3.3 Path Integrals for Spin Systems
(and sums)

No lecture Thursday April 28

Quantum Spin System:

$$\mathcal{H} = \bigotimes_{j=1}^{N} \mathcal{H}_j \qquad \dim \mathcal{H} = 2^N$$

$$\mathcal{H}_j = \text{span}\{ |\uparrow_j\rangle, |\downarrow_j\rangle \}$$

one qbit: hermitian operators on $\mathcal{H}_j$ $= \text{span}\{ \mathbb{1}, \sigma^x, \sigma^y, \sigma^z \}$

$= \text{span}\{ \mathbb{1}, X, Y, Z \}$

$$\begin{cases} XZ = -ZX \\ XY = iZ \end{cases} \qquad (\sigma^\alpha)^2 = \mathbb{1} \quad \alpha = x, y, z.$$

+ cyclic perms $X \to Y \to Z$.

Multiple Qbits: $X_j = \mathbb{1} \otimes \dots \otimes \underbrace{X}_{\text{site } j} \otimes \mathbb{1} \dots \mathbb{1}$

$$\Rightarrow [\sigma_j^\alpha, \sigma_\ell^\beta] = 0 \text{ for } j \neq \ell. \qquad \underline{X_j Z_\ell = (-1)^{\delta_{j\ell}} Z_\ell X_j}.$$

path integrals $\longleftrightarrow$ bases of $\mathcal{H}$

one choice of basis: $Z|\uparrow\rangle = |\uparrow\rangle$
$Z|\downarrow\rangle = -|\downarrow\rangle$.

$$\mathbb{1} = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$$

$$|\uparrow\rangle = |s=1\rangle \qquad |\downarrow\rangle = |s=-1\rangle$$

Countable $\longrightarrow$ path 'sum'.

eg: one qubit: $\hat{H} = E_0 - \frac{\Delta}{2}\hat{X} - \bar{h}\hat{Z}$

If $\bar{h}=0$ then eigenstates are $|\rightarrow\rangle$ and $\langle\leftarrow|$

$\rightsquigarrow$ $\Delta E = \Delta$.

E_1 ——— Δ $\updownarrow$
E_{gs} ———

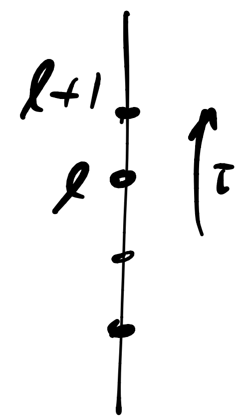
$$Z_Q(T) = \mathrm{tr}\, e^{-\hat{H}/T} = \sum_{s=\pm} \langle s| \underbrace{e^{-\hat{H}/T}}_{e^{-\Delta\tau\hat{H}} \cdots\cdots e^{-\Delta\tau\hat{H}}} |s\rangle$$

$$\mathbb{1} = \sum_s |s\rangle\langle s|$$

$$\Rightarrow Z_Q = \sum_{s_1 \ldots s_{M_\tau}} \langle s_{M_\tau} | e^{-\Delta\tau H} | s_{M_\tau - 1} \rangle\langle s_{M_\tau - 1} | e^{-\Delta\tau H} | s_{M_\tau - 2} \rangle$$

$$\ldots \langle s_1 | e^{-\Delta\tau H} | s_{M_\tau} \rangle$$

$$\hat{T} = e^{-\Delta\tau \hat{H}} \quad \text{transfer matrix}$$



$$= Z_1 = \sum_{\substack{\{s_l = \pm 1\} \\ l = 1..M_\tau}} e^{-S[s]}$$

$$\text{w/ } S[s] = -K \sum_{l=1}^{M_\tau} s_{l+1} s_l - h \sum_{l=1}^{M_\tau} s_l$$

$$\langle s_{l+1} | e^{-aX} | s_l \rangle = e^{K s_{l+1} s_l}$$

$$e^{-2K} = \tanh a \Leftarrow e^{-aX} = \cosh a \, \mathbb{1} - \sinh a \, X .$$

$$Z_1 = \mathrm{tr}\, T^{M_\tau} = \lambda_+^{M_\tau} + \lambda_-^{M_\tau} = \lambda_+^{M_\tau}\left(1 + \left(\frac{\lambda_-}{\lambda_+}\right)^{M_\tau}\right)$$

$$\lambda_\pm = e^{K}\cosh h \pm \sqrt{e^{2K}\sinh^2 h + e^{-2K}} \xrightarrow{h \to 0} \begin{cases} 2\cosh K \\ 2\sinh K . \end{cases}$$

Note: M_τ is up to us.

many classical systems $\longrightarrow$ same Quantum system.

energy gap Δ of $\hat{H}$ $\leftrightsquigarrow$ correlation time ξ of S.

<u>Correlation F'ns:</u>

$$C(\ell,\ell') \equiv \langle s_\ell s_{\ell'} \rangle = \frac{1}{Z} \sum_{\{s_\ell\}} e^{-S[s]} s_\ell s_{\ell'}$$

$$\overset{\text{transl. sym}}{=} C(\ell-\ell')$$

assume: $\ell' > \ell$

$$= \frac{1}{Z} \operatorname{tr}\left(T^{M_\tau - \ell'} Z\, T^{\ell'-\ell} Z\, T^{\ell}\right)$$

<u>eg:</u> $h=0$ $\quad T|\rightarrow\rangle = \lambda_+ |\rightarrow\rangle$

$T|\leftarrow\rangle = \lambda_- |\leftarrow\rangle$.

$$\rightarrow \langle \alpha | Z | \beta \rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\alpha\beta} \quad \text{if } \alpha,\beta = \leftarrow, \rightarrow$$

$$= \frac{\lambda_+^{M_\tau - \ell' + \ell} \lambda_-^{\ell'-\ell} + \lambda_-^{M_\tau - \ell' + \ell} \lambda_+^{\ell'-\ell}}{\lambda_+^{M_\tau} + \lambda_-^{M_\tau}} \xrightarrow{M_\tau \to \infty} \tanh^{\ell'-\ell} K.$$

$S_\ell \equiv Z(\tau) \qquad \tau = \Delta\tau\, \ell$

$$C(\tau) = \langle T\, Z(\tau) Z(0) \rangle = \underline{\tanh^\ell K = e^{-|\tau|/\xi}}$$

Correlation time ξ $\quad \underline{\frac{1}{\xi}} = \frac{1}{\Delta\tau} \ln \coth K \;\; \underline{= \Delta}$

If $e^{-2K} = \tanh X$ then $e^{-2X} = \tanh K$

$$X = \Delta\tau \cdot \Delta = \frac{\Delta\tau}{\xi}.$$

$$\Longleftrightarrow \quad \underline{1 = \sinh 2X \sinh 2K}.$$

To get back continuous-time quantum spin system:

$$\frac{\xi}{\Delta\tau} \overset{K \gg 1}{\simeq} \underline{\frac{1}{2} e^{2K} \gg 1}.$$

Continuum limit & universality

physical quantities of quantum system

correlation time $\xi \simeq \Delta\tau \frac{1}{2} e^{2K}$

length of chain $L_\tau \equiv \Delta\tau M_\tau$ (inverse temp)

physical separations betw. operators $\tau = (\ell-\ell')\Delta\tau$

applied field in $\hat{z}$ $\bar{h} = h/\Delta\tau$

take $\Delta\tau \to 0$, $K \to \infty$, $M_\tau \to \infty$

fixing physics.

eg: $E_\pm = E_0 \pm \sqrt{(\Delta/2)^2 + \bar{h}^2}$

$$F = -T \ln Z_{\mathrm{cl}} = E_0 - T \ln\left(2\cosh \frac{1}{T}\sqrt{\left(\frac{\Delta}{2}\right)^2 + \bar{h}^2}\right)$$

$$\lambda_\pm \overset{*}{\simeq} \sqrt{\frac{2\xi}{\Delta\tau}}\left(1 \pm \frac{\Delta\tau}{2\xi}\sqrt{1+4\tilde{h}^2\xi^2}\right)$$

$$\Rightarrow F = L_\tau\left(\underbrace{-\frac{K}{\Delta\tau}}_{\to\infty} - \frac{1}{L_\tau}\ln\left(2\cosh\frac{L_\tau}{2}\sqrt{\xi^{-2}+4\tilde{h}^2}\right)\right)$$

cutoff-dependent vac. energy

$$\begin{cases} \xi = \dfrac{1}{\Delta} \\ L_\tau = \dfrac{1}{T} \end{cases}$$

matches. ✓

$$C(\tau) = \frac{1}{Z_\Theta}\,\mathrm{tr}\, e^{-H/T}\Big(\theta(\tau)\, Z(\tau)\, Z(0) + \theta(-\tau)\, Z(0)\, Z(\tau)\Big)$$

$$Z(\tau) = e^{H\tau}\, Z(0)\, e^{-H\tau}$$

$$C(\tau)\Big|_{T\to 0} = \sum_n |\langle \underset{\text{g.s.}}{0}|Z|n\rangle|^2\, e^{-(E_n-E_0)|\tau|}$$

time ordering

$$H|n\rangle = E_n|n\rangle.$$

if $\langle 0|Z|1\rangle \neq 0$ and $\langle 0|Z|0\rangle = 0$.

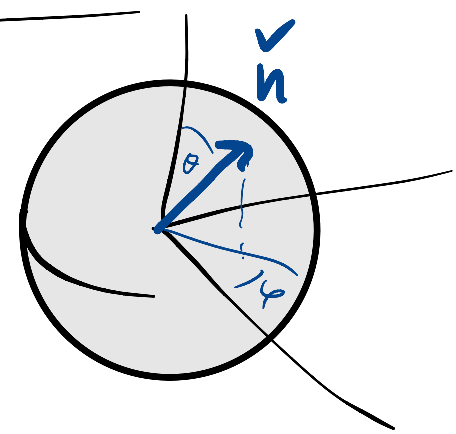
$$\underset{\tau\gg\ldots}{\sim} e^{-\Delta|\tau|}.$$

$$C(\tau) \overset{\tau \gg \dots}{=} (\text{disconnected}) + e^{-|\tau|/\xi}$$

(def of ξ)

$$\Rightarrow \quad \underline{\xi = \frac{1}{\Delta}}.$$

Coherent-State Path integrals for Spin Systems & Geometric "Quantization"



Round 2sphere S^2 has an area element

$$\omega = s \, d\cos\theta \wedge d\varphi$$

w/ $\displaystyle\int_{S^2} \omega = 4\pi s.$

Q: If S^2 = phase space, ω = symplectic form

what quantum system?

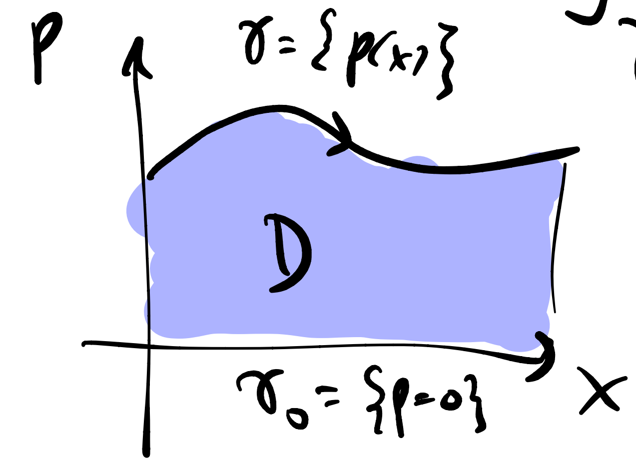
Recall: Phase-space formulation of cl. mech

$$A[x(t), p(t)] = \int_{t_1}^{t_2} dt\,\left(p\dot{x} - H(x,p)\right)$$

(2d phase space)

$$= \int_\gamma \underline{\underline{p(x)\,dx}} - \int H\,dt$$

↑ area under the curve



$$\int p(t)\,\dot{x}(t)\,dt = \int_{\partial D} p\,dx \overset{\text{Stokes}}{=} \int_D dp \wedge dx$$

$$= \int_D \omega$$

More generally:

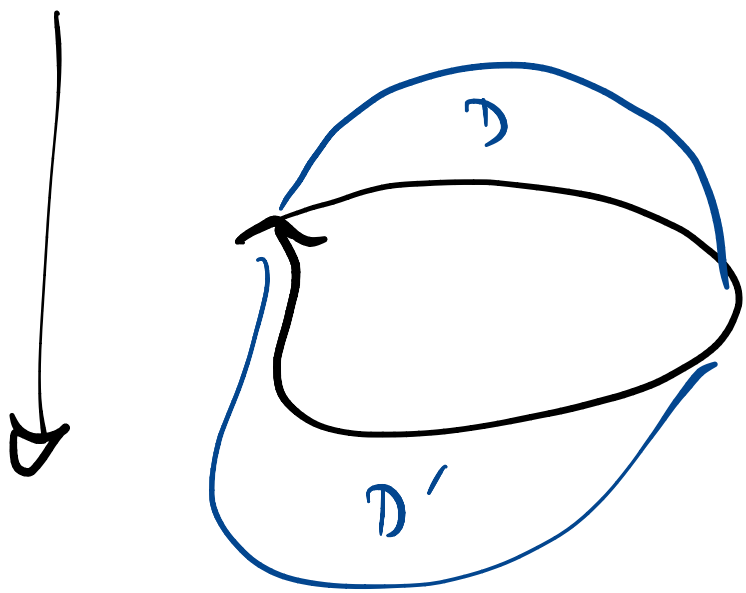
2n-dim'l phase space w/ coords u_α, $\alpha = 1..2n$

w/ symplectic form $\omega = \omega_{\alpha\beta}\, du^\alpha \wedge du^\beta$.

and action $A[u] = \int_D \omega - \int_{\partial D} dt\, H(u,t)$

It's important that $d\omega = 0$

$\Rightarrow 0 = \frac{\delta A'}{\delta u}$ depends only on $\gamma = \partial D$ and not D itself.



$$\omega_{\alpha\beta}\,\dot{u}^{\beta} = \frac{\partial H}{\partial u^{\alpha}}.$$

Locally we can find coords w/ $\omega = d(p_\alpha dx^\alpha)$

globally ω need not be exact.

eg: on S^2 $d\omega = 0$ since no 3-forms
in 2d.

But $\int_{S^2} \omega = 4\pi s \neq 0$

$\Rightarrow \omega \neq d\alpha$ for some globally defined α.

if it were then $\int_{S^2} \omega = \int_{S^2} d\alpha = \int_{\partial(S^2)=\phi} \alpha$

locally: $\alpha = s \cos\theta \, d\varphi$

But φ is singular at the poles $\theta = 0, \pi$.

$\omega = s \, d\cos\theta \wedge d\varphi$

has $O(3)$ symmetry.

[a symplectic form has to be:

- non singular
- non degenerate $\sim \omega \neq 0$ everywhere
- closed $d\omega = 0$.

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what is the corresponding quantum system?

$$Z = \int [d\theta \, d\varphi] \, e^{\frac{i}{\hbar} A[\theta, \varphi]}$$

$$[dx] = N \prod_{i=1}^{M_\tau} dx(t_i)$$

Hints : • It has $O(3)$ sym. (if $H=0$).

• could choose $H = -s\,\underline{\vec{h}}\cdot\check{n}$.

$$\check{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

choose $\check{z} \propto \vec{h}$. ie $\vec{h} = h\check{z}$.

eom : $$0 = \frac{\delta A}{\delta\theta(t)} = -s\sin\theta(\dot{\varphi} - h)$$

$$0 = \frac{\delta A}{\delta\varphi(t)} = -\partial_t(s\cos\theta)$$

$$\Longleftrightarrow \boxed{\partial_t\check{n} = \vec{h}\times\check{n}.}$$

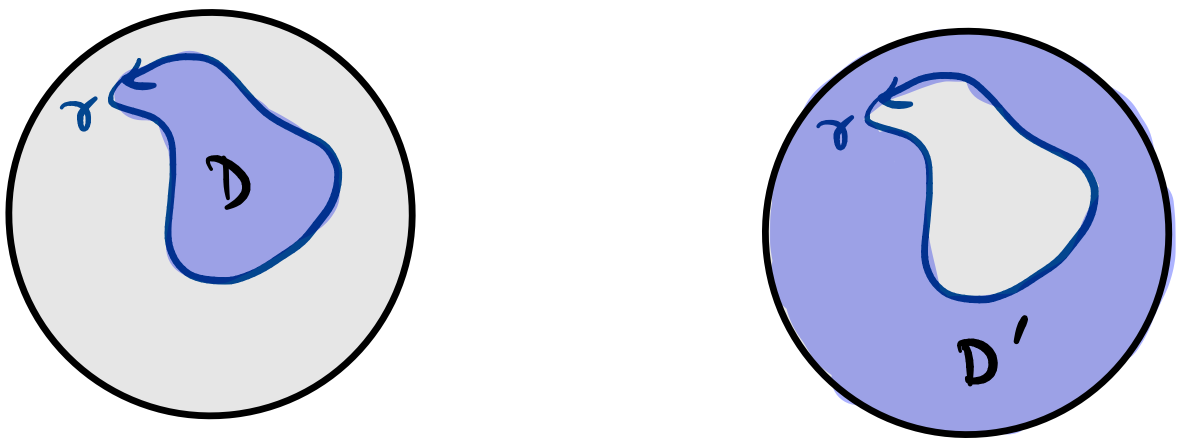
Landau-Lifschitz eqn for spin precession.

• semiclassical expectations.

A patch of phasespace of area $2\pi\hbar$ contributes one quantum state.

here $\text{Area(phasespace)} = 4\pi s \longrightarrow \frac{4\pi s}{2\pi\hbar} = \frac{2s}{\hbar}$ states. (at large s.)

- $e^{i\frac{A}{\hbar}}$ must be indep of choice of D



$$A[\gamma] = \underline{\int_D \omega} \neq \underline{\int_{D'} \omega}$$

with $\partial D = \gamma$

$$s\left(\int_D - \int_{D'}\right) \text{area} = s\int_{S^2} \text{area}$$

$$= 4\pi s.$$

In order for

e^{iA} to depend only on γ and not D

we require $1 \overset{!}{=} e^{i\left(\int_D \omega - \int_{D'} \omega\right)} = e^{4\pi s i}$

$$\Longleftrightarrow \boxed{2s \in \mathbb{Z}.}$$

<u>Goal</u>: write $\int_D \omega$ in a rot.-inv't way.

in terms of n^a.

<u>claim</u>:

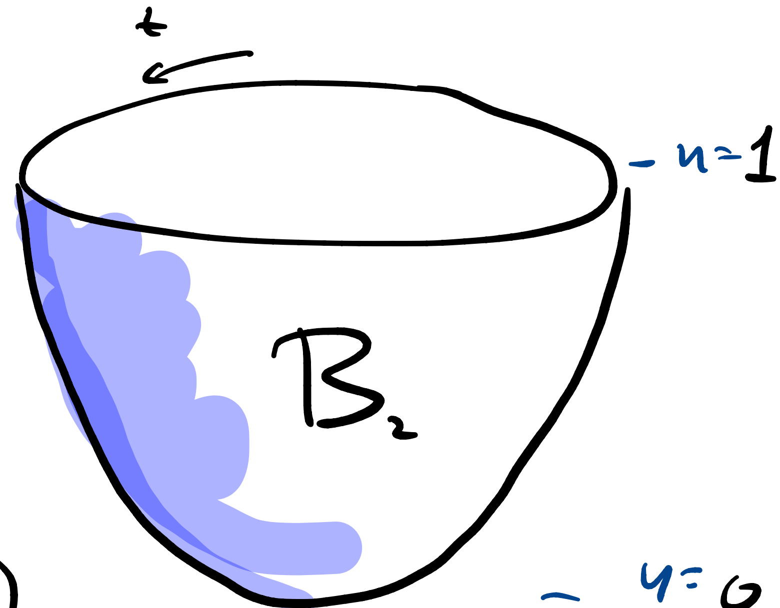
$x^\mu = (t, u)^\mu$

$$\frac{1}{4\pi}\int_D \omega = \frac{1}{4\pi}\int dt \cos\theta\, \dot\phi =$$

$$\frac{1}{8\pi}\int du \int dt\, \epsilon_{\mu\nu}\, n^a\, \underline{\partial_\mu} n^b\, \underline{\partial_\nu} n^c\, \epsilon^{abc} \equiv W_0[\check{n}]$$

extend the $\check{n}(t)$ field into an extra dim.

μ



$$\vec{n}(t, u=1) = \vec{n}(t)$$

$$\vec{n}(t, u=0) = (0,0,1)$$

(in the picture: $\vec{n}(t+\beta) = \vec{n}(t)$)

∂B_2 = actual spacetime.

$\equiv$ WZW term.

<u>claim</u>: δW_0 depends only on $\vec{n}(t)$

not on $\vec{n}(t,u)$.

0

$$\delta W_0 = \iint \delta n \; \partial n \; \partial n$$

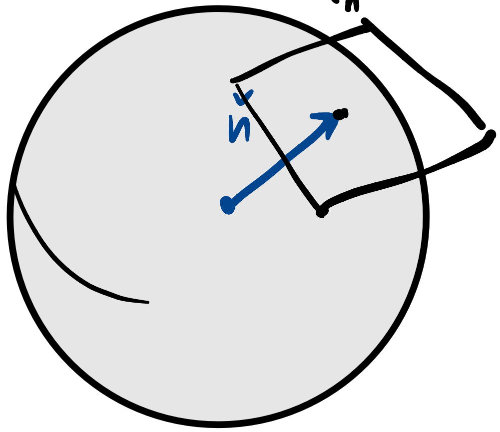
$$+ \iint n \; \partial \delta n \; \partial n$$

$$+ \iint n \; \partial n \; \partial \delta n$$

same by

$\epsilon^{abc} = - \epsilon^{acb}$

$\epsilon^{\mu\nu} = - \epsilon^{\nu\mu}$.



$T_{\vec{n}}S^2 \ni \delta n^a$ and $\partial_t n^a$ and $\partial_u n^a$.

$$\Downarrow$$

$$\delta \vec{n} \cdot \partial_t \vec{n} \times \partial_u \vec{n} = 0$$

$$\delta W_0 = \frac{1}{4\pi} \int_B n^a \, d(\delta n^b) \wedge dn^c \, \epsilon^{abc}$$

$$= \int_B d\left(\frac{1}{4\pi} n^a \delta n^b dn^c \epsilon^{abc} \right)$$

$$\overset{\text{Stokes}}{=} \frac{1}{4\pi} \int_{\partial B} n^a \, \delta n^b \, dn^c \, \epsilon^{abc}$$

$$= \frac{1}{4\pi} \int dt \; \delta\vec{n} \cdot \left(\dot{\vec{n}} \times \vec{n} \right)$$

$$\left(\epsilon^{abc} n^a m^b \ell^c = \vec{n} \cdot (\vec{m} \times \vec{\ell}) \right)$$

$$0 = \frac{\delta}{\delta \vec{n}(t)} \left(4\pi W_0[n] + s\vec{h}\cdot\vec{n} + \lambda(\vec{n}-1) \right)$$

$$= s\, \partial_t \vec{n} \times \vec{n} + s\vec{h} + 2\lambda\vec{n}$$

$$\vec{n} \times (\ast) \Longrightarrow \partial_t \vec{n} = \vec{h} \times \vec{n} .$$

e^{iA} is well-def'd

$\Longrightarrow$ coeff. of $4\pi W_0$ is quantized.