Pions, cont'd

$$SU(2)_L \times SU(2)_R \xrightarrow{\langle \bar{q} q \rangle} SU(2)_{\text{isospin}}$$

goldstones: $U \longrightarrow g_L U g_R^\dagger$.

$$U(x) = e^{2i \frac{\pi^a(x) \tau^a}{F_\pi}} \qquad U^\dagger U = \mathbb{1}.$$

$$\mathcal{L}_\chi = \frac{F_\pi^2}{4} \operatorname{tr} D_\mu U D^\mu U^\dagger + \text{terms w/ } > 2 \text{ derivatives.}$$

$$= \frac{(D\pi)^2}{2} + \frac{1}{F_\pi^2}\left(-\frac{1}{3}(\pi^0)^2 D_\mu \pi^+ D^\mu \pi^- + \ldots\right)$$

$$+ \ldots$$

$$F_\pi \sim V = \langle \bar{q} q \rangle.$$

$SU(2)_L$ is gauged:

$$\mathcal{L}_{\text{weak}} \supset g W_\mu^a \underbrace{(J_\mu^a - J_\mu^{5a})}_{\text{"V - A"}} = g W_\mu^a \left[V_{ij} \bar{Q}_i \gamma^\mu \left(\frac{1-\gamma^5}{2}\right) \tau^a Q_j \right.$$

$Q_1 = \begin{pmatrix} u \\ d \end{pmatrix}$ are doublets

$L_1 = \begin{pmatrix} e \\ \nu_e \end{pmatrix}$ of $SU(2)_L$.

$$+ \bar{L}_i \gamma^\mu \tau^a \frac{1-\gamma^5}{2} L_i \Big].$$

"A pion is a Goldstone boson for axial SU(2)":

$$\langle 0 | J^{5a}_\mu(x) | \pi^b(p) \rangle = i F_\pi \, p_\mu \, e^{-ip\cdot x} \delta^{ab}$$

means: acting w/ the current on the vacuum we can make a pion

1-pion state w/ momentum p

$F_\pi \to 0$ if symmetry is unbroken.

p_μ: global rotation costs no energy

$e^{-ip\cdot x}$: transl sym.

$$\int d^4p \, e^{ip\cdot x} p^\mu \left(\text{BHS} \right) = F_\pi p^2 = F_\pi m_\pi^2$$

$$\text{BHS} = \partial^\mu J^{5a}_\mu \sim 0$$

$\Rightarrow$ pions are massless if $\partial^\mu J^{5a}_\mu = 0$.

charged pion decay :

$$\xleftarrow{\quad t \quad}$$

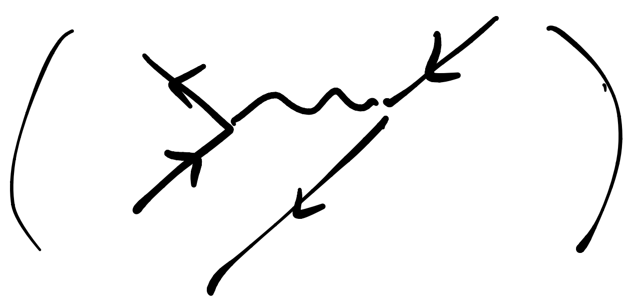
$$\text{leptons} \xleftarrow[W]{\substack{\text{electroweak}\\ \text{interaction}}} \underline{\underline{J^{5a}_\mu}} \xleftarrow{\text{Goldstone}} \pi$$

$$\frac{G_F}{\sqrt{2}} = \frac{g_w^2}{M_W^2}$$

$$\mathcal{M}\left(\pi^+ \to \mu^+ \nu_\mu\right) = \frac{G_F}{\sqrt{2}} F_\pi\, p_\mu\, \bar{v}_{\nu_\mu} \gamma^\mu (1-\gamma^5) u_\mu$$

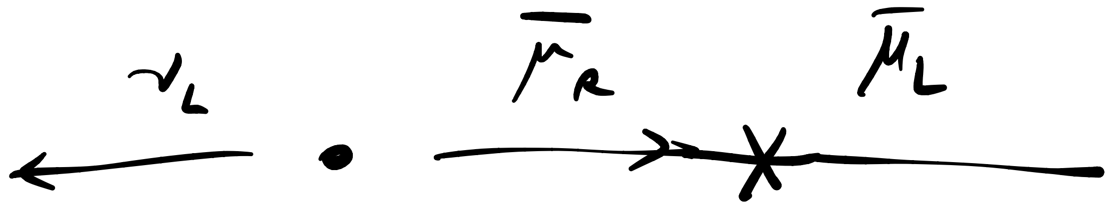
we know $G_F \sim 10^{-5}\ \mathrm{GeV}^{-2}$ from $\mu^- \to e^- \bar{\nu}_e \nu_\mu$

$$\Gamma\left(\pi^+ \to \mu^+ \nu_\mu\right) = \frac{G_F^2 F_\pi^2}{4\pi} m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$



Helicity suppression : $\frac{m_\mu}{m_e} \sim 200$.

Initial state has spin 0 $\Rightarrow$ helicity 0



In order for the final state to have angular momentum 0, we need an insertion of the $m \bar{\mu}_R \mu_L$ operator

$$\tau_{\pi^+} = \Gamma^{-} = 2.6 \cdot 10^{-8}\, s.$$

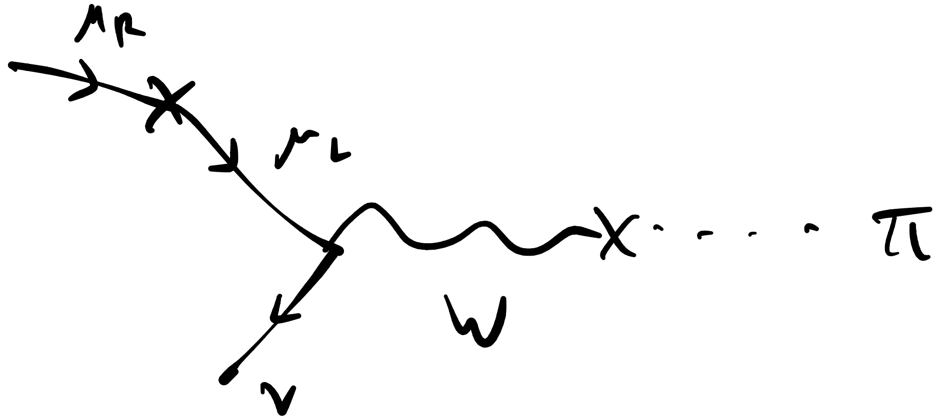
we know : G_F, m_μ, m_π $\Rightarrow$ $F_\pi = 92$ MeV.

(m_μ = 106 MeV)

"pion decay constant"

$\rightarrow$ predictions for many other processes

eg : $\pi^0 \pi^0 \rightarrow \pi^+ \pi^-$ ...



$$\underbrace{\partial^\mu J_\mu^{5a}}_{SU(2)_L} \propto \underbrace{(m_u - m_d) + \alpha}_{\ll \Lambda_{QCD}}$$

<u>π^0 decay</u>:
$$\partial_\mu J^{\mu 5 a} = -\frac{e^2}{16\pi^2}\epsilon^{\mu\lambda\alpha\beta}F_{\mu\nu}F^{\alpha\beta} \times \mathrm{tr}(\tau^a Q^2)$$

$$Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix}$$

$\Uparrow$

$J_e J_e SU(2)_{axial}$ anomaly

$\Downarrow$ $\qquad\qquad\qquad \downarrow$

$$q_\mu \langle p_1 \epsilon_1 ; p_2 \epsilon_2 | J_\mu^{5, a=3}(q) | 0 \rangle = -c\, \frac{e^2}{4\pi^2}\epsilon^{\nu\lambda\alpha\beta} p_1^\nu \epsilon_1^\lambda p_2^\alpha \epsilon_2^\beta \,.$$

2 photons w/ momentum p_α, polariz'n ϵ_α.

$\propto \mathcal{A}(\pi^0 \to \gamma\gamma)$

- $J^{\mu 5, a=3} \subset SU(2)_L$: $\begin{cases} u \to e^{i\theta\gamma^5} u \\ d \to e^{-i\theta\gamma^5} d \end{cases}$

  <u>makes π^0.</u> (like σ^3)

  <u>vs</u>: anomalous $U(1)_A$: $\{ q_i \to e^{i\theta\gamma^5} q_i$

  (η')

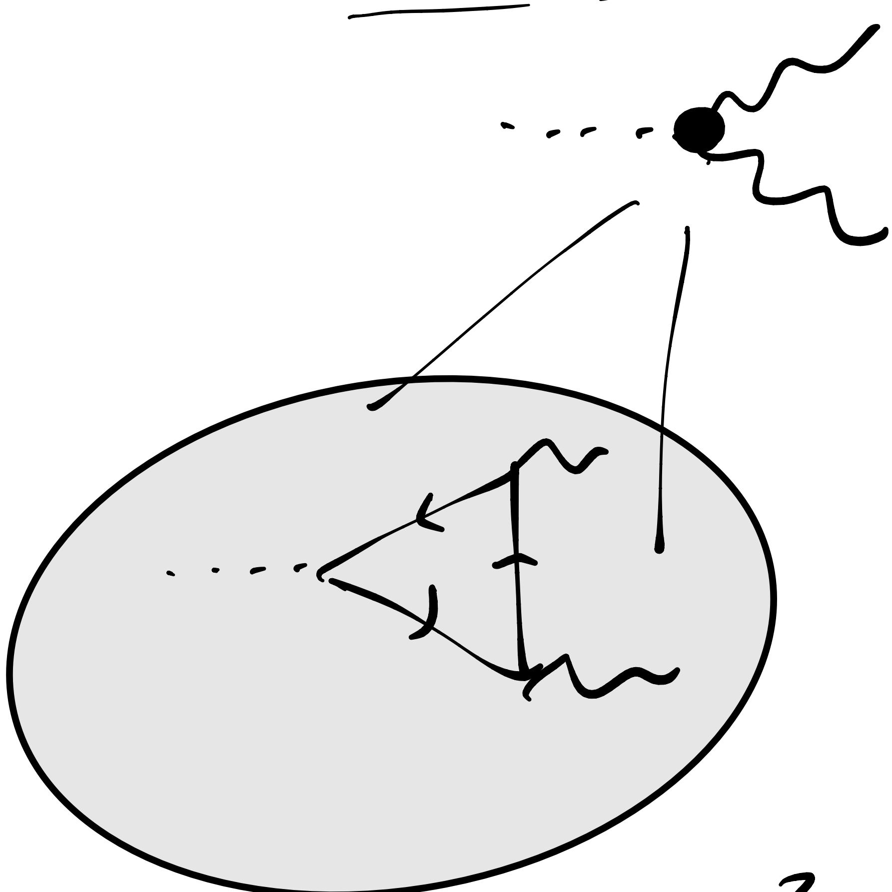
  <u>vs</u>: isospin $\begin{cases} u \to e^{i\theta} u \\ d \to e^{-i\theta} d. \end{cases}$

- $\Gamma(\pi^0 \to \gamma\gamma) \propto \left| \mathrm{tr}_{\text{quarks}} T^a Q^2 \right|^2$

  $\propto N_c^2$.

- This effect is not included in $\mathcal{L}_\chi$ above!

$$\Delta\mathcal{L} = \underline{N_c \frac{e^2}{16\pi^2}}\ \underline{\pi^0}\ \underline{\epsilon^{\mu\nu\rho\sigma}}\ \underline{F_{\mu\nu} F_{\rho\sigma}}$$



$$\delta_{U(1)_{5,\,q=3}} S = \lambda N_c \frac{e^2}{16\pi^2} \int \epsilon^{\cdots} F_{\cdot\cdot} F_{\cdot\cdot}$$

is accomplished by $\underline{\pi^0 \to \pi^0 + \lambda}$

't Hooft anomaly matching ✓

SU(3), baryons : $m_s \sim 95$ MeV

and $\langle \bar{s} s \rangle \sim V^3$

$$SU(3)_L \times SU(3)_R \xrightarrow{\langle \bar{q} q \rangle} SU(3)_{\text{flavor}}$$

(diagonal combination)

$\to 16 - 8 = \underline{8}$ Pseudo goldstones

$\pi^\pm \pi^0$, $K^{\pm 0}$ and η

(only $SU(2)_L$ is gauged.)

$$\underline{\underline{B}}^{ABC} = \epsilon_{\alpha\beta\gamma}\, q^A_\alpha\, q^B_\beta\, q^C_\gamma$$

color indices

$A \in \underline{3}$ of $SU(3)$

$q = (u, d, s)$

Bn_3

$$3 \otimes 3 \otimes 3 = (6 \oplus \bar{3}) \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

Couple nucleons to pions to give them a symmetric mass:

$$N_{L/R} = \begin{pmatrix} p \\ n \end{pmatrix}_{L/R} \xrightarrow{SU(2)_L \times SU(2)_R} g_{L/R} N_{L/R}$$

$$\mathcal{L} \ni \lambda_{NN\pi} \bar{N}_L \Sigma N_R = m_\pi \bar{N}N + \lambda \bar{N}N\pi + \dots$$

$$\Sigma \sim F_\pi U$$

$$\Rightarrow \underline{\underline{m_N}} = \underline{\underline{\lambda_{NN\pi}}} F_\pi. \checkmark$$

scatter π off N

(Goldberger-Treiman relⁿ.)

WZW terms and chiral Lagrangian

L_χ is invt under $\pi \to -\pi$.

$$A(\pi\pi \to \pi\pi\pi) \neq 0.$$

We wrote all terms in L that are MANIFESTLY symmetric.

L_χ is an example of a NLSM (non-linear sigma model):

$$U : \underset{M_D}{\text{spacetime}} \longrightarrow \text{target space}$$

with target space $G/H = SU(N_f)_L \times SU(N_f)_R / SU(N_f)_{\text{diag.}}$

G ← full symmetry

H ← unbroken subgroup

label a point on G/H by

$$U(x)\phi_0 \qquad \text{if } U(x) \in H$$

ϕ_0 ← ref. vacuum

$$U\phi_0 = \phi_0$$

$$U = e^{i\pi^a T^a \frac{2}{F_\pi}} \quad \text{w/ } \{T^a\} = \text{broken generators}$$

π^a are coords on the target sp.

WZW term = a term in $S[U]$ that's symmetric but

$$S[U] \neq \int \mathcal{L}\, d^Dx \text{ w/ } \mathcal{L} \text{ symmetric.}$$

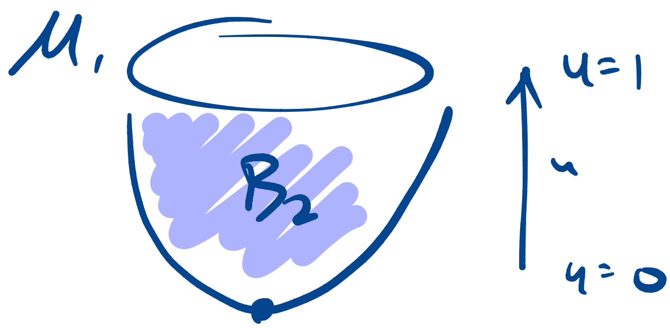
eg $D = 0+1$. $\check{n} : S^1 \to S^2$

S^1 = spacetime $= \mathcal{M}_1$

$S^2 = G/H = SU(2)/U(1)$

$\check{n}(t, u=1) = \check{n}(t)$

$\check{n}(t, u=0) = (0,0,1)$



$$W_0[\check{n}] = \frac{2\pi}{\Omega_2} \int_{B_2} \check{n}^a\, d\check{n}^b \wedge d\check{n}^c\, \epsilon_{abc}$$

manifestly SU(2) symmetric but not local in D dims

$$= \frac{1}{4\pi} \int_{\mathcal{M}_1} dt\, (1-\cos\theta)\, \partial_t \varphi$$

Local in D dims but not Manifestly Symmetric

• $e^{ikW[\hat{n}]}$ depends only on n on $\mathcal{M}_D$ if $k \in \mathbb{Z}$.

$*$
$$\delta W[\check{n}] = \int_{B_2} d(\cdots)$$
$$= \int_{\mathcal{M}} (\quad)$$
depends only $\check{n}(+)$

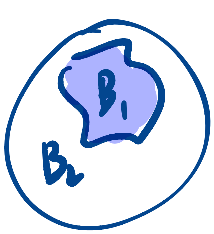
In $D=d+1$ dims $\check{n} \in S^{d+2}$

$$W_d[\check{n}] = \frac{2\pi}{\Omega_{d+2}} \int_{B_{d+2}} n^{a_0}\, dn^{a_1} \wedge dn^{a_2} \cdots \wedge dn^{a_{d+2}}\, \epsilon_{a_0 \cdots a_{d+2}}$$

• $O(d+3)$ symmetric

• $e^{ikW_d[\tilde{n}]}$ depends only on $\tilde{n}\big|_{\mathcal{M}}$

— $\delta W = \int d(\ldots)$



— $\frac{1}{\Omega_{d+2}} \int_{B_1 - B_2} n\, dn \wedge \ldots dn = \frac{1}{\Omega_{d+2}} \int_{S^{d+2}} n\, dn \ldots \in \mathbb{Z}$.

this is the winding #.

$$U \in G/H$$

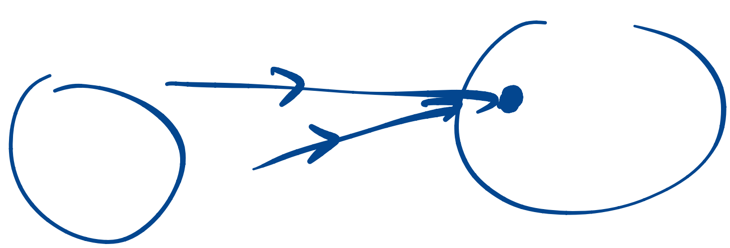
$$\mathcal{W}_{D+1}[U] = c\int_{B_{D+1}} \mathrm{tr}\, \overbrace{U^{-1}dU\wedge \cdots \wedge U^{-1}dU}^{D+1 \text{ of these}}$$

$$\partial B_{D+1} = M_D \qquad = c\int_{B_{D+1}} \mathrm{tr}\,(U^{-1}dU)^{D+1}$$

Interesting when $\exists$ nontrivial

maps: $S^{D+1} \longrightarrow G/H$.

can't be continuously deformed to



classified by $\pi_{D+1}(G/H)$.

claim: • $\delta W_{D-1} = c\,(D+1)\int_{B_{D+1}} d\left[\mathrm{tr}\,(U^{-1}dU)^D U^{-1}\delta U\right]$

Stokes $= c\,(D+1)\int_{M_D} \mathrm{tr}\,(U^{-1}dU)^D U^{-1}\delta U$

$U^\dagger U = 1 \Rightarrow \begin{cases} d(U^\dagger U) = 0. \\ \delta(U^\dagger U) = 0 \end{cases}$

for even D

$\epsilon^{\mu_1 \cdots \mu_{D+1}} = -(-1)^{D+1}\epsilon^{\mu_{D+1}\mu_1\cdots\mu_D}$

$\Rightarrow W_{D-1} = (-1)^D W_{D-1} = 0$

if D odd.

• $c\int_{S^{D+1}} \mathrm{tr}\,(U^{-1}dU)^{D+1} \in \mathbb{Z}$

for some $c_{D=4} = \dfrac{i}{240\pi^2}$.

theta term

eg: $\frac{\mathrm{tr}\, F \wedge F}{8\pi^2}$

$$\mathcal{H} = \int_{M_D} h$$

$$h = dq \quad \text{locally}$$

Doesn't affect eom.

pert. thy

$\mathcal{H} \in \mathbb{Z}$ if $\partial M_D = 0$.

Coeff of $\mathcal{H}$ is periodic

$$e^{i\theta\mathcal{H}} = e^{i(\theta+2\pi)\mathcal{H}}$$

if $\partial M_D = \emptyset$.

WZW term

$$W_{D-1} = \int_{B_{D+1}} n\, dn \wedge dn \cdots$$

$$W_{D-1} = \int_{B_{D+1}} w$$

$$\partial B_{D+1} = M_D$$

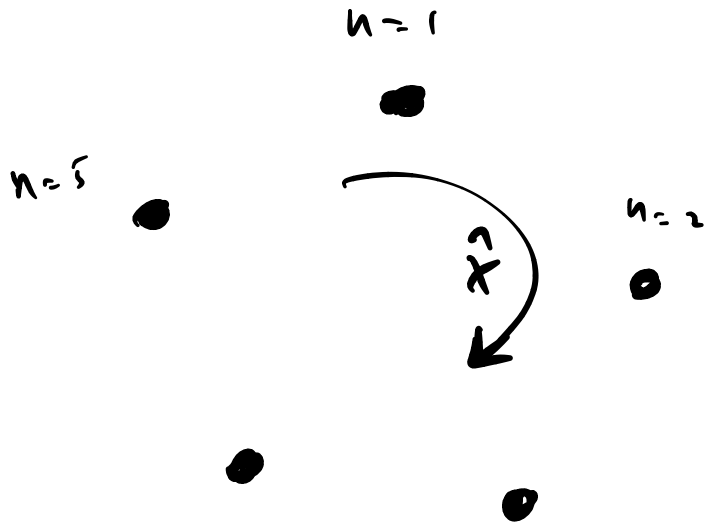
$$\delta w = dv$$

affects eom & pert. thy.

Coeff. of $W \in \mathbb{Z}$

$[x, p] = i$

Hint:



$N = 5$

$\hat{T} = e^{i\hat{p}} = \sum_{n=1}^{N} |n+1\rangle\langle n|$. $\quad \hat{T}|n\rangle = |n+1\rangle$ "shift"

$\hat{Z} = \sum_{n} \underline{\underline{|n\rangle\langle n|}}\, e^{2\pi i n/N}$ "clock"

$= e^{i\frac{2\pi}{N}\hat{x}}$ position $\qquad \hat{Z}|n\rangle = \underline{\underline{e^{\frac{2\pi i n}{N}}}}|n\rangle$

Compare $\hat{Z}\hat{T}$ with $\hat{T}\hat{Z}$