5.2 T-duality

Context: Bosons w a $U(1)$ symmetry.
($D=1+1$)

. $\mathcal{H}$ represents $[b_i, b_j^\dagger] = \delta_{ij}$, $[b_i, b_j] = 0$.

$$H_{BH} = -\tilde{J}\sum_{\langle ij\rangle}\left(b_i^\dagger b_j + h.c.\right) + \frac{U}{2}\sum_i n_i(n_i - 1)$$

$$-\mu\sum_i n_i$$

$= 0$ if $n_i = 0$ OR 1.

$n_i \equiv b_i^\dagger b_i$. $\rightsquigarrow U(1)$ sym.

[when $U\to\infty$

$$\mathcal{H} = \bigotimes_j \mathcal{H}_{1/2}$$

$$\begin{cases} S_j^+ = b_j^\dagger \quad S_j^- = b_j \\ S^z = -2b^\dagger b + 1. \end{cases}$$

$$H = H_{XY} = -\frac{w}{i}\sum_{\langle ij\rangle}\left(X_iX_j + Y_iY_j\right) + \frac{\mu}{2}\sum_j Z_j$$

$$U_\theta = \prod_j e^{i\frac{\theta}{2}Z_j}$$

can be solved by JW!]

Polar Coords:

$$\begin{cases} [n_i, \phi_j] = -i\delta_{ij} \\ \phi_i \equiv \phi_i + 2\pi \quad , \quad n \in \mathbb{Z} \end{cases}$$

$$b_i = e^{-i\phi_i}\sqrt{n_i} \quad , \quad b_i^\dagger = \sqrt{n_i}\, e^{+i\phi_i}$$

$$H_{BH} = -\tilde{J} \sum_{\langle ij \rangle} \left(\sqrt{n_i}\, e^{i(\phi_i - \phi_j)} \sqrt{n_j} + h.c. \right)$$

$$+ \frac{U}{2} \sum_i n_i (n_i - 1) - \mu \sum_i n_i$$

If $\langle n_i \rangle = n_0 \gg 1$ $\qquad \hat{n}_i = n_0 + \hat{\Delta n}_i$

$\Delta\hat{n} \ll n_0$. $\implies b_i = e^{-i\phi_i}\sqrt{n_i}$

$$\simeq e^{-i\phi_i}\sqrt{n_0}.$$

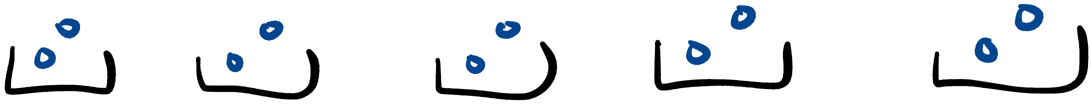
$$\implies H_{BH} \simeq -J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j)$$

$$\left(J \equiv 2\tilde{J} n_0. \right) \qquad + \frac{U}{2} \sum_j (\Delta n_j)^2$$

$n_0 = \frac{\mu}{U} \gg 1.$ $\qquad$ "Rotor model".

2 phases: $\boxed{U \gg J} \Longrightarrow \Delta n = 0.$

(Mott Insulator)



$[n, \phi] = i \longrightarrow \Delta\phi$ big.

$\boxed{U \ll J}$ minimize $-\cos(\phi_i - \phi_j) \to \phi_i = \phi_j = \phi \ \forall i$

(Superfluid)

$$H_{rotor} = U \sum n_i^2 - J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j)$$

$$\simeq U \sum n_i^2 - J \sum_{\langle ij \rangle} \left(1 - \frac{1}{2}(\phi_i - \phi_j)^2 + \cancel{\cdots}\right)$$

$$\simeq \sum_k \left(U \pi_k \pi_{-k} + J \sum_{\mu=1}^{d} (1 - \cos k_\mu a) \phi_k \phi_{-k}\right)$$

goldstone mode (phonon) $\omega \sim k$.

$$L_{eff} = \frac{\rho_s}{2}\left(\frac{(\partial_\tau \phi)^2}{c} + c(\vec{\nabla}\phi)^2\right) + (\partial\phi)^4 + \dots$$

$\phi \to \phi + \epsilon$ is $U(1)$ sym.. $\rho_s = \sqrt{J/U}$, $c = \sqrt{JU}$.

In $D=1+1$: $\langle \phi(x,t)\,\phi(0,0)\rangle \sim \log(x^\mu x_\mu)$

GROWS at large x.

$$\langle b^\dagger(x)\, b(0)\rangle \longrightarrow 0 \quad \text{as } x\to\infty.$$

vs $d>1$: $\langle b^\dagger(x)\, b(0)\rangle \longrightarrow \langle b^\dagger(x)\rangle\langle b(0)\rangle \neq 0$

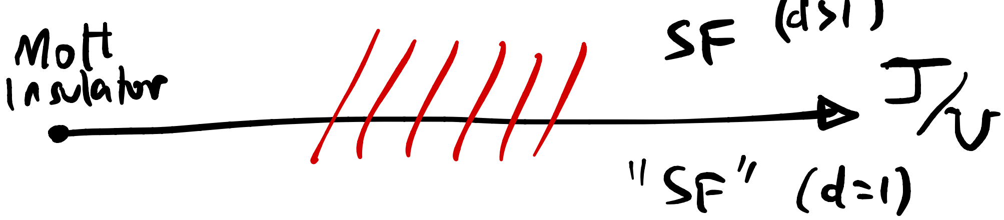
[Hohenberg-Mermin-Wagner-Coleman]

In the $D=1+1$ "SF" phase :

$$\langle b^\dagger(x)\, b(0)\rangle = \langle e^{i\phi(x)}\, e^{-i\phi(0)}\rangle$$

$$= \frac{c_0}{|x|^\eta} \qquad \eta = \frac{1}{2\pi\rho_s}.$$

"algebraic long range order".



$\langle b^\dagger_x b_0\rangle \sim e^{-x/\xi}$

$\langle b^\dagger_x b_0\rangle \sim \frac{1}{x^\eta}$

Massless $\boxed{\text{Compact}}$ relativistic scalar in $D = 1+1$:

$$S[\phi] = \frac{T}{2} \int dt \int_0^L dx \left[(\partial_0 \phi)^2 - (\partial_x \phi)^2 \right]$$

$$= 2T \int dx\, dt \quad \partial_+ \phi\, \partial_- \phi$$

$$\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)$$

on a circle $\underline{x \cong x + L}$.

$$\phi(x,t) \cong \phi(x,t) + 2\pi . \qquad \forall x, t .$$

This is a NLSM w/ target space S^1.

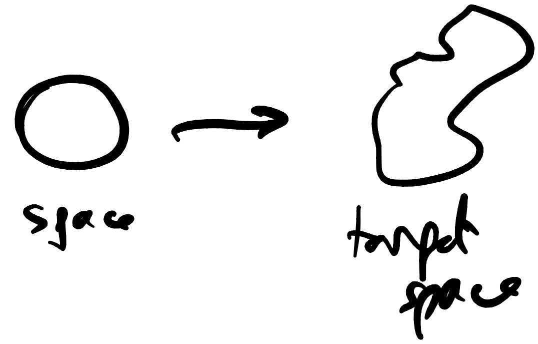
$$S_{NLSM}[\phi^I] = \frac{1}{\alpha'} \int dx\, dt\; g_{IJ}(\phi)\, \partial_\mu \phi^I \partial^\mu \phi^J$$

metric on target sp.

ϕ^I = coords on " ".

$\underline{\text{on } S^1}$: $\quad ds^2 = \underline{\underline{R^2}} d\phi^2 \qquad$ Radius of target space S^1 $= R$.

$$R = T = \sqrt{\ell_s} .$$



$$\phi \simeq \phi + 2\pi$$

$\Rightarrow$ only $e^{in\phi}$ $\quad n \in \mathbb{Z}$

is well-defined.

$$0 = \frac{\delta S}{\delta \phi(x,t)} \propto \partial^\mu \partial_\mu \phi \propto \partial_+ \partial_- \phi$$

$$\Rightarrow \underline{\phi}(x,t) = \phi_L(x^+) + \phi_R(x^-)$$

$$= \phi_L(z) + \phi_R(\bar{z})$$

$$\left(z = x + i\tau \; , \bar{z} = x - i\tau \right)$$

$$\tilde{\phi} = R\phi \cong \tilde{\phi} + 2\pi R . \Rightarrow e^{i\frac{n}{R}\tilde{\phi}}$$

is well-def'd.

$$\mathcal{L} = R^2 (d\phi)^2$$

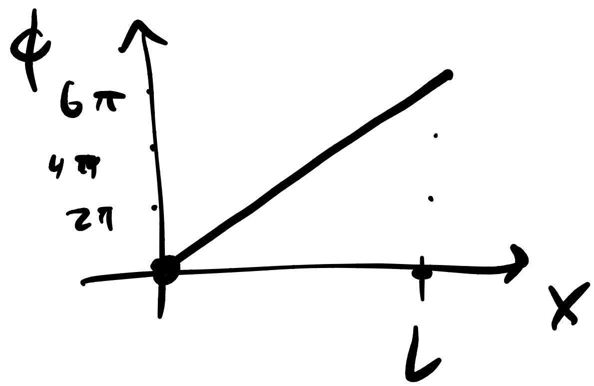
$$= (d\tilde{\phi})^2$$

<u>Symm:</u> $\phi \to \phi + \epsilon$ transl. in the <u>target space</u>

$$\Rightarrow \quad j_\mu = T\,\partial_\mu \phi$$

"momentum" (Boson # n.)

$$\phi(x,t) = \phi(x,t) + 2\pi m, \quad m \in \mathbb{Z}$$



is single valued as long as

$$\phi(x+L,t) = \phi(x,t) + 2\pi m$$

<u>$m \in \mathbb{Z}$</u>.

m = winding # of the string

$$\mathbb{Z} \ni m = \frac{1}{2\pi}\,\phi(x,t)\Big|_{x=0}^{x=L} \overset{\text{FTC}}{=} \frac{1}{2\pi}\int_0^L dx\,\partial_x\phi = \int_0^L \tilde{j}^0\,dx$$

$$\tilde{j}_\mu = \frac{1}{2\pi}\left(\partial_x\phi, -\partial_0\phi\right) = \frac{1}{2\pi}\epsilon_{\mu\nu}\,\partial^\nu\phi\,.$$

$$\partial^\mu \tilde{j}_\mu \propto \epsilon_{\mu\nu}\,\partial^\mu\partial^\nu\phi = 0\,.$$

$$\phi_L(t+x) = \hat{q}_L + (\hat{p}+\hat{w})(t+x) - i\sqrt{\frac{L}{4\pi T}}\sum_{n\neq 0} \frac{\hat{\rho}_n}{n} e^{in(t+x)\frac{2\pi}{L}}$$

$$\phi_R(t-x) = \hat{q}_R + (\hat{p}-\hat{w})(t-x) - i\sqrt{\frac{L}{4\pi T}}\sum_{n\neq 0} \frac{\hat{\tilde{\rho}}_n}{n} e^{in(t-x)\frac{2\pi}{L}}.$$

$$\phi = \phi_L + \phi_R. \qquad = \phi^\dagger$$

$$\Rightarrow \rho_n^\dagger = \rho_{-n}.$$

$$q = q_L + q_R = \frac{1}{L}\int_0^L dx\, \phi(x,t)$$

c.o.m position of string

$$\pi = T\partial_0\phi = T(\partial_+\phi_L + \partial_-\phi_R)$$

$$\mathbb{Z} \ni j = \pi_0 = \int_0^L dx\, \pi(x,t) = T\int_0^L dx\, \dot{\phi} = LT2p$$

$$p = \frac{j}{2LT} \qquad w = \frac{\pi m}{L} \qquad \text{w/}\ \underline{\underline{j, m \in \mathbb{Z}}}.$$

$$[\phi(x), \pi(y)] = i\,\delta(x-y) \quad \longleftrightarrow$$

$$[q_L, p_L] = i = [q_R, p_R]$$

$$[\rho_n, \rho^\dagger_{n'}] = n\,\delta_{nn'} \qquad \rho^\dagger_n = \rho_{-n}$$

$$\text{OR} \quad \underline{[\rho_n, \rho_m] = n\,\delta_{n+m}}. \qquad [\rho, \tilde{\rho}] = 0.$$

$$H = \int dx\, \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\int dx \left(\frac{\pi^2}{T} + T(\partial_x \phi)^2 \right)$$

$$= L\frac{1}{4T}(p_L^2 + p_R^2) + \pi \sum_{n=1}^{\infty} (\rho_{-n}\rho_n + \tilde{\rho}_{-n}\tilde{\rho}_n) + \text{cst}$$

$$= \frac{1}{2L}\left(\frac{j^2}{T} + T(2\pi m)^2 \right) + \pi \sum_{n=1}^{\infty} n(N_n + \tilde{N}_n) + \text{cst}$$

$\underline{\underline{j, m \in \mathbb{Z}}}$ $\qquad$ $\underline{g_0}$: $\rho_n|0\rangle = 0 = \tilde{\rho}_n|0\rangle$. $\quad n>0$.

$$[\hat{p}_L - \hat{p}_R, w) = \# w$$

$U(1)_{\text{boson}\,\#} \overset{IR}{\rightsquigarrow} U(1)_{\substack{\text{boson}\\ \#}} \times U(1)_{\substack{\text{winding}\\ =\text{vortex}\#}}$

$= U(1)_L \times U(1)_R$

$$\begin{cases} j_L^\mu = (j_L^z, j_L^{\bar z})^\mu = (j_+, 0)^\mu \\ j_R \quad (\qquad) = (0, j_-)^\mu \end{cases}$$

are conserved $\partial_+ j_- = 0 = \partial_- j_+$.

<u>Observation</u>: the spectrum is invt under

$m \leftrightarrow j \qquad T \leftrightarrow \frac{1}{(2\pi)^2 T}$.

<u>T-duality</u>. $\phi_L + \phi_R \longleftrightarrow \phi_L - \phi_R$.

<u>Vertex Operators</u>: <u>like</u> $e^{i\phi} \sim b$

$\langle \phi_L(x)\, \phi_L(0)\rangle = -\frac{1}{\pi T} \log \frac{z}{a} \qquad \langle \phi_R(x)\, \phi_R(0)\rangle = -\frac{1}{\pi T}\log\frac{\bar z}{a}$

$\langle \phi_L\, \phi_R \rangle = 0$.

$$V_{\alpha\beta}(z,\bar z) \equiv \; : e^{i\left(\alpha\phi_L(z)+\beta\phi_R(\bar z)\right)} :$$

$$\phi_L(z) = q_L + p_L z + i\sum \frac{\rho_n}{n} w^n \qquad w \equiv e^{2\pi i z/L}$$

$$: e^{i\alpha\phi_L(z)} : \; \equiv e^{i\alpha q_L} e^{i\alpha p_L z} e^{i\alpha\sum_{n<0}\frac{\rho_n}{n}w^n} e^{i\alpha\sum_{n>0}\frac{\rho_n}{n}w^n}$$

$\boxed{q \; p \; - \; +}$ (sim for ϕ_R)

$\boxed{e^{ip\hat{x}} \text{ inserts momentum } p.}$

$$\psi(p_0) = \langle p_0|\psi\rangle$$

$$\langle p_0| e^{ip\hat{x}} |\psi\rangle = \psi(p_0+p)$$

In order for $V_{\alpha\beta}$ to be single valued under $\phi\to\phi+2\pi$:

$$\tilde{\phi}_0 \equiv q_L - q_R .$$

$$p = \frac{\alpha+\beta}{2} \in \mathbb{Z}$$

$$V_{\alpha\beta}(0)|w,p\rangle = e^{i\left(\frac{\alpha+\beta}{2}\right)\hat{q}_0} e^{i\left(\frac{\alpha-\beta}{2}\right)\tilde{\phi}_0} e^{i\alpha\sum_{n<0}\rho_n} \cancel{e^{i\alpha\sum_{n>0}\rho_n}}^{\,1} |w,p\rangle$$

oscillator gs $\rho_n|w,p\rangle = 0 \quad n>0$

$$= e^{i\alpha \sum_{n<0} \rho_n} \left| w + \frac{\alpha-\beta}{2} \;,\; p + \frac{\alpha+\beta}{2} \right\rangle$$

$$\left\{ \begin{array}{l} \alpha+\beta \in 2\mathbb{Z} \\ \alpha-\beta \in 2\mathbb{Z} \end{array} \right. \Longrightarrow \alpha, \beta \text{ both odd or both even.}$$

$$\langle V_{\alpha\beta}(z,\bar z)\, V_{\alpha'\beta'}(0,0) \rangle = \frac{D_0 \; a^{\#}}{z^{\alpha^2/\pi T}\; \bar z^{\beta^2/\pi T}}$$

$$D_0 = \left\langle e^{i(\alpha+\alpha')q_L + i(\beta+\beta')q_R} \right\rangle_0 \leftarrow \text{just zms.}$$

$$= \delta_{\alpha+\alpha'}\, \delta_{\beta+\beta'} \Longleftarrow \text{charge conservation}$$

$$= \frac{\#}{z^{2h_L}\, \bar z^{2h_R}}$$

$$(h_L, h_R) = \frac{1}{2\pi T}(\alpha^2, \beta^2).$$

$$\Delta = h_L + h_R.$$

Special values of T = radius of ϕ or BH coupling J/U.

- SU(2) radius when $2\pi T = 1$.

  fixed by T-duality.

ops $V_{\alpha\beta} = V_{1,1}$ are marginal

$V_{\alpha\beta} = V_{\pm 1 0}$ and $V_{0 \pm 1}$ are conserved currents

$\rightsquigarrow SU(2) \times SU(2)$ symmetry.

(same as $SU(2)_1$ WZW model.)

- free fermion radius: when $2\pi T = 2$.

$$\langle V_{10}(z,\bar{z})\, V_{10}(0,0) \rangle \sim \frac{1}{z}$$

$$= \langle \psi_+(z)\, \psi_+(0) \rangle$$

w/ $\mathcal{L} = \bar{\psi}_+ \partial_- \psi_+$

bosonization :
$$\partial_\mu \phi \longleftrightarrow \bar{\psi}\gamma_\mu\psi$$
$$\epsilon_{\mu\nu}\partial^\nu\phi \longleftrightarrow \bar{\psi}\gamma_\mu\gamma^5\psi$$
$$? \longleftrightarrow \psi$$
$$\delta R^2 \partial_\mu\phi\partial^\mu\phi \longleftrightarrow \bar{\psi}\psi\bar{\psi}\psi$$

- Supersymmetric radius : $2a\,T = \frac{2}{3}$

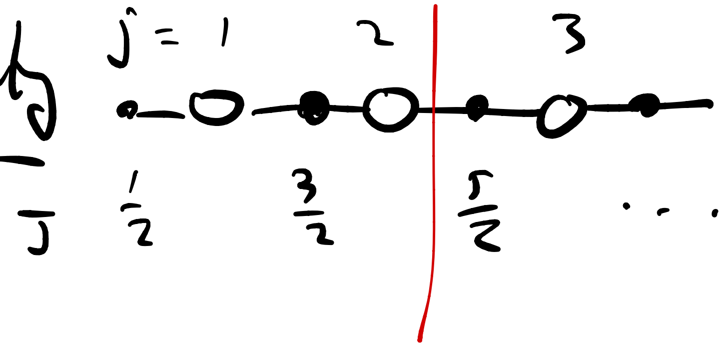
V_{10} has dim $(\frac{3}{2}, 0)$

is a supersymmetry current

$$j_{\mu\alpha} . \quad Q_\alpha = \int_0^L j_{0\alpha}\, dx$$

$$\{Q, Q\} \propto H$$

Lattice T-duality



$$m_J = \frac{\phi_{j+1}-\phi_j}{2\pi} \qquad \Theta_J \equiv \sum_{j<J} 2\pi n_j$$

= (# of bosons to the left of J) 2π

$$\Rightarrow [m_J, \Theta_{J'}] = -i\,\delta_{JJ'}.$$

$$e^{i\Theta_J} = e^{i\sum_{j<J} 2\pi n_j}$$

rotates the phase of all bosons to the left of J by 2π.

$\Rightarrow$ inserts a vortex between j and j+1.

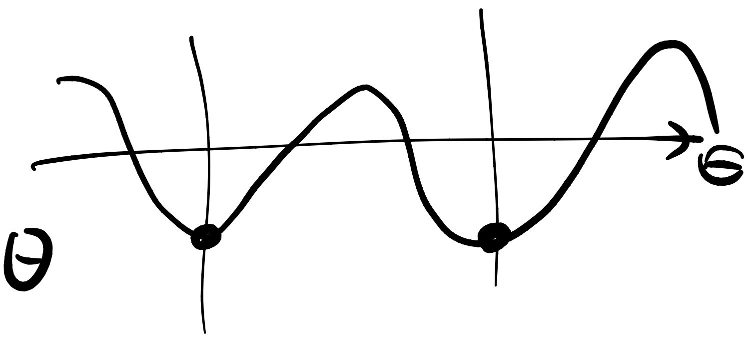
$$H_{rotor} = \frac{U}{2}\sum_J \left(\frac{\Theta_{J+1}-\Theta_J}{2\pi}\right)^2 - J\sum_J \cos 2\pi m_J$$

$$\overset{\text{'SF'}}{\simeq} \sum \left(\frac{U}{2}\left(\frac{\Theta_{J+1}-\Theta_J}{2\pi}\right)^2 + \frac{J}{2}\left(2\pi m_J\right)^2 \right)$$

Harmonic chain? but $\Theta \in 2\pi\mathbb{Z}$.

$$H_{\text{rotor}} = U \sum \left(\frac{\Theta_{j+1}-\Theta_j}{2\pi}\right)^2 + \frac{J}{2}\left(2\pi n_j\right)^2$$

$$- \lambda \sum_j \cos\Theta_j$$



$$\rightsquigarrow \mathcal{L}_{\text{eff}} = \frac{(\partial_\mu \Theta)^2}{2(2\pi)^2 \rho_s} - \lambda\cos\Theta$$

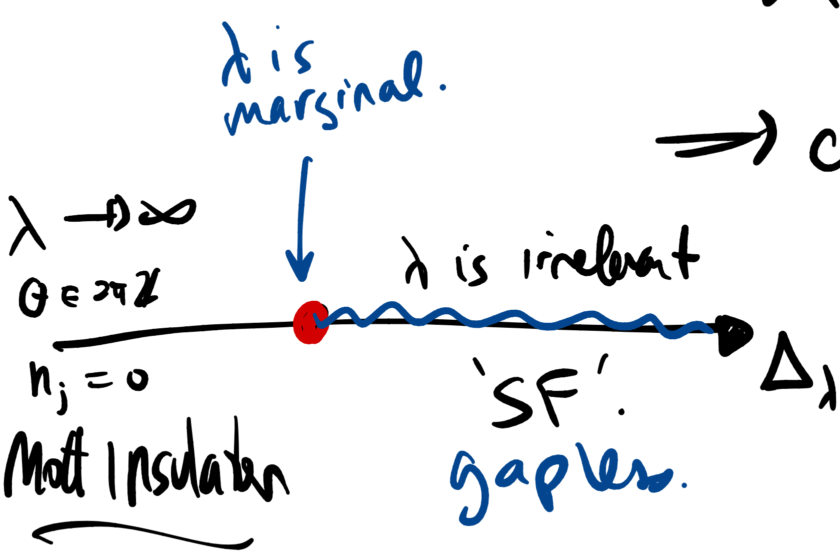
$$\rho_s \rightsquigarrow \frac{1}{(2\pi)^2 \rho_s}$$

$\Theta = \phi_L - \phi_R$ is the T-dual variable!

$$[\phi(x), \Theta(y)] = 2\pi i\, \mathrm{sign}(x-y)$$

$\longrightarrow$ $\cos\Theta$ inserts a vortex

Kosterlitz-Thouless transition



$$\langle e^{i\theta(x)} e^{-i\theta(0)} \rangle = x^{\frac{c}{2\pi\rho_s}}$$

$$\Delta = \pi \rho_s \overset{!}{=} 2$$

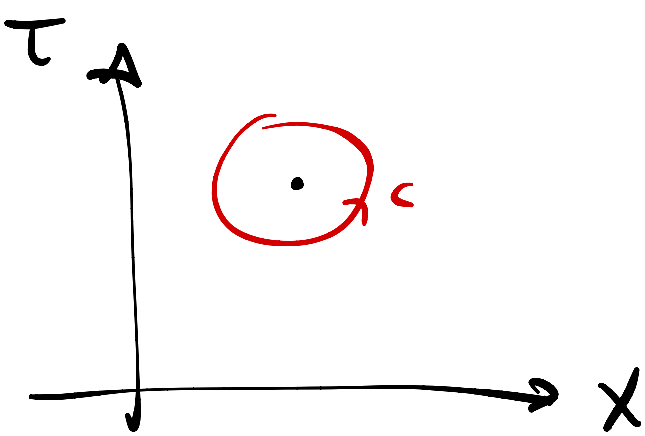
$$\langle b^+ \; b \rangle \sim x^{-\eta} \quad \text{w/} \quad \eta = \frac{1}{2\pi\rho_s}$$

$$= 1/4$$

at the trans.

Mott Insulator =

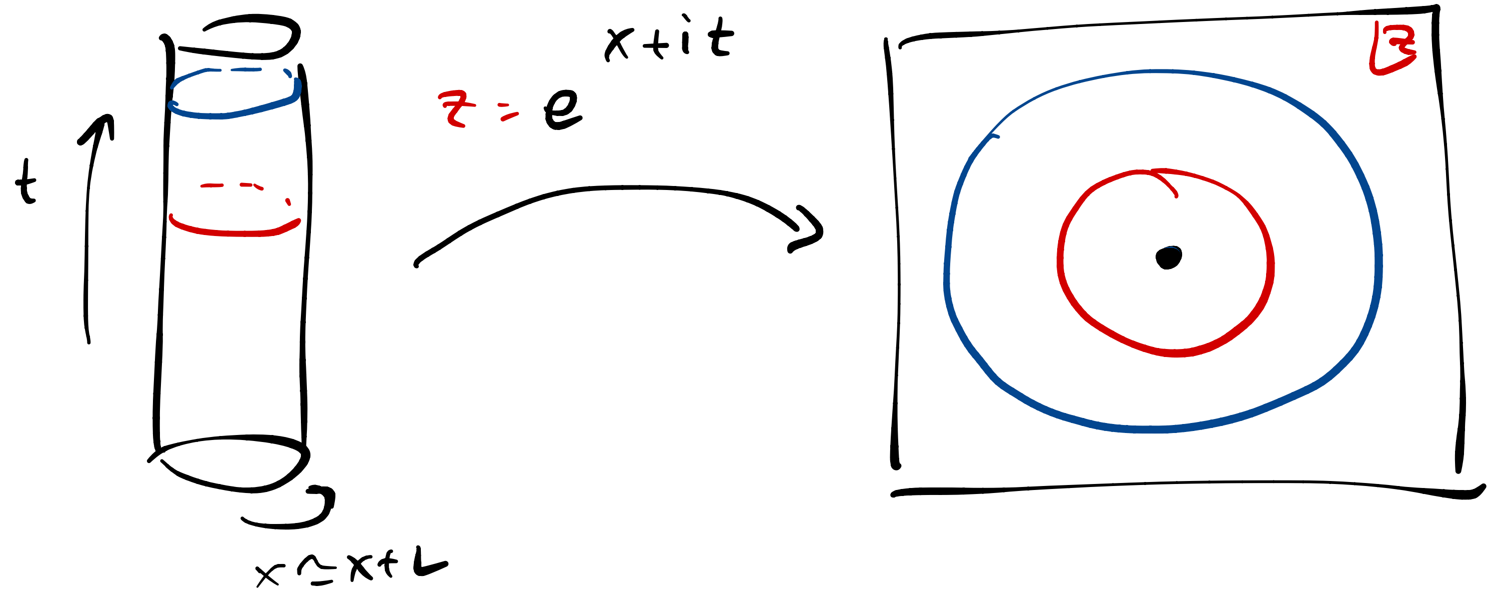
vortex condensate.



$$\underline{\oint_C d\phi = 2\pi m}$$

$$\langle V_{\alpha,\beta}(z) \dots \rangle = \langle \quad \dots \rangle_{\oint d\phi = 2\pi m(\alpha,\beta)}$$

state-operator corresp



$$|\mathcal{O}\rangle \longrightarrow \mathcal{O}(0)$$

energy eval $\longleftrightarrow$ scaling dim. of $\mathcal{O}$