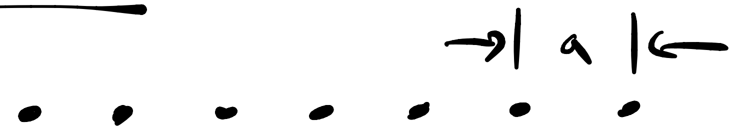
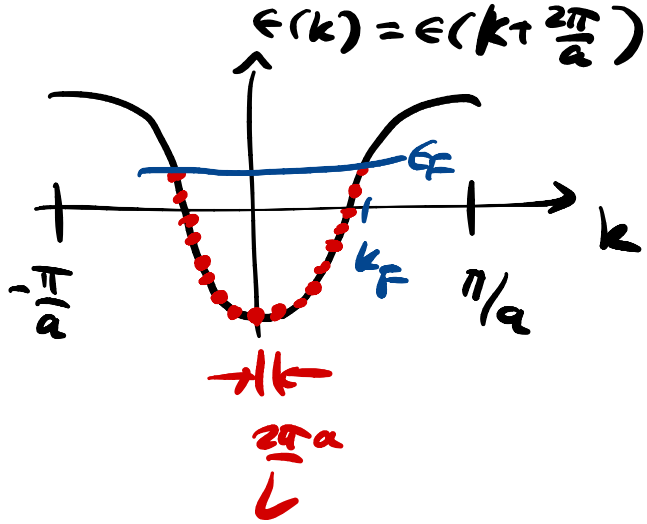
Physics of the Anomaly

Last time:



$$H = -t \sum c^{\dagger}_{n} c_{n+1} + h.c.$$



for low energies:

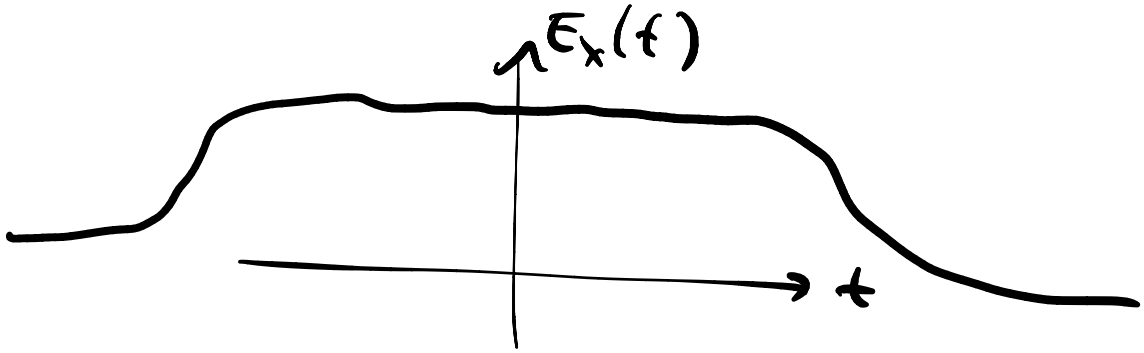
$$\rightsquigarrow S[\Psi] = \int d^2x \; \bar{\Psi} i \not{\partial} \Psi$$

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad c_n \sim e^{ink_F} \psi_L(n) + e^{-ink_F} \psi_R(n)$$

Couple to EM field:

$$H \to -t \sum_{n} c_n^{\dagger} e^{-iA_x(t)} c_{n+1} + h.c.$$

$$\rightsquigarrow \quad S[\Psi] = \int d^2x \, \bar{\Psi} \, i \not{D} \Psi$$

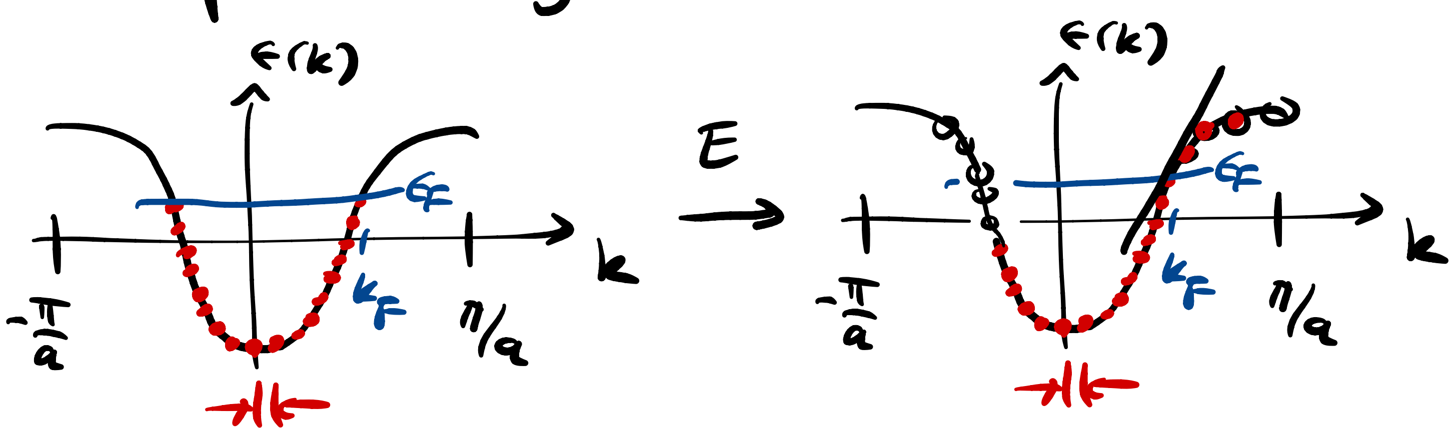


each particle experiences force

$$\partial_t p = e E_x$$

net change in $p = \hbar k$ of each particle

$$\Delta p = e \int dt \, E_x(t) = 3 \cdot \frac{2\pi}{L}$$



From the POV of $S[\psi] = \int d^2x\, \bar{\Psi}\, i\not{D}\, \Psi$:

$$\Delta Q_A = \Delta(N_R - N_L) = 2 \underbrace{\frac{\Delta p}{2\pi a/L}}_{3} = \frac{L}{\pi a} \Delta p$$

$$L = a\sum^{N} = \int dx$$

$$= \frac{L}{\pi} e \int dt\, E_x(t)$$

$$= \frac{1}{\pi} e \int_0^L dx \int dt\, E_x(t)$$

$$= \frac{e}{2\pi} \int d^2x\, \epsilon^{\mu\nu} F_{\mu\nu}$$

$$\Longleftarrow \quad \partial_\mu J^\mu_A = \frac{e}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu}$$

't Hooft Anomaly Matching:

Anomalies are RG invariant.

Pf: $$Z \longrightarrow e^{i\int \alpha \mathcal{A}} Z$$

under sym. w/ anomaly $\partial^\mu j_\mu = \mathcal{A}$.

But RG preserves Z.

$\blacksquare$

Why care? Much of physics is

UV model

$\rightsquigarrow$??

IR physics

Another way to have $\mathcal{A} \neq 0$:

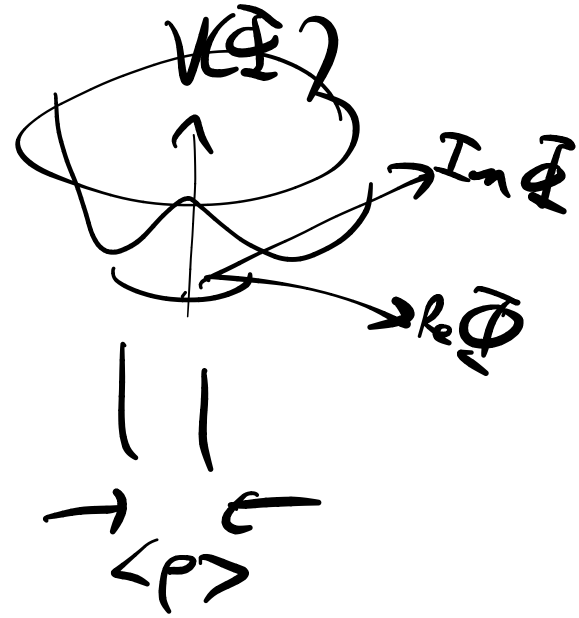
If $\exists \; \phi \longrightarrow \phi + \underline{\alpha} \iff$ Sym is SSB.

w/ $$S \ni \int \phi \frac{F \wedge F}{8\pi^2} \implies \delta_\alpha S = \int \alpha \mathcal{A}.$$

$$\Phi = \rho e^{i\phi} \to e^{i\alpha}\Phi . \quad \text{linear.}$$

$$\Longleftrightarrow \quad \phi \to \phi + \alpha . \quad \text{nonlinear.}$$

$$\langle \rho \rangle \neq 0 .$$



$$S \ni \int \phi F \wedge F$$

is an example of Wess-Zumino-Witten term.

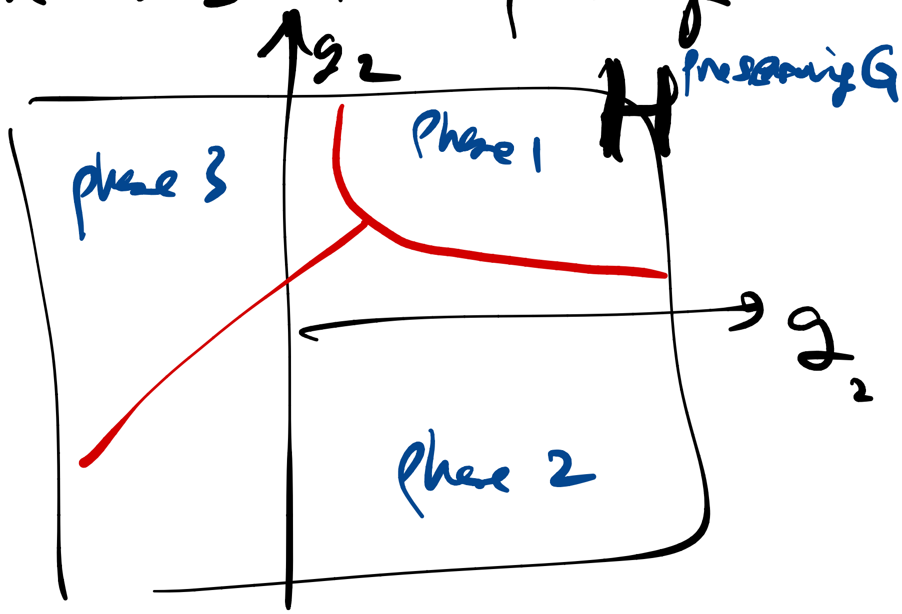
Comments: ① Discrete symms can also be anomalous.

Pf: $\underset{\rightsquigarrow \text{ anomaly}}{G^{\text{continuous}}} \supset \Gamma^{\text{discrete}}$

② Anomaly = obstruction to gauging the symmetry.

③ Anomaly is more basic than phase of matter

all these phases have the same G anomaly



• Other anomalies.

$$F_{\mu\nu} = [D_\mu, D_\nu]$$ curvature of connection

$$R_{\mu\nu}{}^{\rho}{}_{\tau} V^\tau = [D_\mu, D_\nu] V^\rho$$ curvature of spacetime.

[Tong lectures on Gauge theory ch 3]

To couple spinors to curved spacetime:

$$g_{\mu\nu}(x) = e_\mu{}^a(x)\, e_\nu{}^b(x)\, \eta_{ab}$$

Mink. metric.

four bone → vierbein

many → viel bein

Local Lorentz transf $\Lambda_a{}^b(x)$ preserve $g_{\mu\nu}$
$SO(3,1)$

required connection is $\omega_\mu{}^{ab}$ spin connection.

<u>Def</u>: $$D_\mu e_\nu^a \equiv \partial_\mu e_\nu^a - \Gamma^\rho_{\mu\nu} e_\rho^a + \omega_\mu{}^a{}_b e_\nu^b \stackrel{!}{=} 0.$$

Field strength of ω:

$$(R_{\mu\nu})^a{}_b = \partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b - [\omega_\mu, \omega_\nu]^a{}_b$$

$$= (R_{\mu\nu}{}^\rho{}_\sigma)\, e^a_\rho e^\sigma_b .$$

Spinor covariant deriv:

$$D_\mu \psi_\alpha = \partial_\mu \psi_\alpha + \frac{1}{4}\omega_\mu{}^{ab} (\Sigma_{ab})^\beta_\gamma \psi_\beta$$

$\gamma^\mu(x) = \gamma^a e_a^\mu(x)$ $\qquad \Sigma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$

Cordinary gammas

$$S[\psi] = \int d^D x \underbrace{\sqrt{g}} \; \bar{\Psi} \, i \slashed{D} \, \Psi$$

$$\sqrt{g} \equiv \sqrt{|\det g|}$$

$$(i\slashed{D})^2 = \dots + \Sigma^{\mu\nu} R_{\mu\nu}^{\dots}$$

same calc. as before:

$$D_\mu j^\mu_A = -\frac{1}{384\pi^2} \boxed{\sum_f Q_f (-1)^f} R_{\mu\nu\lambda\tau} R_{\rho\sigma}{}^{\lambda\tau} \epsilon^{\mu\nu\rho\sigma}$$

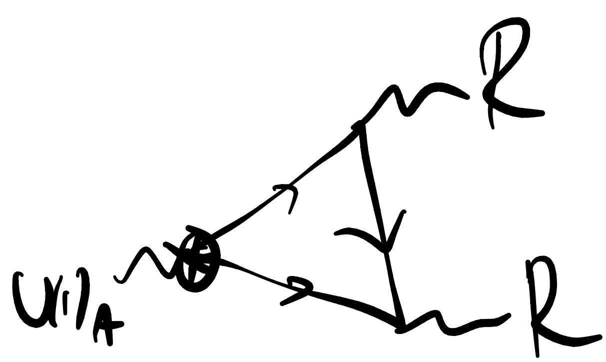
Weyl

-1 for L
$+1$ for R

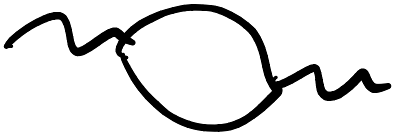
Requires $\sum_f Q_f (-1)^f \overset{!}{=} 0$

for $U(1)_Y$.

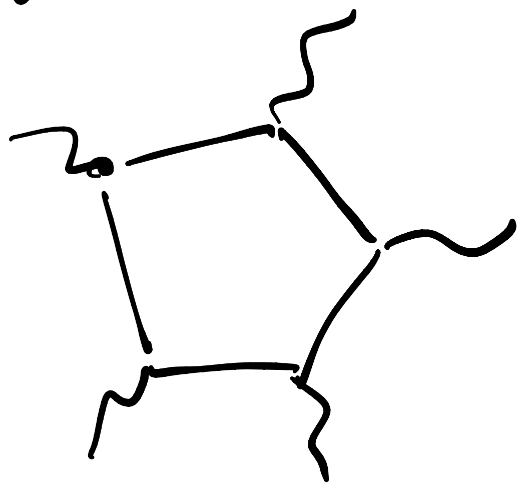
" grav. Mixed anomaly "
" grav. chiral anomaly "



- Purely gravitational anomalies happen in $D = 8k+2$



$D=2$



$D=10$.

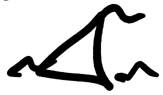
[Alvarez-Gaumé - Witten 83]

$\propto$ chiral central charge.

- $SU(2)$ anomaly. in $D=3+1$.

consider $SU(2)$ gauge theory w/ a Dirac fermion Ψ in the $\underline{2}$ representation. $\begin{pmatrix}\cdot\\ \cdot\end{pmatrix}$

no perturbative anomalies because $\underline{\underline{2 \simeq \bar{2}}}$

()

$$Z = \int DA\, D\Psi\, D\bar{\Psi}\, e^{-S_{YM}[A] - \int d^4x\, \bar{\Psi} i \not{D} \Psi}$$
$$= \int DA\, \det(i\not{D})\, e^{-S_{YM}[A]}$$

$$\det(i\not{D}) = \prod_n \epsilon_n$$

regulate in gauge inv. way ✓

Instead consider just ψ_L in the $\underline{2}$ of $SU(2)$.

$$Z_L = \int DA \, D\psi_L \, D\bar{\psi}_L \times e^{-S_{YM}[A] - \int d^4x \, \bar{\psi}_L i\sigma^\mu D_\mu \psi_L} \qquad \psi_L = \frac{(1+\gamma_5)}{2}\Psi$$

$$= \int DA \, \underbrace{\det\left(i\not{D}\left(\frac{1+\gamma^5}{2}\right)\right)} e^{-S_{YM}}$$

$\sigma^\mu D_\mu : L \to R$

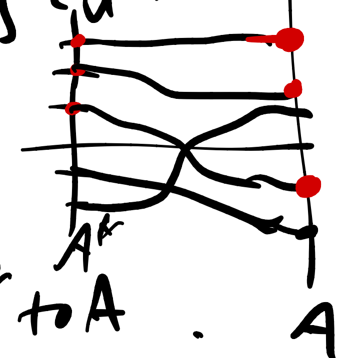
evecs w $\epsilon_n = 0$ come in L/R pairs.

$$= \pm\sqrt{\det(i\not{D})}$$

Try to define sign of $\sqrt{\ }$:

pick some A^*_μ and take only $\epsilon_n > 0$

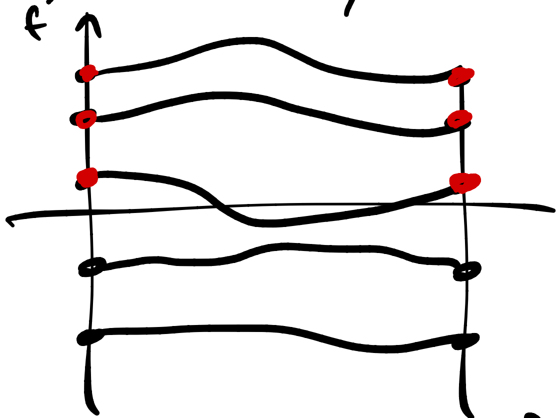
so $\sqrt{\det i\not{D}_{A^*}} > 0$.



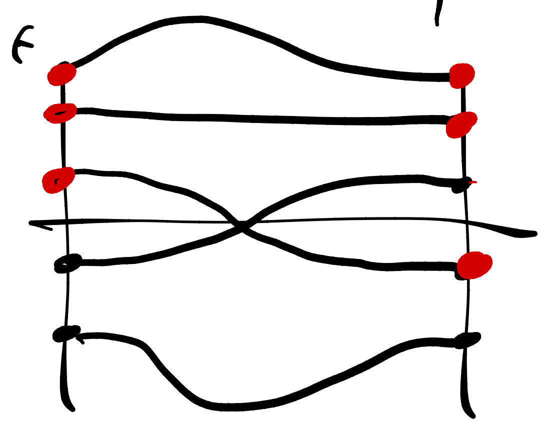
to define $\sqrt{\det i\not{D}_A}$: find a path from A^* to A.

Q: is this def gauge inv't?

$$A_\mu \longmapsto A_\mu^\Omega = \Omega(x) A_\mu \Omega^{-1}(x) + i\,\Omega(x) A_\mu \Omega^{\dagger}(x)$$



OR



A even # of crossings A^Ω $\qquad$ A $\qquad$ A^Ω

odd #

$$\sqrt{\det i\not{D}_A} = \sqrt{\det i\not{D}_{A^\Omega}} \qquad \sqrt{\det i\not{D}_A} = -\sqrt{\det i\not{D}_{A^\Omega}}$$

✓ A: (YES) $\qquad$ ✗ A: NO.

what is $\Omega : \mathbb{R}^4 \to G = SU(2)$

$$\Omega \to 1 \quad \text{at } x \to \infty$$

$$\mathbb{R}^4 \cup \infty = S^4. \qquad \Omega : S^4 \to G.$$

<u>claim:</u> ① $\pi_4(\underset{\simeq S^3}{SU(2)}) = \mathbb{Z}_2$.

② if Ω is nontrivial
then $\det^{1/2} i\not{D}_{A^\Omega} = -\det^{1/2} i\not{D}_A$.

if: an odd # of
spin odd-half-integer
L weyl fermions.

$$S[\phi] = \int d^D x \sqrt{g} \left[D_\mu \phi D_\nu \phi \, g^{\mu\nu}(x) - V(\phi) \right]$$

$$a \prod \gamma^\mu = \gamma^5 \quad (\gamma^5)^2 = 1.$$

$$\mathrm{tr}\, \gamma^5 \underbrace{\Sigma^{\mu\nu} \Sigma^{\rho\sigma} \Sigma^{\lambda\alpha}}_{= \# \epsilon^{\mu\nu\rho\sigma\lambda\alpha} \gamma^5 + \dots}$$

$$\partial^\mu j_\mu^A \overset{\text{anomaly}}{=} \mathcal{A} = \delta_\theta S_{eff}$$

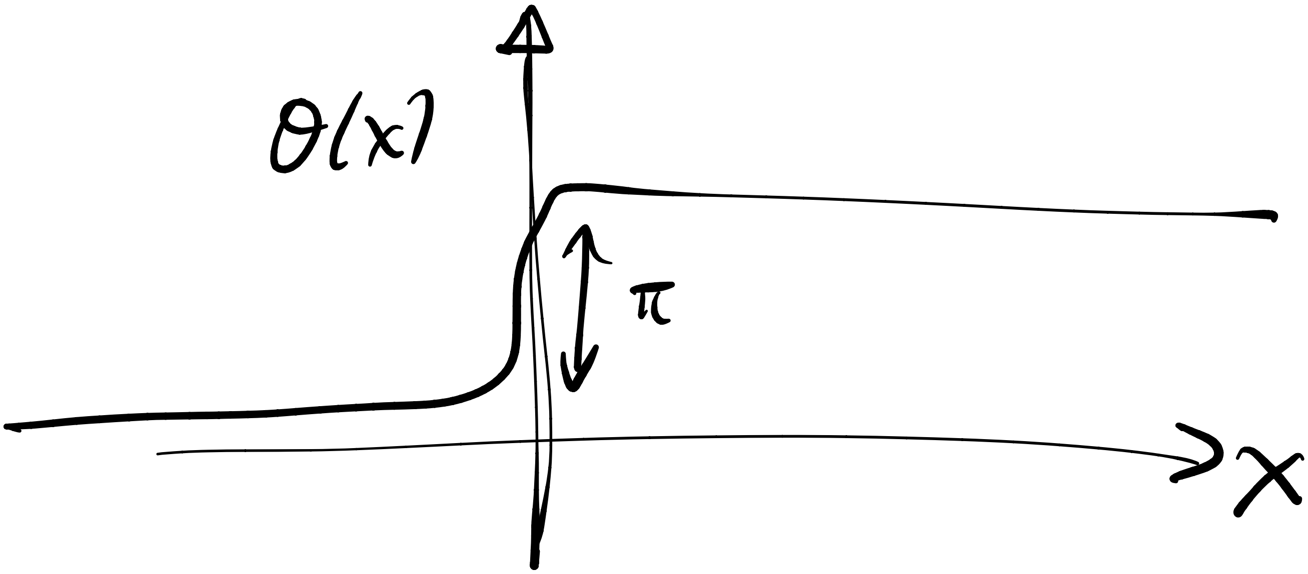
$$\underline{\underline{\langle j^\mu \rangle}} = \frac{\delta S_{eff}}{\delta A_\mu(x)}$$ ← know from anomaly.

$$S_{eff} \to S_{eff} + \int \frac{\theta}{2\pi} F_{\mu\nu} \epsilon^{\mu\nu}.$$

$$Z \xrightarrow{\alpha} \underbrace{\int (D\psi D\bar\psi) e^{i\int \alpha \mathcal{A}}}\, e^{iS[\psi] + i\int \alpha m \bar\psi \gamma^5 \psi}$$

$$= Z\, e^{i\int \alpha (\underline{\underline{\mathcal{A}}} + 2m\, \underline{\bar\psi \gamma^5 \psi})}$$

$$\overset{\text{Noether}}{=} Z\, e^{i\int \alpha\, \partial^\mu j_\mu}$$



$\forall t$.

$\theta(x>0) = \pi + \theta(x<0)$.