + 2\pi, y + 2\pi)$; use $\frac{N}{2\pi}\mathrm{d}x \wedge \mathrm{d}y$ as the symplectic form. Show that the resulting Hilbert space represents the Heisenberg algebra

   $$e^{\mathbf{ix}}e^{\mathbf{iy}} = e^{\mathbf{iy}}e^{\mathbf{ix}}e^{\frac{2\pi\mathbf{i}}{N}}.$$

   (I am using boldface letters for operators.) The irreducible representation of this algebra is the same Hilbert space as a particle on a periodic one-dimensional lattice with N sites.

   Notice that the area of the phase space (using this symplectic form) is

   $$\int_{T^2} \frac{N}{2\pi}\mathrm{d}x \wedge \mathrm{d}y = \frac{N}{2\pi}\int_0^2 \pi dx \int_0^{2\pi} dy = 2\pi N \in 2\pi\mathbb{Z}$$

   if N is an integer, so the WZW term gives a well-defined contribution to the path integral $e^{\mathbf{i}\int_D \frac{N}{2\pi}dx\wedge dy}$. We conclude that $p_x = \frac{N}{2\pi}y$ so that

   $$[x, y] = \frac{2\pi\mathbf{i}}{N}$$

   and the single-valued operators $e^{\mathbf{ix}}$ and $e^{\mathbf{iy}}$ satisfy the stated algebra (using the BCH formula).

   Notice that the fact that $x \equiv x + 2\pi$ means that $p_x \in \mathbb{Z}$, so $y \in 2\pi\mathbb{Z}/N$ takes discrete. Then $y \equiv y + 2\pi$ says that only the N values $y = 1..N$ are independent.

4. **Particle on a sphere with a monopole inside.**

   Consider a particle of mass m and electric charge e with action

   $$S[\vec{x}] = \int \mathrm{d}t \left(\frac{1}{2}m\dot{\vec{x}}^2 + e\dot{\vec{x}} \cdot \vec{A}(\vec{x})\right)$$

   constrained to move on a two sphere of radius r in three-space, $\vec{x}^2 = r^2$. Suppose further that there is a *magnetic monopole* inside this sphere: this means that $4\pi g = \int_{S^2} \vec{B} \cdot \mathrm{d}\vec{a} = \int_{S^2} F$, where $F = \mathrm{d}A$. (Since the particle lives only at $\vec{x}^2 = r^2$, the form of the field in the core of the monopole is not relevant here.)

   (a) Find an expression for $A = A_i\mathrm{d}x^i = A_\theta\mathrm{d}\theta + A_\varphi\mathrm{d}\varphi$ such that $F = \mathrm{d}A$ has flux $4\pi g$ through the sphere.

   (b) Show that the Witten argument gives the Dirac quantization condition $2eg \in \mathbb{Z}$.

(c) Take the limit $m \to 0$. Count the states in the lowest Landau level. Compare with the calculation in lecture for coherent state quantization of a spin-s.

For $m \to 0$, we can ignore the second-order kinetic term. (Recall (for the case of electrons in the plane) that the cyclotron frequency goes like $\omega_c \sim \frac{B}{m} \overset{m\to 0}{\to} \infty$, in this limit, leaving behind just the lowest Landau level.) One way to find the states is simply to do canonical quantization on $S = \int dts(1+\cos\theta)\dot{\phi}$, with $s = 2eg \in \mathbb{Z}$. From this we conclude

$$p_\phi = s(1+\cos\theta). \tag{4}$$

But $\phi \equiv \phi + 2\pi$ means $p_\phi \in \mathbb{Z}$. The range of the RHS of (4) is $[0, 2s]$. There are $2s+1$ integers in this range, in agreement with our result from the study of the spin system. Relative to the expectation from Landau level degeneracy in flat space (one state per flux quantum) there is an extra $+1$.