§5.7 Holographic Duality

- Quantum gravity is different.

QFT ≡ q. system w/ extensive dofs

$$\#\text{ of possible states} \propto \left(\#\text{ of states/site}\right)^{\#\text{ of sites}} \sim 2^V$$

$$\log(\#\text{ of states}) \sim V \log 2$$

$$S(F) = \log\left(\begin{matrix}\#\text{ of states} \\ \text{w/ } F\end{matrix}\right) \leq S_{max} = V \log 2.$$

QG = a system w/ dynamical metric

① BH ≡ region from which no escape (classically)

QG = a system w/ black holes.

② BHs are formed by grav. collapse of dense enough matter.

③ Thermodynamics w/ gravity requires that a BH has an entropy.

$$S_{BH} = \frac{\text{area of horizon}}{4 \ell_p^2}$$

$$\left(\ell_p \equiv \sqrt{\frac{G_N \hbar^2}{c^3}} \right)$$

energy $\longleftrightarrow$ mass

temp $\longleftrightarrow$ surface gravity

$$\Delta \left(S_{stuff} + \underline{\underline{S_{BH}}} \right) \geq 0$$

④ claim:

$$S_{max}\begin{pmatrix} \text{a region of space w/} \\ \text{surface area } A \\ \text{in a gravitating system} \end{pmatrix} \leq S_{BH}\begin{pmatrix} \text{biggest} \\ \text{BH that} \\ \text{fits} \end{pmatrix} = \frac{A}{4\ell_p^2}$$

Pf: suppose $\exists$ $E_{stuff} < M_{BH}$, $S_{stuff} > S_{BH}$ $\xrightarrow[\text{waste heat}]{\text{dumping}}$ forms a BH w/ $S \leq S_{BH}$

$$S_{max}(QG) \propto \# \text{ of dof} \propto \frac{\text{area}}{l_p^2} \ll \frac{\text{volume}}{l_p^3}.$$

<u>not</u> <u>extensive</u> !!

(holographic principle)

<u>Temperature of a BH</u>

$$ds^2_{Sch} = -f(r)\,dt^2 + \underbrace{\frac{dr^2}{f(r)}} + r^2 d\Omega^2$$

$\uparrow$ on S^2

$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$

For a <u>generic</u> BH

$$f(r)\Big|_{r\sim r_H} \sim (r-r_H)\,2\kappa$$

$\uparrow$ "surface gravity"

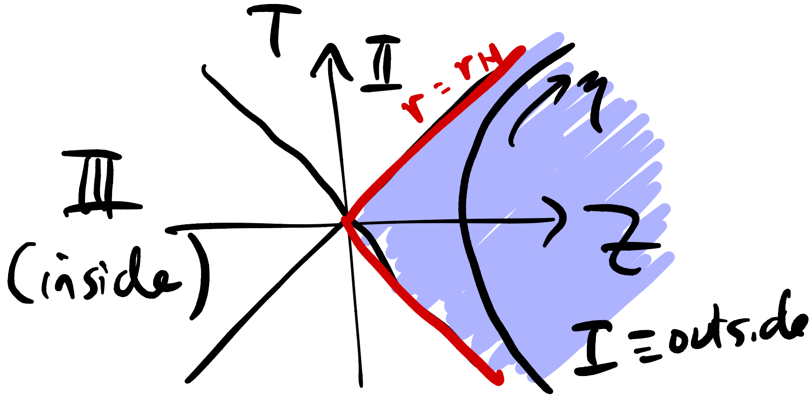
$$\frac{dr^2}{r-r_H} \equiv dR^2 \implies R \sim \sqrt{(r-r_H)}$$

$$\implies ds^2 \overset{r\to r_H}{\simeq} -\kappa^2 R^2 dt^2 + dR^2 + r_H^2 d\Omega^2$$

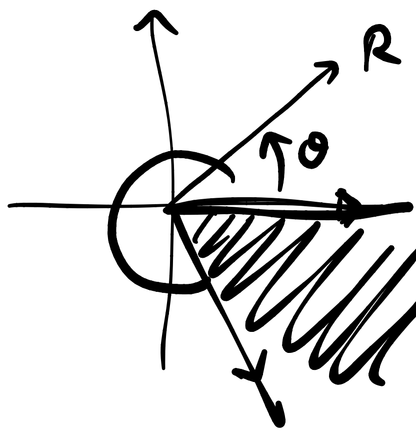
$\eta = \kappa t$

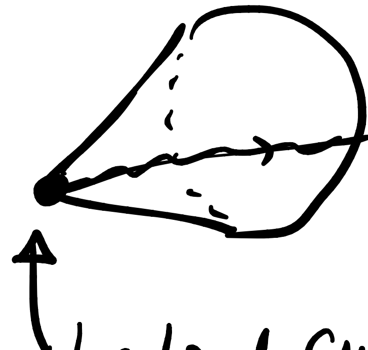
$\theta = -i\eta$

$$= R^2 d\theta^2 + dR^2 + \ldots$$



$$T = R\sinh\eta$$
$$Z = R\cosh\eta$$
$$ds^2 = -dT^2 + dZ^2 + \dots$$





localized curvature unless $\theta \cong \theta + 2\pi$.

Solving eom requires $\theta \cong \theta + 2\pi$

$\Longrightarrow$ euclidean time $\tau = -it$ is periodic $\cong \tau + 2\pi K$.

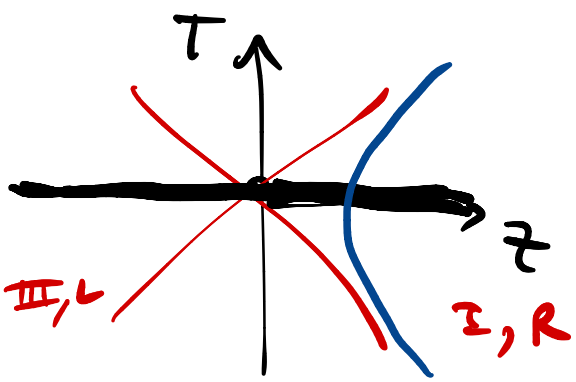
$$\Longrightarrow T = \frac{\kappa}{2\pi} = T_{BH}.$$

<u>Claim:</u> let $|gs\rangle \equiv$ g.s. of any QFT in Minkowski space $(T, Z \ldots)$

$T_{\mu\nu} \equiv$ its stress tensor

$$\rho_R \equiv \mathrm{tr}_L\, |gs\rangle\langle gs| = \frac{1}{Z} e^{-2\pi H_R}$$

thermal state for

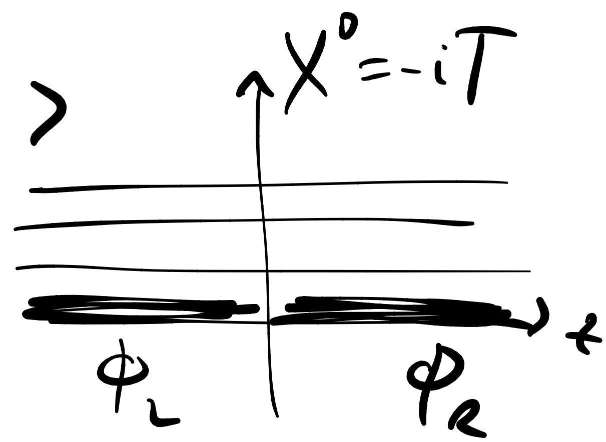


$$H_R = \int_{\text{const } T} d^d x \; T_{\eta 0}$$

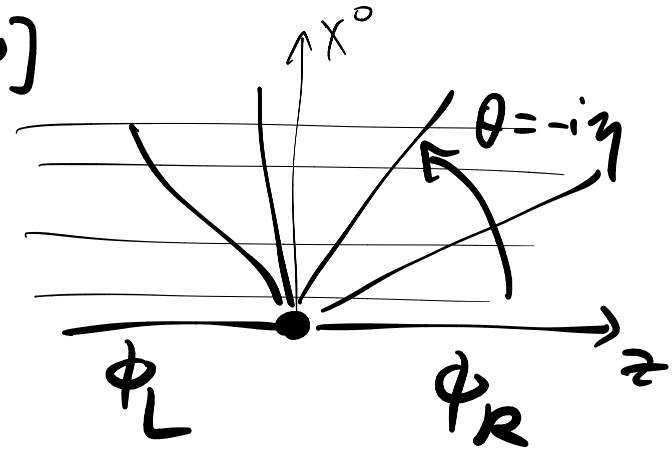
<u>pf:</u>
$$\phi(x,y,z) = \begin{cases} \phi_R(x,y,z) & \text{for } z>0 \\ \phi_L(x,y,z) & \text{for } z<0. \end{cases}$$

$$\Psi[\phi_L, \phi_R] = \langle \phi_L \phi_R | gs \rangle$$

$$= \frac{1}{\sqrt{Z}} \int_{x^0>0,\ \phi(\vec{x}, x^0=0) = (\phi_L, \phi_R)} [d\phi]\, e^{-S_{Eucl}[\phi]}$$



$$= \frac{1}{\sqrt{Z}} \int_{\substack{0 \le \theta < \pi \\ \phi(\theta=0)=\phi_R \\ \phi(\theta=\pi)=\phi_L \\ \partial_R \phi|_{R=0} = 0}} [d\phi]\; e^{-S_{eucl}[\phi]}$$



$$= \frac{1}{\sqrt{Z}} \langle \phi_L | e^{-\pi H_R} | \phi_R \rangle$$

H_R = generator of rotations

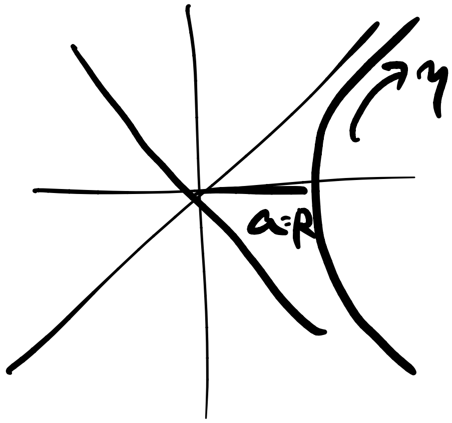
$$\langle \phi_R | \rho_R | \phi_R' \rangle = \int [d\phi_L]\, \Psi^*[\phi_L, \phi_R]\, \Psi[\phi_L, \phi_R']$$

$$= \left(\frac{1}{\sqrt{Z}}\right)^2 \int [d\phi_L] \langle \phi_R | e^{-\pi H_R} | \phi_L \rangle \langle \phi_L | e^{-\pi H_R} | \phi_R' \rangle$$

$$\mathbb{1} = \int [d\phi_L]\, |\phi_L\rangle\langle\phi_L|$$

$$= \frac{1}{\sqrt{Z}} \langle \phi_R | e^{-2\pi H_R} | \phi_R' \rangle$$

$\blacksquare$



$$\begin{cases} Z = R\cosh\eta \\ T = R\sinh\eta \end{cases}$$

• a uniformly accelerating observer in $\mathbb{R}^{d,1}$ sees a thermal state!

$$\rightsquigarrow T = \frac{R}{2\pi} = \frac{a}{2\pi}.$$

[Unruh]

• $t = \eta/\kappa \longrightarrow T_{BH} = \frac{\kappa}{2\pi}.$

$$dE = T\,dS + \ldots$$

AdS/CFT:

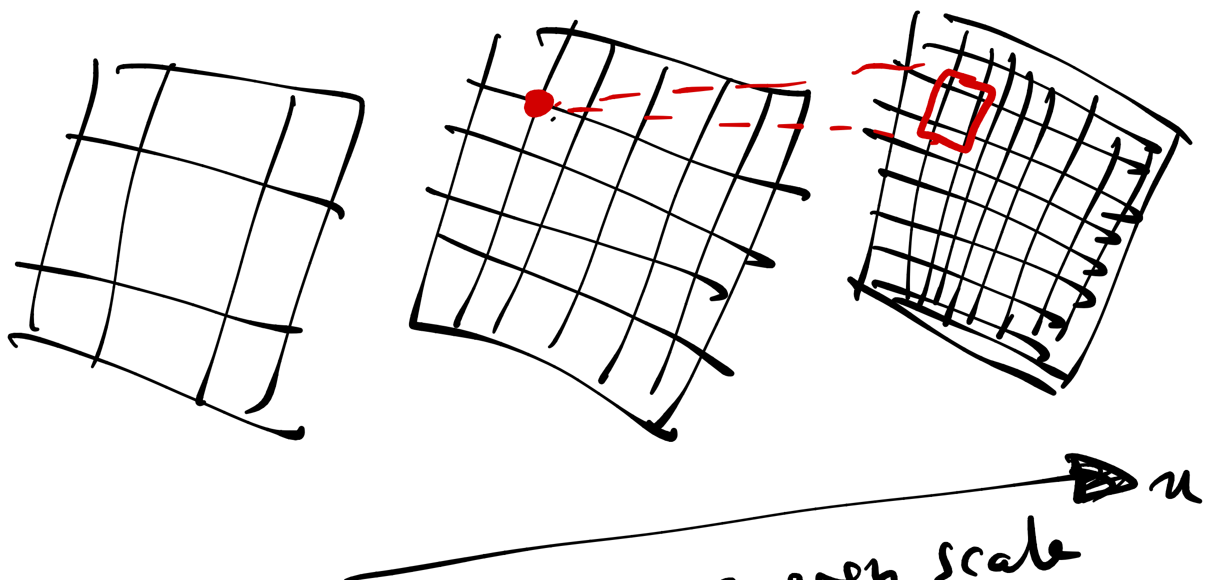
CLAIM: Some ordinary QFTs are QGs.

linearized gravity $\xrightarrow{D \geq 4}$ massless spin 2 particle (graviton)

Thm (Weinberg–Witten): A Poincaré inv't QFT w/ $\partial^\mu T_{\mu\nu} = 0$ can't have massless spin $j>1$ particles w/ $P^\mu = \int T^{0\mu} \neq 0$.

loophole: the QG can live elsewhere.

holographic principle: QG should have an extra dimension.



RG eqns

$$u \partial_u g = \beta(g(u))$$

are local in u.

$\Rightarrow$ maybe the extra dim is the RG scale.

<u>Specialize</u>: $\beta = 0$. ~~Lorentz~~ Poincaré sym.

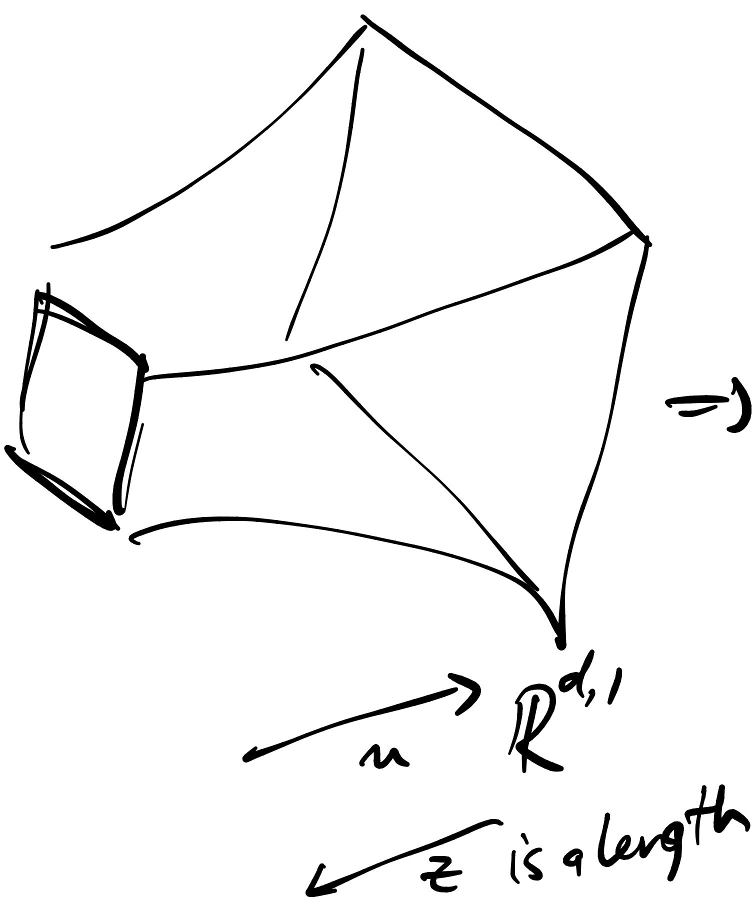
$$\Longrightarrow \begin{cases} x^\mu \to \lambda x^\mu \\ u \to \lambda^{-1} u \end{cases} \quad \text{symmetry.}$$

Find a metric w/ this sym:

$$ds^2 = \left(\frac{\tilde{u}}{\tilde{L}}\right)^2 \eta_{\mu\nu}\, dx^\mu dx^\nu + \frac{d\hat{u}^2}{\hat{u}^2} L^2$$

$$\hat{u} \equiv \frac{\tilde{L}}{L} u. \quad \Longrightarrow ds^2 = \left(\frac{u}{L}\right)^2 \eta_{\mu\nu}\, dx^\mu dx^\nu + \frac{du^2}{u^2} L^2.$$

is AdS_{d+1}



$$z = L^2/u$$

$$\Longrightarrow ds^2 = \left(\frac{L}{z}\right)^2 \left(\eta_{\mu\nu}\, dx^\mu dx^\nu + dz^2\right)$$

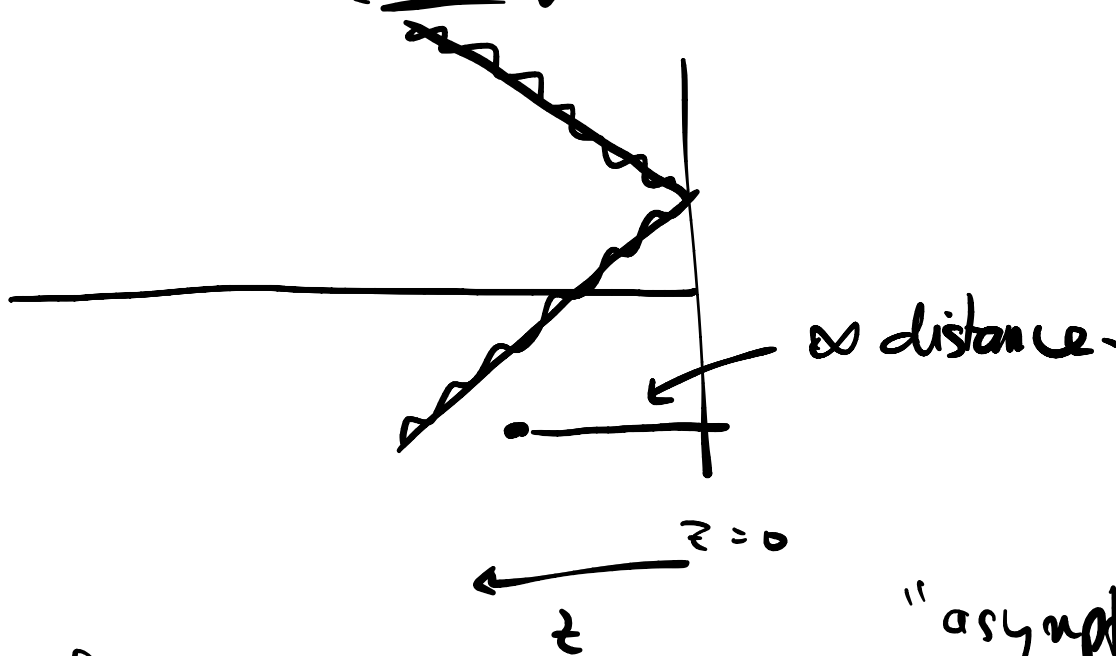
$$S_{bulk}[g, \dots] = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{g} \left[-2\Lambda + R + \dots \right]$$

$$0 = \frac{\delta S_{bulk}}{\delta g^{AB}} \Rightarrow R_{AB} + \frac{d}{L^2} g_{AB} = 0$$

$$-2\Lambda = \frac{d(d-1)}{L^2} \qquad \underline{\Lambda < 0}.$$

is classical when $L \gg \ell_{pl} \sim \sqrt{G_N}$.

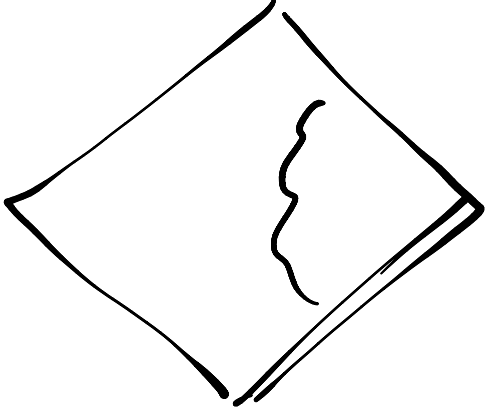
AdS has a boundary at $z = 0$.



"asymptotically AdS"

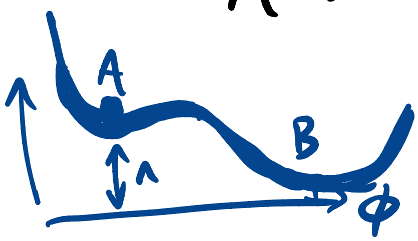
Metric is frozen at $z = 0$.

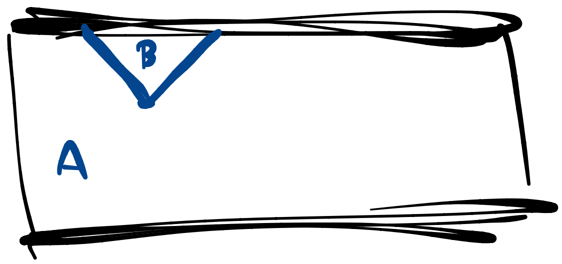
vs $\Lambda = 0$



<u>bcs on null ∞.</u>

we $\Lambda > 0$





bcs in future.

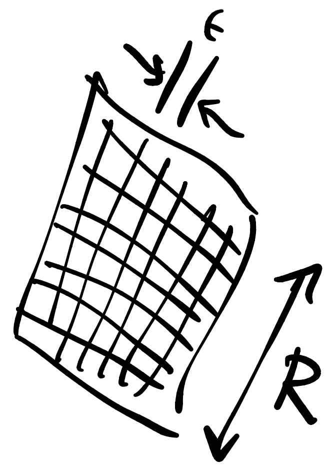
<u>Counting of dof.</u>

$$\frac{\text{Area of bdy}}{4\, l_{pl}} \stackrel{!}{=} \#\text{ of dofs in QFT} \equiv N_d$$

$$\infty \stackrel{\checkmark}{=} \infty .$$

$$N_d = \left(\frac{R}{\epsilon}\right)^d N^2$$

$\uparrow$ # of dof/site.



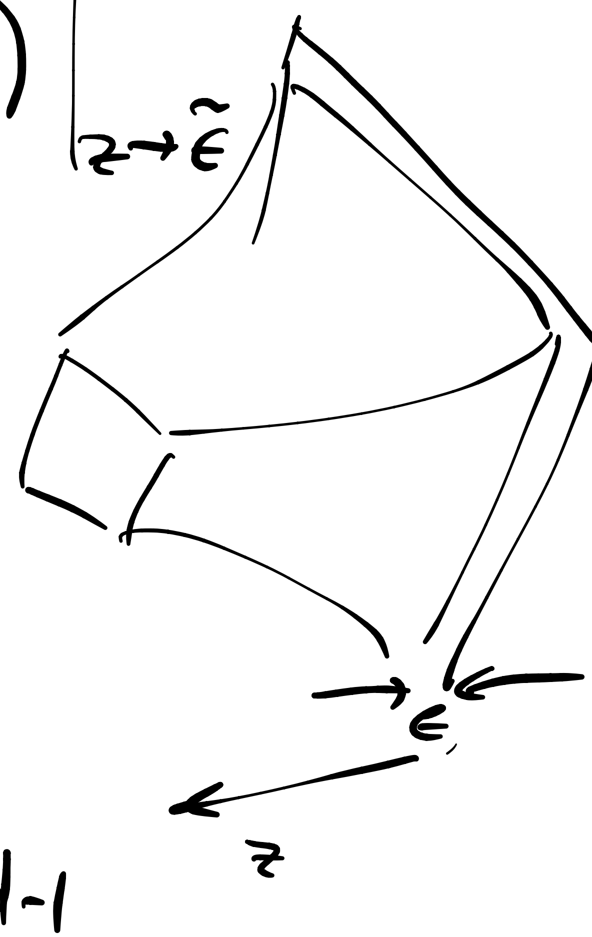
$$\text{Area of bdy} \equiv A = \int_{\substack{\mathbb{R}^d \text{ at} \\ z \to 0 \\ \text{fixed } t}} \sqrt{g}\, d^d x = \int_{\mathbb{R}^d} d^d x \left(\frac{L}{z}\right)^d$$

$$\sqrt{g} \equiv \sqrt{\det_{ij} g_{ij}}$$

$$ds^2 = \ldots + \left(\frac{L}{z}\right)^2 d\vec{x}^2$$

$$A = \int_0^R d^d x \cdot \left(\frac{L}{z}\right)^d \Bigg|_{z \to \tilde{\epsilon}}$$

$$= \left(\frac{RL}{\tilde{\epsilon}}\right)^d .$$



$$\frac{A}{4G_N} = \frac{L^d}{4G_N} \cdot \left(\frac{R}{\epsilon}\right)^{d-1}$$

$$\overset{!}{=} N^2 \left(\frac{R}{\tilde{\epsilon}}\right)^{d-1}$$

$$\boxed{\frac{L^d}{G_N} = N^2}$$

QG is classical $\Longleftrightarrow$ N^2 is large.

<u>Preview</u>: fields in AdS $\longleftrightarrow$ local ops in CFT

spin — spin

mass — scaling dim Δ

$$m^2 L^2 = \Delta(\Delta - D).$$

simple bulk $\longleftrightarrow$ CFT w/ few low-dim ops.

<u>want</u>:

$$Z[J] = \left\langle e^{-\int J_A(x) O_A(x) d^D x} \right\rangle_{CFT}.$$

UV part of the CFT.

$$Z[J] = Z_{QG}\left[\begin{matrix}\text{b.c. depend} \\ \text{on } J\end{matrix}\right]$$

$$\text{"} = \int [\mathcal{D}g_{AB}\cdots]\, e^{-S_{grav}[g_{AB}\cdots]}\text{"}$$

$$\underset{N^2 \gg 1}{\sim} e^{-S_{grav}[\underline{g}_{AB},\, \underline{\phi}\cdots]}$$

$\underline{g}, \underline{\phi}$ solves eom

w/ bc depend on J.

BH thermo (in AdS) $=$ thermo of CFT.

$$\mathrm{tr}_{CFT}\, e^{-\beta H_{CFT}} = Z_{QG}\left[\text{bdy} = \mathbb{R}^3 \times S^1_\beta\right]$$

$\tau \simeq \tau + \beta = \tau + \frac{1}{T}$.

$$\simeq e^{-S_{grav}|_{\text{BH of temp } T}} = e^{-\beta F(T)}$$

$$S = -\frac{\partial F}{\partial T} \overset{\text{einstein gravity}}{=} \frac{\text{area of horizon}}{4G_N}.$$

In a CFT: $$\frac{F}{R^d} = \# \, T^{d+1}$$

"central charge" $\propto N^2$.

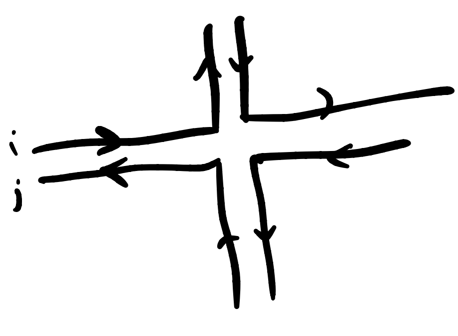
vectorlike large N: $\phi^{i=1..N}$ $\qquad N \frac{\text{dofs}}{\text{site}}$

simple.

matrix large N: Φ^{ij}

$$\mathcal{L} = \mathrm{tr}(\Phi)^4 + \dots \qquad \sim N^2 \frac{\text{dofs}}{\text{site}}.$$

$$= \Phi_{ij}\Phi_{jl}\Phi_{lm}\Phi_{mi}$$

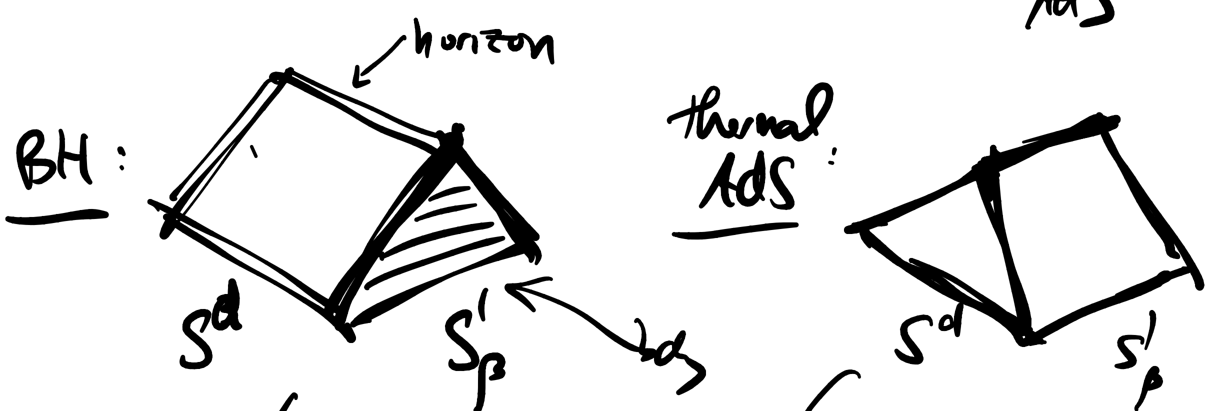


$\longrightarrow$ planar diagrams dominate.

['t Hooft].

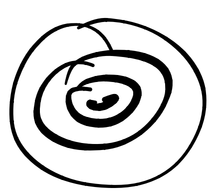
$$\mathrm{tr}_{\substack{CFT \\ \text{on } S^d}} e^{-\beta H} = Z_{QG}[\text{bdy} = S^1_\beta \times S^d]$$

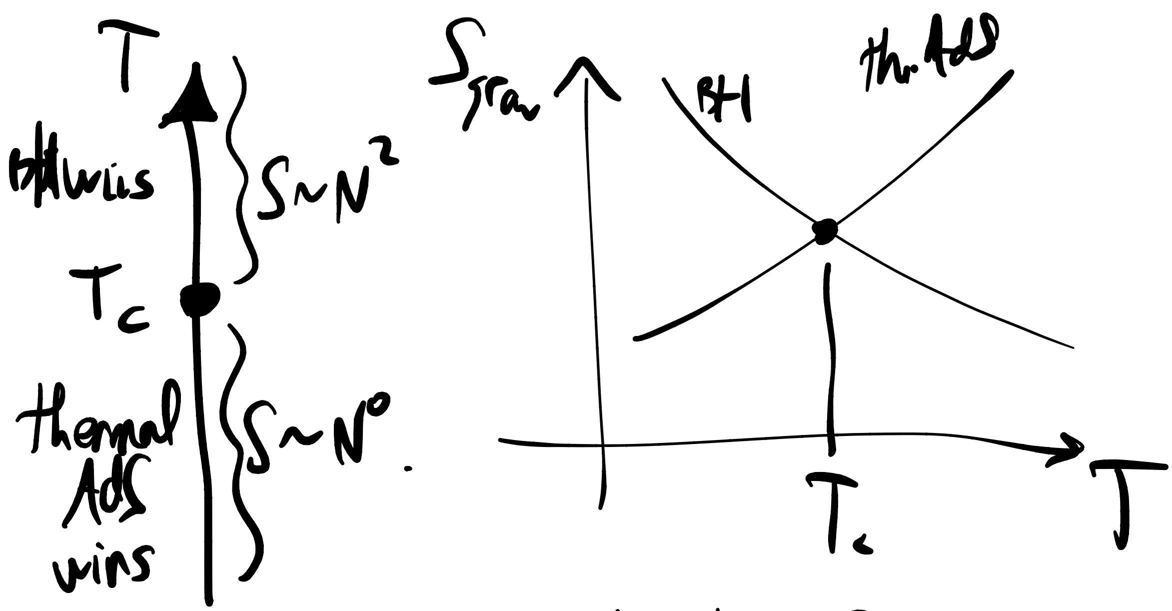
$$\cong e^{-S_{grav}|_{BH}} + e^{-S_{grav}|_{\text{thermal AdS}}}$$



$S^d \times (\text{Disk})$

$= \underline{S^1_\beta \times \text{Ball}_{d+1}}$





Hawking-Page transition.

$$S[g_{\mu\nu}] = \frac{1}{G_N}\int d^Dx \sqrt{g}\,\left[\underline{-2\Lambda} + R + \underline{\ldots}\right]$$

$$g_{\mu\nu} = g^0_{\mu\nu} + \#\,\underline{h_{\mu\nu}}$$

$\Lambda = 0$, g^0 = mink.

$$h_{\mu\nu} = \int d^dp\, e^{ip\cdot x}\, \zeta^s_{\mu\nu}\, \underline{a^s(p)} + h.c.$$

$$[G_N] = M^{-2} \Rightarrow$$ interactions are irrelevant.

('non-renormalizable')

$$dM = T\, dS_{BH} \qquad [\text{Smarr}]$$