The conformal bootstrap in $D \geq 3$

Simon Martin[1]

[1] *Department of Physics, University of California at San Diego, La Jolla, CA 92093*

The conformal bootstrap is a highly powerful non-perturbative technique to study conformal field theories (CFTs). This paper constitutes a brief introduction to the subject in $D \geq 3$. After a presentation of key notions of conformal field theories in $D \geq 3$, we detail how the conformal bootstrap can be used to constrain the scaling dimension of operators in a CFT. Impressive results obtained in the $3D$ Ising CFT are presented, while a few other theories are also mentioned.

I. INTRODUCTION

Conformal field theories (CFTs) are a cornerstone of modern theoretical physics. Indeed, these quantum field theories (QFTs) invariant under the conformal group (transformations preserving angles) play an important role in various areas of physics. For example, they describe critical points of continuous phase transitions [1], fixed points of renormalization group flows and also appear in quantum gravity via the AdS/CFT correspondence [2].

However, most of the interesting CFTs are strongly coupled theories that cannot be studied with the standard expansion in Feynman diagrams. The conformal bootstrap constitutes a highly promising non-perturbative program to constrain and solve conformal field theories. This approach, which was originally proposed in the early 1970s [3, 4], has been very successful to solve 2D CFTs. However, during many decades, little progress was made in the study of CFTs in $D \geq 3$ until in 2008, where a set of new numerical techniques was presented in [5], which allowed to efficiently apply the conformal bootstrap to higher-dimension CFTs.

The goal of this paper is to give a short overview of the modern conformal bootstrap in $D \geq 3$. In the first section, important notions of CFTs are reviewed, followed by a presentation of the main ideas of the conformal bootstrap in section 2. Finally, impressive results obtained in the last years are mentioned in section 3, while we conclude in section 4.

II. CONFORMAL FIELD THEORIES

We consider a spacetime in D dimensions with a metric $g_{\mu\nu}$. The conformal transformations are defined as the set of operations on coordinates $x \to x' = x'(x)$ respecting $g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}$. Note that for the rest of the paper, we work in Euclidean signature, with $g_{\mu\nu} = \delta_{\mu\nu}$. There are four types of transformations that respect this condition: $\Omega(x) = 1$ corresponds to translations $x'^\mu = x^\mu + a^\mu$ and rotations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. The case $\Omega(x) \neq 1$ corresponds to the dilatation $x'^\mu = \lambda x^\mu$ ($\lambda \in \mathbb{R}$). Finally, special conformal transformations (SCTs)

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}\,, \tag{1}$$

have a local $\Omega(x)$. The four transformations presented above, generated respectively by the operators P_μ, $M_{\mu\nu}$, D and K_μ, constitute the conformal group. Appendix A presents the differential operator representation of the generators as well as the conformal algebra.

The first step to study CFTs is to find how field operators $\mathcal{O}^a(x)$ transform under conformal transformations. The analysis is done in Appendix B and reveals crucial points. First, from the action of D and $M_{\mu\nu}$ at the origin

$$\begin{aligned} [D, \mathcal{O}^a(0)] &= \mathrm{i}\,\Delta\,\mathcal{O}^a(0) \\ [M_{\mu\nu}, \mathcal{O}^a(0)] &= \mathrm{i}(S_{\mu\nu})^a{}_b \mathcal{O}^b(0)\,, \end{aligned} \tag{2}$$

we learn that operators in a CFT are labeled by their scaling dimension Δ and their spin l (the rank of the SO(D) representation under which $\mathcal{O}^a(x)$ transforms). Secondly, P_μ and K_μ are respectively raising and lowering operators: they generate operators with an increased/decreased scaling dimension by 1. Finally, the spectrum of a CFT is composed of two types of operators: primaries, which are annihilated by SCTs at the origin and descendants, generated by acting with P_μ's on primaries. A primary and its descendants form a conformal family, an irreducible representation of the conformal group.

One of the major consequences of conformal invariance is that it imposes strong constraints on correlation functions. For example, it can easily be shown that the 2-point function of two real scalar operators of scaling dimension Δ_1 and Δ_2 is

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{\delta_{\Delta_1\Delta_2}}{|x_1 - x_2|^{2\Delta_1}} \tag{3}$$

while the 3-point function of three scalars is

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{|x_{12}|^{\Delta-2\Delta_3}|x_{23}|^{\Delta-2\Delta_2}|x_{31}|^{\Delta-2\Delta_1}}\,, \tag{4}$$

where $x_{ij} = x_i - x_j$ and $\Delta = \sum_{i=1}^{3}\Delta_i$, while λ_{123} is a numerical coefficient which depends on the three operators. General expressions for 2 and 3-point functions of operators with spins, involving conformally invariant tensor structures, are presented in [6]. Note that the great simplicity of the correlation function's structure stops at three points. Indeed, when dealing with four spacetime points, the following crossing-ratios are invariant under all four conformal transformations

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,. \tag{5}$$

Hence, for four identical scalars $\mathcal{O}$ of dimension Δ, the 4-point function is

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{G(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}}\,, \tag{6}$$

where $G(u,v)$ is a function of u and v. We will see in the next section that 4-point functions play a crucial role in the conformal bootstrap.

Next, we mention a few words on one of the most important tool to study correlators in CFTs: the operator product expansion, which states that the product of two operators $O_i^a(x_1)$ and $O_j^b(x_2)$ (i,j labels the operator, while a,b are the spin indices) can be expressed as

$$\mathcal{O}_i^a(x_1)\mathcal{O}_j^b(x_2) = \sum_k f_{ijk}\, C^{abc}(x_{12},\partial_2)\mathcal{O}_k^c(x_2)\,, \tag{7}$$

where the sum is performed over the primaries in the CFT. The differential operator $C_{ijk}^{abc}(x_{12},\partial_2)$, which is entirely constrained by conformal invariance, generates descendants in the series. The numbers f_{ijk}, which are called OPE coefficients, can be related with the coefficient(s) of the 3-point function λ_{ijk} by multiplying both sides of the OPE by an operator appearing on the right-hand side and by taking the expectation value on both sides. The OPE can be derived in radial quantization by inserting two operators inside a sphere and then by using the state-operator correspondence. These two notions are presented in Appendix C. Note that in a CFT, the OPE has a finite radius of convergence as long as no other operators are inserted between x_1 and x_2 [7].

The OPE is extremely powerful. Indeed, it can be used to express any n-point correlation function as a series of $(n-1)$-point functions of other operators in the CFT. Therefore, by applying the OPE multiple times, any correlation function can be reduced to a sum of simple 2-point functions.

Let's now go back to the evaluation of the 4-point function (Eq. 6). Note that we will stick with four identical scalars for simplicity. The OPE can then be applied to the product $\phi(x_1)\phi(x_2)$ and $\phi(x_3)\phi(x_4)$, which yields

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}}\sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}}^2\, g_{\Delta_{\mathcal{O}},l_{\mathcal{O}}}(u,v)\,, \tag{8}$$

where we have used a basis of operators in which the 2-point function is diagonal and the sum is performed over primary operators. Moreover, the following quantity has been defined

$$g_{\Delta_{\mathcal{O}},l_{\mathcal{O}}}(u,v) = |x_{12}|^{2\Delta}|x_{34}|^{2\Delta} \times C^a(x_{12},\partial_2)C^b(x_{34},\partial_4)\,\langle \mathcal{O}^a(x_2)\mathcal{O}^b(x_4)\rangle\,, \tag{9}$$

called a conformal block, which is only a function of u and v. A given conformal block encodes the presence of a single conformal family in the expansion of the 4-point function and is thus labeled by the scaling dimension and the spin of the primary. Note that conformal blocks can be computed for any theories using a procedure relying only on the conformal algebra, which is presented in [8].

III. THE CONFORMAL BOOSTRAP

Since any n-point correlator can be reduced to a series of 2-point functions with the OPE, we see that any CFT is entirely described by the scaling dimension and the spin of its primary operators and the OPE coefficients (up to some additional coefficients such as the central charge, the normalization of the energy-momentum tensor 2-point function). The knowledge of all these numbers, called the CFT data, allows to compute any correlation function. However, not every set of number represents a consistent CFT. Indeed, for a CFT to be well-defined, the application of the OPE in the 4-point function must be associative: applying the OPE on $\phi(x_1)\phi(x_2)$ and $\phi(x_3)\phi(x_4)$ must yield the same result as applying the OPE for $\phi(x_1)\phi(x_4)$ and $\phi(x_2)\phi(x_3)$, which is equivalent to the swap $x_1 \leftrightarrow x_3$. Therefore, the function $G(u,v)$ appearing in the 4-point function must respect the crossing symmetry $G(u,v) = \frac{u^\Delta}{v^\Delta}G(v,u)$, which can be rewritten in terms of the conformal blocks as

$$\sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}}^2 F_{\Delta_{\mathcal{O}} l_{\mathcal{O}}}^{\Delta}(u,v) = 0\,, \qquad (10)$$

which is called the conformal bootstrap equation, where we have defined

$$F_{\Delta_{\mathcal{O}} l_{\mathcal{O}}}^{\Delta} = v^{\Delta} g_{\Delta_{\mathcal{O}} l_{\mathcal{O}}}(u,v) - u^{\Delta} g_{\Delta_{\mathcal{O}} l_{\mathcal{O}}}(v,u)\,. \qquad (11)$$

Eq. (10) is a strong constraint on the CFT, since it must be verified for any CFT data and any four spacetime points. The goal of the modern conformal bootstrap is to use this equation to establish constraints on the CFT data for physically interesting operators.

Before discussing about the bootstrap algorithm, it is important to assume the unitarity of the theory (which is often the case for physically interesting theories). This has two important effects on the CFT data: it puts lower bounds on the scaling dimensions and it ensures that OPE coefficients are real. More details are presented in Appendix D. The latter is particularly powerful, since it forces $f_{\phi\phi\mathcal{O}}^2 \geq 0$. In this case, it is useful to treat $F_{\Delta_{\mathcal{O}} l_{\mathcal{O}}}^{\Delta}(u,v)$ as a vector, labeled by $\Delta_{\mathcal{O}}$ and $l_{\mathcal{O}}$, in an infinite-dimensional vector space of functions of (u,v). The conformal bootstrap equation can thus be seen as a linear combination of vectors with positive coefficients that must add to 0.

Knowing this, let's detail the strategy of the numerical conformal bootstrap to put constraints on scaling dimensions:

1. We first assume a CFT spectrum (set of Δ and l) respecting the lower bounds established by unitarity.

2. We then look for a linear functional α acting on the vectors $F_{\Delta_{\mathcal{O}} l_{\mathcal{O}}}^{\Delta}(u,v)$, which must respect $\alpha[F_{00}^{\Delta}(u,v)] > 0$ for the identity operator (since $\mathbb{1}$ has $\Delta = 1$ and $l = 0$) and $\alpha[F_{\Delta_{\mathcal{O}} l_{\mathcal{O}}}^{\Delta}(u,v)] \geq 0$ for all the other operators in the spectrum.

3. If such an α exists, then applying it on both sides of the bootstrap equation leads to a contradiction. This means that the spectrum initially assumed is not consistent and can be discarded. If no α is found, nothing can be concluded.

Since the vectors $F_{\Delta_{\mathcal{O}} l_{\mathcal{O}}}^{\Delta}(u,v)$ are infinite-dimensional, it is of course impossible to find a functional for the entire vector-space. However, the search for α can be restricted to finite-dimensional subspaces and if a functional is found in this case, the assumed spectrum can still be discarded. More details on this subject can be found in [9]. Note that a similar algorithm exists to put bounds on the OPE coefficients.

Finally, stronger constraints can be put on CFT spectra by adding inputs from known theories. For example, we can restrict ourselves to theories with a certain symmetry, certain types of operators that are known to appear, etc. This will play an important role in the next section.

IV. RESULTS

We now mention some interesting results obtained with the conformal bootstrap in $D = 3$. Let's start by illustrating the most spectacular results, presented in [10, 11]. In these papers, the authors considered a CFT with a $\mathbb{Z}_2$ symmetry which has two operators of low scaling dimensions: a pseudoscalar σ (odd under $\mathbb{Z}_2$) and a scalar ϵ. By applying the conformal bootstrap algorithm presented in the previous section to the 4-point function $\langle\sigma\sigma\sigma\sigma\rangle$, it was possible to put an upper-bound on the scaling dimension Δ_ϵ in terms of Δ_σ. The exclusion plot is shown below

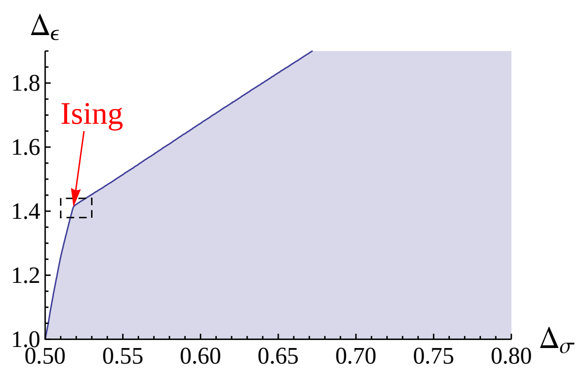

Figure 1: Exclusion plot for the scaling dimension of the smallest scalar ϵ in terms of the scaling dimension of the smallest pseudoscalar σ. The white region above the curve is ruled-out by the conformal bootstrap. Plot taken from [10].

The white region above the blue curve is ruled-out by the conformal bootstrap, while the blue region corresponds to allowed sets $(\Delta_\sigma, \Delta_\epsilon)$ of scaling dimensions. The most interesting feature of this figure is the kink, which occurs near the scaling dimensions of a famous CFT: the 3D Ising critical point. Therefore, it seems that this CFT saturates the upper bound.

To obtain stronger constraints on the Ising CFT, it is necessary to add additional information to the bootstrap analysis. This was performed in [11], where the authors have established crossing relations for the two additional 4-point functions $\langle\sigma\sigma\epsilon\epsilon\rangle$ and $\langle\epsilon\epsilon\epsilon\epsilon\rangle$. Furthermore, by requiring that σ and ϵ are the only two relevant operators ($\Delta < 3$) in the CFT, the impressive exclusion plot presented in Figure 2 has been obtained.

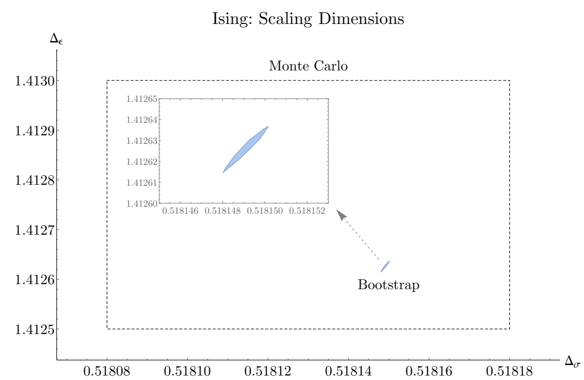

Figure 2: Allowed values of the scaling dimensions $(\Delta_\sigma, \Delta_\epsilon)$ in the $3D$ Ising CFT, compared with the most precise Monte Carlo results. Plot taken from [11].

Therefore, having added additional information to the bootstrap analysis has constrained the Ising CFT to a tiny island in the $(\Delta_\sigma, \Delta_\epsilon)$ space. This leads to the most precise values of the scaling dimensions ever computed [11]

$$\begin{aligned} \Delta_\sigma &= 0.5181489(10)\,, \\ \Delta_\epsilon &= 1.412625(10)\,. \end{aligned} \tag{12}$$

The conformal boostrap has also been applied to a generalization of the Ising CFT in $D = 3$ with N real scalar fields, called the O(N) model. Scaling dimensions and OPE coefficients are obtained for small $N > 1$ in [11], improving upon results obtained with quantum Monte Carlo simulations, the ϵ-expansion and the $1/N$ expansion.

Finally, the conformal bootstrap has also been applied to fermionic CFTs [12, 13]. In $D = 3$, the strongly interacting fixed point of the Gross-Neveu-Yukawa model is an interesting example

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2}\bar{\psi}_i(\not{\partial} + g\phi)\psi_i - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \\ &-\frac{1}{2}m^2\phi^2 - \lambda\phi^4\,, \end{aligned} \tag{13}$$

where i labels the N flavors of (real) Majorana fermions ψ_i, while ϕ is a parity-odd scalar. Scaling dimensions of the smallest operators have been computed in [13] for various N. For example, for $N = 2$, it was found that

$$\Delta_\psi = 1.067\,, \quad \Delta_\sigma = 0.660\,, \quad \Delta_{\sigma^2} = 2.14\,. \tag{14}$$

V. CONCLUSION

The conformal bootstrap is a particularly powerful method to constrain and solve conformal field theories. It is a highly general approach, since it does not require a Lagrangian description or a UV version of the theory. It only relies on conformal invariance, unitarity, crossing relations and some inputs from CFTs of interest. This paper served as a brief overview of the conformal bootstrap in $D \geq 3$. We started by introducing some essential notions of conformal field theories. We then presented the program of the conformal bootstrap and an example of algorithm to constrain the scaling dimension of operators. The last section was dedicated to a review of some interesting results obtained during the last years.

Although the conformal bootstrap is an extremely promising avenue that has led to various impressive outcomes, there are still plenty of open questions that are yet to be answered. As an example, few results have been obtained for critical gauge theories [14].

Acknowledgements

I would like to thank Prof. John McGreevy for his amazing 3-quarter quantum field theory course.

[1] A. M. Polyakov, JETP Lett. **12**, 381 (1970).
[2] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998), arXiv:hep-th/9711200 .
[3] S. Ferrara, A. F. Grillo, and R. Gatto, Annals of Physics **76**, 161 (1973).
[4] A. M. Polyakov, Zh. Eksp. Teor. Fiz. **66**, 23 (1974).
[5] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, JHEP **12**, 031 (2008), arXiv:0807.0004 [hep-th] .
[6] H. Osborn and A. C. Petkou, Annals Phys. **231**, 311 (1994), arXiv:hep-th/9307010 .

[7] D. Pappadopulo, S. Rychkov, J. Espin, and R. Rattazzi, Phys. Rev. D **86**, 105043 (2012), arXiv:1208.6449 [hep-th] .

[8] F. A. Dolan and H. Osborn, Nucl. Phys. B **678**, 491 (2004), arXiv:hep-th/0309180 .

[9] D. Poland, S. Rychkov, and A. Vichi, Rev. Mod. Phys. **91**, 015002 (2019), arXiv:1805.04405 [hep-th] .

[10] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Phys. Rev. D **86**, 025022 (2012).

[11] F. Kos, D. Poland, D. Simmons-Duffin, and A. Vichi, JHEP **08**, 036 (2016), arXiv:1603.04436 [hep-th] .

[12] L. Iliesiu, F. Kos, D. Poland, S. S. Pufu, D. Simmons-Duffin, and R. Yacoby, JHEP **03**, 120 (2016), arXiv:1508.00012 [hep-th] .

[13] L. Iliesiu, F. Kos, D. Poland, S. S. Pufu, and D. Simmons-Duffin, JHEP **01**, 036 (2018), arXiv:1705.03484 [hep-th] .

[14] Y.-C. He, J. Rong, and N. Su, SciPost Phys. **11**, 111 (2021), arXiv:2101.07262 [hep-th] .

[15] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics (Springer-Verlag, New York, 1997).

[16] L. F. Alday, (2016).

[17] S. M. Chester, (2019), arXiv:1907.05147 [hep-th] .

[18] S. Rychkov, *EPFL Lectures on Conformal Field Theory in D>= 3 Dimensions*, SpringerBriefs in Physics (2016) arXiv:1601.05000 [hep-th] .

[19] D. Simmons-Duffin, in *Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings* (2017) pp. 1–74, arXiv:1602.07982 [hep-th] .

Appendix A: Conformal group and algebra

To obtain the Lie algebra of the conformal group, we first need to find its generators. To do so, it suffices to consider an infinitesimal conformal transformation on the coordinates $x^\mu \to x^\mu + \epsilon^\mu(x)$. One can then show that in the differential operator representation acting on functions of spacetime, the generators are [15]

$$\begin{aligned}
&\text{Translation:} && P_\mu = -\mathrm{i}\,\partial_\mu \\
&\text{Rotation:} && M_{\mu\nu} = \mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
&\text{Dilatation:} && D = -\mathrm{i}\,x^\mu\partial_\mu \\
&\text{Special conformal transformation:} && K_\mu = -\mathrm{i}(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)
\end{aligned} \tag{A1}$$

Knowing this, it is then possible to obtain the commutation relations between the generators, which forms the conformal algebra

$$\begin{aligned}
[D, P_\mu] = \mathrm{i}\,P_\mu, \quad [D, K_\mu] = -\,\mathrm{i}\,K_\mu, \quad [K_\mu, P_\nu] = 2\,\mathrm{i}(\eta_{\mu\nu}D - M_{\mu\nu}) \\
[M_{\mu\nu}, P_\rho] = -\,\mathrm{i}(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \quad [M_{\mu\nu}, K_\rho] = -\,\mathrm{i}(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \\
[M_{\mu\nu}, M_{\rho\sigma}] = -\,\mathrm{i}(M_{\mu\rho}\eta_{\nu\sigma} - M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\rho}\eta_{\mu\sigma} + M_{\nu\sigma}\eta_{\mu\rho}), \quad [D, M_{\mu\nu}] = 0\,.
\end{aligned} \tag{A2}$$

Note that by expressing the generators in the more compact form

$$\begin{aligned}
J_{\mu,\nu} = M_{\mu\nu}, \qquad J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu)\,, \\
J_{-1,0} = D, \qquad J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu)\,,
\end{aligned} \tag{A3}$$

with $J_{a,b} = -J_{b,a}$, for $a = -1, 0, 1, ..., D$, one can show that these new generators respect the $\mathrm{so}(D+1,1)$ algebra

$$[J_{mn}, J_{pq}] = \mathrm{i}(\eta_{mq}J_{np} + \eta_{np}J_{mq} - \eta_{mp}J_{nq} - \eta_{nq}J_{mp})\,, \tag{A4}$$

which shows that the conformal group is isomorphic to $\mathrm{SO}(D+1,1)$.

Appendix B: Action of generators on field operators

To study CFTs, we need to know how field operators transform under the conformal group. To do so, it is sufficient to determine the action of the generators on the fields. The procedure (presented notably in [16]), is to start with the operator $\mathcal{O}(0)$ (spin indices implicit) at the origin, to translate it at position x with the Heisenberg equation

$$\mathcal{O}(x) = \mathrm{e}^{-\,\mathrm{i}\,P\cdot x}\,\mathcal{O}(0)\,\mathrm{e}^{\mathrm{i}\,P\cdot x}\,, \tag{B1}$$

and to finally compute the commutator $[G, \mathcal{O}(x)]$ (where G is any of the generators). In an irreducible representation of $\mathrm{SO}(D)$, the result for the first three generators is

$$\begin{aligned} [P_\mu, \mathcal{O}(x)] &= \mathrm{i}\,\partial_\mu \mathcal{O}(x)\,, \\ [D, \mathcal{O}(x)] &= \mathrm{i}(\Delta + x^\mu \partial_\mu)\mathcal{O}(x)\,, \\ [M_{\mu\nu}, \mathcal{O}(x)] &= \mathrm{i}(S_{\mu\nu} - x_\mu \partial_\nu + x_\nu \partial_\mu)\mathcal{O}(x)\,, \end{aligned} \tag{B2}$$

where $S_{\mu\nu}$ are $\mathrm{SO}(D)$ matrices acting on the spin indices of $\mathcal{O}$: $S_{\mu\nu}\mathcal{O}(x) = (S_{\mu\nu})^a{}_b \mathcal{O}^b(x)$. The rank l of the representation of $\mathrm{SO}(D)$ is the spin. Moreover, Δ is a real number called the scaling dimension of the operator. Since D and $M_{\mu\nu}$ commute, we see that operators in a CFT are labeled by Δ and l.

Next, using the conformal algebra and the action of D on $\mathcal{O}(0)$, we see that [17]

$$\begin{aligned} [D, [P_\mu, \mathcal{O}(0)]] &= \mathrm{i}\,\Delta P_\mu \mathcal{O}(0) + \mathrm{i}\,P_\mu \mathcal{O}(0) - \mathrm{i}\,\mathcal{O}(0) P_\mu - \mathrm{i}\,\Delta \mathcal{O}(0) P_\mu = \mathrm{i}(\Delta + 1)[P_\mu, \mathcal{O}(0)] \\ [D, [K_\mu, \mathcal{O}(0)]] &= \mathrm{i}\,\Delta K_\mu \mathcal{O}(0) - \mathrm{i}\,K_\mu \mathcal{O}(0) + \mathrm{i}\,\mathcal{O}(0) K_\mu - \mathrm{i}\,\Delta \mathcal{O}(0) K_\mu = \mathrm{i}(\Delta - 1)[K_\mu, \mathcal{O}(0)]\,. \end{aligned} \tag{B3}$$

This shows that the action of P_μ/K_μ is to raise/lower the scaling dimension by 1. In fact, we see directly from the conformal algebra that P_μ and K_μ act respectively as a raising/lowering operator.

In fact, in a unitary CFT, scaling dimensions are non-negative and bounded from below. Therefore, K_μ can only act a finite number of times before yielding 0. Operators that are annihilated at the origin by special conformal transformations are called primary operators, which respect

$$[K_\mu, \mathcal{O}(0)] = 0\,. \tag{B4}$$

Primary operators constitute building blocks of the operator spectrum of a CFT. Indeed, given a primary of dimension Δ, operators of higher scaling dimension can be generated by acting with P_μ an arbitrary number of times. These operators are called descendants and they form with their primary a conformal family. It turns out that every operator in a CFT is either a primary or a descendant.

Knowing this, it can be shown that the action of K_μ on the primary $\mathcal{O}(x)$ is

$$[K_\mu, \mathcal{O}(x)] = \mathrm{i}(2\Delta x_\mu + 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu - 2x^\nu S_{\mu\nu})\mathcal{O}(x)\,. \tag{B5}$$

Appendix C: Radial quantization and state-operator correspondence

This appendix presents an alternative way to define CFTs in terms of quantum states [18]. In general QFT (not conformal) invariant under time translation, it is natural to divide spacetime in slices of equal time and to evolve quantum states along the time direction with the Hamiltonian $H = P^0$. In this case, entering states $|\mathrm{in}\rangle$ can be generated by acting with creation operators in the past, while outgoing states $|\mathrm{out}\rangle$ are generated in the future.

However, because of their scale invariance, it is more natural to foliate spacetime for CFTs in terms of hyperspheres of $D-1$ dimensions. The evolution of quantum states is then along the radial direction. In this case, incoming/outgoing states are generated by acting with creation operators inside/outside a hypersphere. This is called the radial quantization. Moreover, scale invariance implies that the Hilbert space is the same on every hypersphere. Therefore, we define quantum states living on these surfaces as $|\Delta, l\rangle_a$, such that

$$D\,|\Delta,l\rangle_a = \mathrm{i}\,\Delta\,|\Delta,l\rangle_a\;, \qquad M_{\mu\nu}\,|\Delta,l\rangle_a = \mathrm{i}(S_{\mu\nu})_{ab}\,|\Delta,l\rangle_b\;. \tag{C1}$$

Like the operators in a CFT, the quantum states living on hyperspheres are also labeled by Δ (eigenvalue of D) and an SO(D) representation. To understand this, let's build the Hilbert space of the theory. The vacuum state is defined as being annihilated by the four generators of the conformal group

$$P_\mu|0\rangle = D|0\rangle = L_{\mu\nu}|0\rangle = K_\mu|0\rangle = 0\,. \tag{C2}$$

Let's now insert a primary scalar (for simplicity) of scaling dimension Δ at the origin, which corresponds to the state $|\mathcal{O}\rangle = \mathcal{O}(0)\,|0\rangle$. Acting with D, we get

$$D\,\mathcal{O}(0)\,|0\rangle = [D,\mathcal{O}(0)]|0\rangle + \mathcal{O}(0)\,D|0\rangle = \mathrm{i}\,\Delta\,\mathcal{O}(0)\,|0\rangle\,. \tag{C3}$$

Since this state is an eigenstate of D with a scaling dimension Δ, we see that $|\mathcal{O}\rangle = |\Delta, l=0\rangle$. On the other hand, acting with K_μ yields

$$K_\mu\,|\mathcal{O}\rangle = [K_\mu,\mathcal{O}(0)]\,|0\rangle + \mathcal{O}(0)\,K_\mu\,|0\rangle = 0\,, \tag{C4}$$

which shows that the state generated by the primary $\mathcal{O}$ is annihilate by the SCTs at the origin. We then see a clear relation between primary operators and the states that they generate. In fact, we can write schematically [19]

$$\begin{aligned}
[D,\mathcal{O}^a(0)] = \mathrm{i}\,\Delta\,\mathcal{O}^a(0) \quad &\longleftrightarrow \quad D|\Delta,l\rangle_a = \mathrm{i}\,\Delta\,|\Delta,l\rangle_a\,,\\
[M_{\mu\nu},\mathcal{O}^a(0)] = \mathrm{i}(S_{\mu\nu})^a{}_b\,\mathcal{O}^b(0) \quad &\longleftrightarrow \quad M_{\mu\nu}|\Delta,l\rangle_a = \mathrm{i}(S_{\mu\nu})^a{}_b\,|\Delta,l\rangle_b\,,\\
[K_\mu,\mathcal{O}^a(0)] = 0 \quad &\longleftrightarrow \quad K_\mu|\Delta,l\rangle_a = 0\,,
\end{aligned} \tag{C5}$$

which is called the state-operator correspondence: there is a one-to-one mapping between field operators in a CFT and states that they generated in radial quantization.

Appendix D: Unitarity consequences

In a unitary CFT, states obtained in radial quantization have a positive-definite norm. With this condition, one can show that this puts lower bounds on the scaling dimension of operators in the CFT spectrum. For example, three important bounds are

$$\Delta \geq \begin{cases} \frac{D-2}{2} & \text{for } \; l=0,\\ \frac{D-1}{2} & \text{for smallest spinor},\\ l+d-2 & \text{for } \; l\geq 1, \text{ traceless symmetric tensor}\,. \end{cases} \tag{D1}$$

Moreover, in a unitary CFT, we are free to choose a basis in which all the operators are real. This implies that all the OPE coefficients are also real. See [9] for more details.