3.4 Topological terms from Integrating out fermions

spins

$$H_K = M \, (c^\dagger \vec{\sigma} c) \cdot \vec{S}$$

$$\rightsquigarrow Z = \int [D\psi D\bar{\psi} \, \underline{D\check{n}}] \, e^{-\int_0^T dt \, \bar{\psi}(\partial_t - M\check{n}\cdot\vec{\sigma})\psi - S_0[\check{n}]}$$

$\psi = (\psi_\uparrow, \psi_\downarrow)$

$\check{n}^2 = 1$

$$S_0[\check{n}] = 4\pi \overset{S}{\vee} W_0[\check{n}] + \int K \dot{n}^2 + \dots$$

__Claim:__ for a fixed slowly varying config of $\check{n}$ the fermions are gapped.

$$\langle T c^\dagger_\alpha(t) \, c_\beta(0) \rangle = \langle \bar{\psi}_\alpha(t) \, \psi_\beta(0) \rangle \sim e^{-Mt}.$$

$$Z = \int [D\check{n}] \, e^{-S_{eff}[\check{n}]} \qquad D \equiv \partial_t - M\check{n}\cdot\vec{\sigma}.$$

$$S_{eff}[\check{n}] = S_0[\check{n}] - \log\det(D) = S_0 + S_1$$

fermion loop

slowly-varying : $\underline{|\dot{\vec{n}}| \ll M}$.

<u>Method 1</u> : $\delta S_1 = -\mathrm{tr}(\delta D D^{-1})$

$$= -\mathrm{tr}\, \delta D D^\dagger (D D^\dagger)^{-1}$$

$$D^\dagger \equiv -\partial_t - M \vec{n}\cdot\vec{\sigma} \qquad \delta D = -M \delta\vec{n}\cdot\vec{\sigma}$$

$$\Rightarrow \delta S_1 = M\, \mathrm{tr}\Big[\delta\vec{n}\cdot\vec{\sigma}\, (\partial_t + M\vec{n}\cdot\vec{\sigma}) \times \underbrace{\left(-\partial_t^2 + M^2 - M\dot{\vec{n}}\cdot\vec{\sigma}\right)}_{DD^\dagger}{}^{-1} \Big]$$

<u>eg</u> : $D = \underline{\underline{M}} \left(\vec{n}\cdot\vec{\sigma} + \frac{\partial_t}{M}\right)$ $\underset{\text{small}}{}$
small

$$\boxed{\begin{array}{l} \underline{\text{use}}: \vec{n}^2 = 1 \\ \Rightarrow \vec{n}\cdot\delta\vec{n} = 0, \\ \vec{n}\cdot\dot{\vec{n}} = 0. \end{array}}$$

$$\delta S_1 = \int dt \left[-\frac{M}{|M|} \frac{1}{2} \delta\vec{n}\cdot(\vec{n}\times\dot{\vec{n}}) + \frac{1}{4M} \delta\dot{\vec{n}}\cdot\dot{\vec{n}} + \dots \right]$$

$$= \delta \left[-2\pi \frac{M}{|M|} W_0[\hat{n}] + \int_0^T dt \frac{\dot{\vec{n}}^2}{8M} + \dots \right]$$

$$\begin{cases} K \to K + \frac{1}{8M} \\ S \to S - \frac{\mathrm{sign}(M)}{2} \end{cases}$$

$M>0$ is (AFM) $\to H_K$ wants to $\left(\substack{\text{minimize} \\ \text{max.}}\right) S_{total}^2$
$M<0$ is (FM)

$$\vec{S}_{total} = \vec{S} + c^\dagger \vec{\sigma} c$$

$$\underline{S} \otimes \underline{\tfrac{1}{2}} = \underline{S - \tfrac{1}{2}} \oplus \underline{S + \tfrac{1}{2}}$$

$M>0$ this is the g.s.

if $M<0$ this is g.s.

Method 2 to calculate $S_1 = -\mathrm{tr} \ln D \overset{?}{=} -\mathrm{tr} \ln \tilde{D}$

$$\tilde{D} \equiv U^\dagger(t) D U(t) \equiv \partial_t - ia - M\sigma^3$$

$$\underline{a \equiv U^\dagger i \partial_t U.}$$

(2×2 matrix)

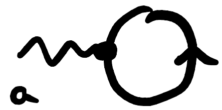
$$U^\dagger_{(t)} \vec{n}_{(t)} \vec{\sigma} U_{(t)} = \sigma^3$$

$$\underline{\tilde{D} = G_0^{-1}(1 - G_0 ia)} \qquad \underline{\underline{G_0^{-1} = \partial_t - M\sigma^3.}}$$

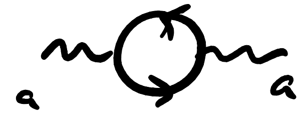
$$S_1[\check n] = -\mathrm{tr}\ln\tilde D = \mathrm{tr}\ln G_0 - \mathrm{tr}\ln(1 - G_0\, i a)$$

$$= \mathrm{tr}\Big(\ln G_0 + G_0\, i a + \tfrac{1}{2}(G_0\, i a)^2 + \dots\Big)$$

$\uparrow$ cst. ind. of $\check n$



$\approx S_{(1)}$



$+\dots$

$\approx S_{(2)}$

$$S_{(1)} = \mathrm{tr}(G_0\, i a) = \mathrm{tr}_\sigma \int d\omega\, \frac{e^{i\omega dt}}{-i\omega - M\sigma^3}\, i\, a_{\omega=0}$$

$$G(t=0) = \langle c^\dagger(dt)\, c(0)\rangle$$

$$= \theta(M\sigma^3) = \frac{1+M\sigma^3}{2}$$

$$= -\mathrm{sign}(M)\int dt\; i a^3(t)$$

$$a = \sum_{\alpha=1,2,3} a^\alpha \sigma^\alpha .$$

$$i a^3 = \frac{1}{2}\mathrm{tr}\, i a \sigma^3 = \frac{1}{4}\cos\theta\,\dot\varphi$$

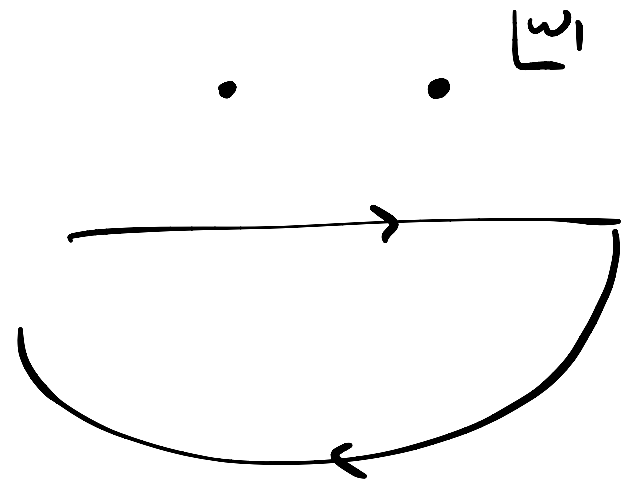
$$\Rightarrow S_{(1)} = -2\pi\,\mathrm{sgn}(M)\, W_0[n].$$

$$S_{(2)} = \frac{1}{2}\,\mathrm{tr}\left(G_0\, i a\right)^2$$

$$= \frac{1}{2} \int d\omega_1 \int d\omega\, \mathrm{tr}_\sigma \left[\frac{1}{-i\omega_1 - M\sigma^3}\, i a_{-\omega} \cdot \frac{1}{-i(\omega_1+\omega) - M\sigma^3}\, i a_\omega \right]$$

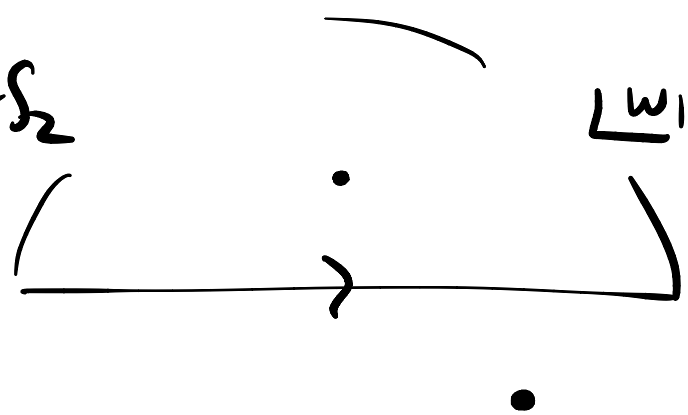
$$I(s_1, s_2) = \int d\omega_1 \quad \frac{1}{-i\omega_1 - M s_1} \quad \frac{1}{-i(\omega_1+\omega) - M s_2}$$

if $s_1 = s_2$



$$= 0.$$

if $s_1 = -s_2$



$$= \frac{1}{2M - i s_1 \omega}$$

$$= \frac{1}{2}\,\frac{1}{M}\left(\frac{1}{1 - i s_1 \frac{\omega}{M}}\right) = \frac{1}{2M}\left(1 + O\left(\frac{\omega}{M}\right)\right)$$

$$S_{(2)} = \frac{1}{2M}\int dt\,\left(a_1^2 + a_2^2\right)\left(1 + \mathcal{O}\left(\tfrac{1}{M}\right)\right)$$

$$= \frac{1}{8M}\int dt\,\left(\partial_t \vec{n}\right)^2\left(1 + \mathcal{O}\left(\tfrac{1}{M}\right)\right)$$

$$\begin{cases} \vec{n}\cdot\vec{\sigma} = u\sigma^3 u^+ \\ 1 = u^+ u \\ u^+\dot{u} = -ia \end{cases}$$

$$\boxed{\begin{array}{l}\text{This is is a local} \\ \text{action up to scale } M \\ \Rightarrow \check{n} \text{ gapped out } \psi.\end{array}}$$

$$\langle \bar{\psi}_\alpha(t)\,\psi_\beta(0)\rangle = \left(\frac{1}{D}\right)_{t,0} = \left(\frac{D^+}{DD^+}\right)_{t,0}$$

$$= \int d\omega\; e^{i\omega t}\,\frac{(\omega + iM\,\vec{n}\cdot\vec{\sigma})}{\omega^2 + M^2} \sim e^{-Mt}.$$

<u>Notice</u>:
$$\begin{cases} u \to u\,e^{i\psi(t)\sigma^3} \\ a = u^+ i\partial_t u \to e^{-i\sigma^3\psi}(a + i\partial_t)\,e^{i\sigma^3\psi} \\ S_1 \to S_1 + \int dt\,\dot{\psi}. \end{cases}$$

3.5 Pions

Physics below the scale of the Higgs phenomenon in EW sector.

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{g^2}\mathrm{tr}\,G_{\mu\nu}G^{\mu\nu}$$

$$+ i \sum_{\alpha = L,R} \sum_{f} \bar{q}_{\alpha f} \not{D} q_{\alpha f} - \underline{\underline{\bar{q} M q}}$$

flavors: u, d, maybe s.

choose q to diagonalize $M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$

If $m_u = m_d$: Isospin symmetry

$$\begin{pmatrix} u \\ d \end{pmatrix} \to U \begin{pmatrix} u \\ d \end{pmatrix} \quad U \in SU(N_f = 2)$$

(non chiral)

If $m_u = m_d = 0$ L, R would decouple

$$q \to e^{i\alpha\gamma_5} q \qquad q_L \to V q_L,\ q_R \to V^{-1} q_R$$

$V \in SU(N_f = 2)$.

Why SU and not U ?

$U(1)_A$ is anomalous

$$\partial_\mu j^\mu_A \propto \underline{\underline{\mathrm{tr}\, G \wedge G}}$$

$\Rightarrow$ not a symmetry of QCD.

(no η' goldstone)

['t Hooft]

$U(1) \subset \underset{\text{Isospin}}{U(N_f)}$ is <u>baryon # B</u>

$q \to e^{i\alpha} q$. is anomalous

$$\partial_\mu j^\mu_B \propto \mathrm{tr}\, \underset{\sim}{W \wedge W}$$

electroweak field str.

<u>also</u>: B-L is not anomalous.

<u>phenomenological Input</u> : $\langle \bar{q}_f q_f \rangle = V^3$ indep of f

$(\forall f)$

g.s. expectation.

"chiral condensate"

Λ spontaneously breaks

$$SU(N_f)_L \times SU(N_f)_R \longrightarrow SU(N_f)_{\text{isospin}}$$

<u>$N_f = 2$</u>. $\begin{pmatrix} u \\ d \end{pmatrix}$ is a doublet of

$$\left.\begin{aligned} p &= u_\alpha u_\beta d_\gamma \epsilon^{\alpha\beta\gamma} \\ n &= u_\alpha d_\beta d_\gamma \epsilon^{\alpha\beta\gamma} \end{aligned}\right\} \Longrightarrow \begin{pmatrix} p \\ n \end{pmatrix} \text{ is a doublet}$$

Isospin is explicitly (weakly) broken by:

① $m_d = 4.7$ MeV $\neq m_u = 2.15$ MeV

② electromagnetism $q_d = -\frac{1}{3} \neq q_u = +\frac{2}{3}$.

$$m_d - m_u \ll \Lambda$$

<u>Use this SB-structure w/ the EFT strategy:</u>

① dof: goldstones = pions

② ✓

③ cutoff $= \Lambda$.

Linear σ-model: $\quad \Sigma_{\alpha\beta} \sim \bar{q}_\alpha q_\beta$

$$SU(2)_L \times SU(2)_R : \begin{cases} \Sigma \to g_L \Sigma g_R^\dagger \\ \Sigma^\dagger \to g_R \Sigma^\dagger g_L^\dagger \end{cases}$$

we can make singlets: $\quad \Sigma_{\alpha\beta}\Sigma^\dagger_{\beta\alpha} = \mathrm{tr}\,\Sigma\Sigma^\dagger \equiv |\Sigma|^2$

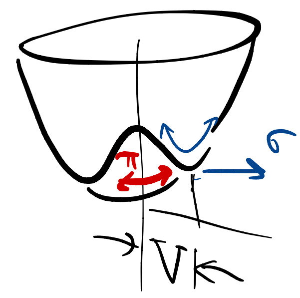
$$\mathcal{L} = |\partial_\mu \Sigma|^2 + \underline{\underline{m^2}}\,\mathrm{tr}\,\Sigma\Sigma^\dagger - \underline{\underline{\frac{\lambda}{4}}}\left(\mathrm{tr}\,\Sigma\Sigma^\dagger\right)^2 - g\,\mathrm{tr}\left(\Sigma\Sigma^\dagger\Sigma\Sigma^\dagger\right) + \dots$$

choose $V(\Sigma)$ to have minima at

$$\langle \Sigma \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \left(\hookrightarrow v = \frac{2m}{\sqrt{\lambda}},\ g = 0 \right)$$

preserves only $SU(2)_{\text{isospin}}$:

$$\Sigma \to g \Sigma g^\dagger .$$



$$\Sigma(x) = \frac{v + \sigma(x)}{\sqrt{2}}\, e^{\frac{2i\pi^a(x)\tau^a}{F_\pi}}$$

π^a parametrize $\{\text{minima}\} = SU(2)\times SU(2)/SU(2)$

$\simeq S^3$.

$$F_\pi = V = \frac{2m}{\sqrt{\lambda}}$$

Under $g_{L/R} = e^{i\theta^a_{L/R}\tau^a}$

$$\begin{cases} \pi^a \longrightarrow \pi^a + \underbrace{\frac{F_\pi}{2}(\theta^a_L - \theta^a_R)}_{\text{nonlinear realization of } SU(2)_{\text{axial}}} - \underbrace{\frac{1}{2} f^{abc}(\theta^a_L + \theta^a_R)\pi^c}_{\text{linear rep (adjoint) of } SU(2)_{\text{isospin}}} \\ \sigma \rightarrow \sigma \end{cases}$$

$\pi^\pm, \pi^0$ create pions

shift sym under $SU(2)_{\text{axial}}$ forbids mass terms (π^2) $\times$

σ is massive & can be removed

by $m \to \infty$, $\lambda \to \infty$ fixing F_π.

$$U(x) \equiv \frac{\sqrt{2}}{V}\Sigma(x)\Big|_{\sigma=0} = e^{\frac{2i\pi^a\tau^a}{F_\pi}}.$$

$$U^\dagger U = U U^\dagger = \mathbb{1}.$$

$$\mathcal{L}_\chi = \frac{F_\pi^2}{4} \operatorname{tr} D_\mu U D^\mu U^\dagger + L_1 \operatorname{tr} \left(D_\mu U D^\mu U^\dagger\right)^2$$

$$+ L_2 \operatorname{tr} D_\mu U D_\nu U^\dagger \operatorname{tr} D^\nu U^\dagger U^\mu U$$

$$+ L_3 \operatorname{tr} D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger + \ldots$$

6 derivs

Leading term

$$\underset{\downarrow}{=} \frac{1}{2} \partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{F_\pi^2}\left(-\frac{1}{3}\pi^0\pi^0 D_\mu \pi^+ D^\mu \pi^- + \ldots\right)$$

$$+ \underbrace{\frac{1}{F_\pi^4}\left(\frac{1}{18}(\pi^-\pi^+)^2 D_\mu \pi^0 D^\mu \pi^0 + \ldots\right)}$$

irrelevant interactions w/ 2 derivs

w/ coeffs are <u>determined</u>

$$\left(\frac{E}{F_\pi} \text{ is a small parameter.}\right)$$

Pion mass & Spurion Method

$m_{\pi^\pm} \sim 140\ \text{MeV}$

$\mathcal{L}_{QCD} \ni \bar{q} M q$ (M is a matrix of couplings.)

AN INVARIANCE of $\mathcal{L}_{QCD}$ is

(*) $\underline{q_{L/R} \to g_{L/R}\, q_{L/R}}$, $\underline{M \to g_L M g_R^\dagger}$.

Pretend that M is a field. (spurion)

then * is a symmetry

this true including QCD interactions.

$\Rightarrow \mathcal{L}_{eff}(U, M)$ must be * symmetric.

$$\Delta \mathcal{L}_\chi = \frac{V^3}{2} \operatorname{tr}\left(M U + M^\dagger U^\dagger\right) + \ldots$$

$$= V^3 (m_u + m_d) - \frac{V^3}{2 F_\pi^2} (m_u + m_d) \sum_a \pi_a^2 + \mathcal{O}(\pi^3)$$

pion mass!

$$\langle \bar{q} M q \rangle = V^3 (m_u + m_d)$$

$$\Longrightarrow \quad m_\pi^2 \simeq \frac{V^3}{F_\pi^2} (m_u + m_d) .$$

[Gell-Mann - Oakes - Renner rel'n.]

<u>Next</u>: Pion decay.