Solitons with fractional fermion number

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We study zero-mode solutions for Fermi fields under a Yukawa interaction allowing for soliton solutions in $1+1$-dimensions and in the potential of Yang-Mills monopoles in $3+1$-dimensions. We review in both cases how the soliton solutions arises in the absence of fermions, and obtain the zero-energy solutions for the Fermi fields. In particular, we calculate the fermion number corresponding to these zero-modes and find that they are fractionalized, carrying $\frac{1}{2}$ units of charge. Moreover, we present a lattice example where zero-modes play an important role: the $1D$ p-wave superconductor.

INTRODUCTION

In this paper, we examine Dirac equation zero-mode solutions under different soliton potentials and dimensions [1]. For the instances reviewed, we see that whenever a Dirac equation posseses a non-degenerate, zero-energy mode implies that the solitons are degenerate doublets with fermion number $\pm\frac{1}{2}$.

Starting in $1+1$-dimensions, we consider the fermion field coupled to a scalar field via a Yukawa interaction. Assuming that soliton solutions for the scalar field exist, we show that the Fermi field under this configuration exhibits zero-energy mode with fermion number $\pm\frac{1}{2}$.

We then proceed to $3+1$-dimensions where we consider Yang-Mills theory with a triplet of mesons coupled to fermionic fields. The theory without the fermions is in itself interesting as soliton solutions arise known as t' Hooft-Polyakov monopoles. We dedicate Appendix 1 to review this important topic. Under this monopole configuration, the Fermi zero-mode solutions are found in two cases: isospinor and isovector fermions. In the former the solution is nondegenerate, while in the latter is doubly degenerate.

Finally, we present a lattice example of the $1+1$-dimensional case: 1D p-wave superconductor. We show how the analysis of the previous sections helps understand and identify the different topological phases of the system. In particular, by trading the fermions with Majorana operators, we arrive to the famous Kitaev chain. In this model, one finds that in the topological phase the zero-modes live at the edges of the chain and that two different topological phases are only connected by the Majorana zero-mode.

FERMIONIC SOLITONS IN 1+1-DIMENSIONS

To warm-up, let us consider theories with a scalar field ϕ and a spinor field ψ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{g^2}U(g\phi) + i\bar{\psi}\gamma^\mu\partial_\mu\psi - G\bar{\psi}V(g\phi)\psi. \quad (1)$$

Suppose that in the absence of the fermion fields, the potential U allows for soliton solutions to exist with $\phi_c(\infty) = -\phi_c(-\infty)$. A convenient example is $U(\phi) = \frac{1}{2}\lambda^2(1-\phi^2)^2$. Static solutions will be the classical ground states $\phi_0 = \pm g^{-1}$ but also interpolations between these two ground states (solitons)

$$\partial_x\phi_c = \pm\sqrt{U(g\phi_c)}, \quad \phi_c = \pm\frac{1}{g}\tanh(\lambda x). \quad (2)$$

Then, the Dirac equation for these configurations is

$$i\gamma^\mu\partial_\mu\psi - GV(g\phi_c)\psi = 0. \quad (3)$$

Furthermore, suppose the Yukawa interaction has the anti-symmetric property $GV(g\phi_c(\infty)) = -GV(g\phi_c(-\infty)) = m > 0$. Taking $\psi(x,t) = e^{-iEt}\psi_E(x)$, we get

$$i\gamma^1\partial_x\psi_E(x) - GV(g\phi_c)\psi_E(x) = -E\gamma^0\psi_E(x). \quad (4)$$

We can find the zero-modes by setting $E=0$ and representing the Dirac matrices $\gamma^0 = \sigma_x$, $\gamma^1 = i\sigma_z$

$$\partial_x\psi_0(x) = -\sigma_z GV(g\phi_c)\psi_0(x), \quad (5)$$

with solution $\psi_0(x) = \exp\left(-\int_0^x GV(g\phi_c(x))\sigma_z\right)\tilde{\psi}_0$. We can expand it in eigenstates of σ_z to obtain the two linearly independent solutions

$$\psi_0(x) = A\, e^{-\int_0^x GV(g\phi_c(x))}\begin{pmatrix}1\\0\end{pmatrix} + B\, e^{\int_0^x GV(g\phi_c(x))}\begin{pmatrix}0\\1\end{pmatrix}. \quad (6)$$

However, as we need our solution to be normalizable when $x\to\infty$, $B=0$ leaving us with a unique non-degenerate zero-mode at the interface where the mass sign changes.

To calculate the fermion number, let us expand the field operator as usual but including the zero-energy modes

$$\psi(x,t) = \hat{a}\,\psi(x) + \int dk\left(e^{-iE_kt}\hat{b}_k\psi_k(x) + e^{iE_kt}\hat{d}_k^\dagger\psi_k^c\right), \quad (7)$$

where $\hat{b}_k^\dagger/\hat{b}_k$ and $\hat{d}_k^\dagger/\hat{d}_k$ create and annihilate fermions and antifermions. The $\hat{a}$ operator is associated to the zero-energy mode satisfying $\{a, a^\dagger\} = 1$ and

$$\begin{aligned} &a\,|+,S\rangle = |-,S\rangle\,, \quad a^\dagger\,|-,S\rangle = |+,S\rangle\,, \\ &a\,|-,S\rangle = a^\dagger\,|+,S\rangle = 0, \end{aligned} \quad (8)$$

where $|\pm, S\rangle$ are the filled and empty states of the zero-mode a. Replacing into the fermion number operator one gets

$$Q = \frac{1}{2}\int dx\,[\psi^\dagger, \psi] = a^\dagger a - \frac{1}{2} + \int dk(b_k^\dagger b_k - d_k^\dagger d_k), \quad (9)$$

which implies that

$$Q\,|\pm, S\rangle = \pm\frac{1}{2}\,|\pm, S\rangle\,, \quad (10)$$

i.e. each of the two solitons carry $\frac{1}{2}$ units of charge.

At first sight, even though the existence of states with fermion number $\pm\frac{1}{2}$ is remarkable, it appears as there is no practical significance. In fact, this is what Jackiw and Rebbi concluded in their paper [1]. Nonetheless, a physical realization is nowadays very well-known in polymers: polyacetylene. It consists of bonded CH groups forming an isomeric long-chain. In the ground state we have alternating electronic single and double bonds. Hence, there exists two inequivalent but degenerate configurations which we label as A and B as shown in the figure below. But we can also have a topologically stable configuration by introducing an imperfection. In one direction the polymer is in the state A while in the other direction is in state B. And in between these two states, where the bond alternation pattern interpolates, is the location of the soliton.

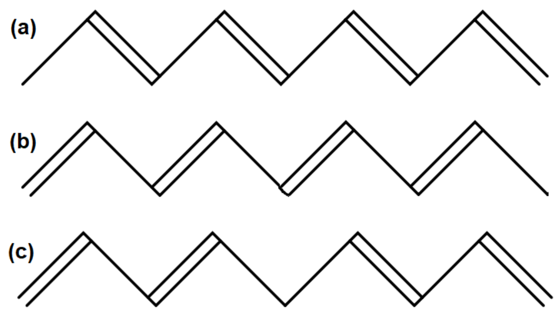

FIG. 1: Adapted from [6]. (a) Ground state configuration A. (b) Ground state configuration B. (c) A topological defect separating the two different states.

FERMIONIC SOLITONS IN 3+1-DIMENSIONS

Let us continue our discussion but now to $3+1$ dimensions. Consider the following Yang-Mills theory coupled to fermionic fields as

$$\begin{aligned}\mathcal{L} =& -\frac{1}{4}F_a^{\mu\nu}F_{a\mu\nu} + \frac{1}{2}(D_\mu\Phi)_a(D^\mu\Phi)_a - \frac{1}{g^2}U(g\Phi) \\ &+ i\bar{\psi}_n\gamma^\mu(D_\mu\psi)_n - Gg\bar{\psi}_n T^a_{nm}\psi_m\Phi_a, \end{aligned} \quad (11)$$

where G is positive dimensionless coupling, and λ, μ carry dimensions of mass.

In the absence of fermions, one can show that monopole solutions exist (See Appendix 1). These are given by

$$\Phi_a = \frac{x^a}{gr}\phi(r), \quad A_a^i = \epsilon^{aij}\frac{x_j}{gr}A(r), \quad A_a^0 = 0, \quad (12)$$

where $\phi(r), A(r)$ vanish at $r \to 0$, and $\phi(r)$ will approach the vacuum value while $A(r) \sim r^{-1}$ as $r \to \infty$.

Dirac equation under this monopole potential is

$$\big(\vec{\alpha}\cdot\vec{p}\,\delta_{nm} + A(r)T^a_{nm}(\vec{\alpha}\times\hat{x})_a + G\phi T^a_{nm}\hat{x}_a\beta\big)\psi_m = E\psi_n,$$

where we will take the following representation for the Dirac matrices

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \beta = -i\begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \quad (13)$$

Decomposing the fermion field ψ_n into its upper and lower components $\psi_n = (\chi_n^+,\ \chi_n^-)^T$ and setting $E = 0$

$$\big(\vec{\alpha}\cdot\vec{p}\,\delta_{nm} + A(r)T^a_{nm}(\vec{\alpha}\times\hat{x})_a \pm iG\phi T^a_{nm}\hat{x}_a\big)\chi_m^\pm = 0. \quad (14)$$

Let us consider the isospinor instance. The generators are $T^a = \frac{1}{2}\tau^a$ and $n, m = 1, 2$. In this case, the lower component vanishes while the upper can be found to be

$$\chi^+_{\nu n} = N e^{-\int_0^r dr'(\frac{1}{2}G\phi(r') - A(r'))}(s_\nu^+ s_n^- - s_\nu^- s_n^+), \quad (15)$$

where ν is the Dirac indices, N is a normalization factor and $s_n^\pm$ are eigenstates of the isospin τ^3 while $s_\nu^\pm$ eigenstates of the spin operator σ_z. The isospinor solution is nondegenerate and corresponds to zero spin. Expanding the fermion field leads again to the conclusion that the doublet has fermion number $\pm\frac{1}{2}$.

A similar procedure can be done for isovectors where $T^a_{nm} = i\epsilon_{nam}$ $(n, m = 1, 2, 3)$. The lower component again vanishes but the upper do not. In this case, the upper part will be proportional to an arbitrary spinor $\chi_n^+ \sim \chi$ meaning that we have two linearly independent solutions s^+, s^- compared to the isospinor. Two operators a_s then will be found in the Dirac field expansion, each of which will carry fermion number $\pm\frac{1}{2}$. But since we have now two independent pairs of operators, the solitons will be product vectors with fermion numbers $+1$ ($|+,+\rangle$), -1 ($|-,-\rangle$) and 0 ($|\pm,\mp\rangle$). Thus, we get four soliton states.

WHAT ABOUT 2+1-DIMENSIONS?

Jackiw and Rebbi studied the occurrence of the zero-modes for the Dirac equation in topologically interesting background fields in $1+1$-D and $3+1$-D. Later in 1981, Jackiw and Rossi filled the dimensional gap analyzing the $2+1$-dimensional case [2]. We present here a brief summary of the results.

Consider massless Dirac fermions in the Abelian gauge theory coupled to charged scalar fields. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{D}\psi - \frac{1}{2}ig\phi\bar{\psi}\psi^c + \frac{1}{2}ig^*\phi^*\bar{\psi}^c\psi, \quad (16)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\not{D} = \gamma^\mu(\partial_\mu + ieA_\mu)$, e is the charge of the fermion and ψ^c is the charge conjugate spinor related to $\bar{\psi}$ via $\psi^c = i\sigma_y\bar{\psi}^T$. Using $\psi = \begin{pmatrix}\psi_1 & \psi_2\end{pmatrix}^T$ we can rewrite the interaction,

$$\mathcal{L}_I = \frac{i}{2}(g\phi\bar{\psi}\psi^c + g^*\phi^*\bar{\psi}^c\psi) = ig\phi\psi_1^*\psi_2^* - ig^*\phi^*\psi_2\psi_1, \quad (17)$$

where we find that the interaction term generates a pairing between the ψ_1 and ψ_2. The Lagrangian in Eq. (16) thus represents a $2+1$-D superconductor model.

Let us consider the following background field

$$\phi(\vec{r}) = e^{in\theta}f(r), \quad qA^i(\vec{r}) = \epsilon^{ij}\hat{r}^j A(r), \quad A^0 = 0.$$

where $\vec{r} = (r\cos\theta, r\sin\theta)$ and $q = 2e$ is the charge of the scalar field. The asymptotic behavior of the functions $f(r)$ and $A(r)$ are

$$f(r) = \begin{cases} f_0 r^{|n|}, & r\to 0 \\ f_\infty, & r\to\infty \end{cases}, \qquad A(r) = \begin{cases} 0, & r\to 0 \\ -\frac{n}{r}, & r\to\infty \end{cases}.$$

Then, Dirac equation is

$$i\partial_t\psi = \vec{\alpha}\cdot(\vec{p} - e\vec{A})\psi - g\phi\sigma_y\psi^*, \quad \vec{\alpha} = (\sigma_x, \sigma_y). \quad (18)$$

One can show that the non-linear equation for the zero-modes under this potential

$$\vec{\alpha}\cdot(\vec{p} - e\vec{A})\psi_0 = g\phi\sigma_y\psi_0^*, \quad (19)$$

can be solved analytically using the ansatz

$$\psi_0 = \begin{pmatrix} e^{\frac{1}{2}\int_0^r A(\rho)}\psi_U \\ e^{-\frac{1}{2}\int_0^r A(\rho)}\psi_L \end{pmatrix}, \quad (20)$$

and that possesses $|n|$ linearly independent zero-modes satisfying $\psi_0 = \psi_0^c$, i.e. Majorana zero-modes. A complete and thorough explanation of this statement can be found in [2].

P-WAVE SUPERCONDUCTOR

At the lattice level, an interesting toy model is given by the $1D$ p-wave superconductor. We stress that while it seems like this model realizes the $1+1-D$ zero-mode studied before, it in fact does not. These are two completely different physical situations. The field theory considered in Eq. (1), ϕ is a neutral scalar field where there is a conserved fermion number. Solitons have a fractional fermion number because they have complex zero-modes carrying integer fermion number, creating a pair of states with fermion number $\pm\frac{1}{2}$. A different field theory is where the scalar ϕ is charged under the fermion number symmetry. After ϕ condensates, the $U(1)$-symmetry is broken to $\mathbb{Z}_2$. Thus, the corresponding Majorana zero-mode found in this theory do not have a fermion number associated.

Let us be explicit. Instead of the Yukawa interaction between the neutral ϕ and the Fermi fields in (1), consider an interaction as in (17). Then, equation (5) would become

$$\partial_x\psi_0^M(x) = -\sigma_z g\phi\,\psi_0^M(x)^*. \quad (21)$$

This non-linear equation can fix the phase of the zero-mode solution and the zero-mode can be set to be real, i.e. a Majorana zero-mode. To see this, let us evaluate the equations for the real/imaginary parts

$$\partial_x\psi_0^{R/I} = \mp\sigma_z g\phi\,\psi_0^{R/I}. \quad (22)$$

The solution is

$$\psi_0^{R/I}(x) = Ae^{\mp\int_0^x g\phi(x)}\begin{pmatrix}1\\0\end{pmatrix} + Be^{\pm\int_0^x g\phi(x)}\begin{pmatrix}0\\1\end{pmatrix}. \quad (23)$$

where again we choose the normalizable solution. Hence,

$$\psi_0^M(x) = \psi_0^R + i\psi_0^I = Ne^{-\int_0^x g\phi(x)}\begin{pmatrix}1\\i\end{pmatrix}. \quad (24)$$

Under charge conjugation, this zero-mode will be self-conjugate

$$\psi_0^c = \sigma_z(\psi_0^M)^* = \psi_0^M \quad (25)$$

as it can be easily seen. These Majorana modes are the ones relevant for the lattice model we will study next: the p-wave superconductor and the Kitaev chain.

The Hamiltonian of the p-wave superconductor is

$$\hat{H} = \sum_j\big(-\mu c_j^\dagger c_j - (tc_{j+1}^\dagger c_j + |\Delta|c_j c_{j+1} + h.c.)\big), \quad (26)$$

where t is the hopping between nearest neighbors, μ the onsite energy and Δ is the superconductiong pairing. This BCS-type model has a momentum dependent order parameter $\Delta(p) = i\Delta\sin(p)$

$$\hat{H} = \sum_p\big(-2t\cos(p) - \mu\big)c_p^\dagger c_p + \big(i|\Delta|\sin(p)c_{-p}c_p + h.c.\big).$$

It is convenient to rewrite the Hamiltonian in the BdG form, $\hat{H} = \frac{1}{2}\Psi^\dagger\mathcal{H}_{BdG}\Psi$ where $\Psi^\dagger = \begin{pmatrix}c_p^\dagger & c_{-p}\end{pmatrix}$ and

$$\mathcal{H}_{BdG}(p) = \big(-2t\cos(p) - \mu\big)\hat{\sigma}_z + 2|\Delta|\sin(p)\,\hat{\sigma}_y. \quad (27)$$

Written this way, we see that the Hamiltonian has particle-hole symmetry implemented by the operator $\mathcal{P} = \sigma_x K$ where K denotes complex conjugation, i.e. $\mathcal{P}\mathcal{H}_{BdG}(-p)\mathcal{P}^{-1} = -\mathcal{H}_{BdG}(p)$. This means that for every eigenvector $|\psi\rangle$ with energy E and momentum p, there is an eigenvector $\mathcal{P}|\psi\rangle$ with energy $-E$ and momentum $-p$.

The energy spectrum is given by

$$E(p) = \pm\sqrt{(2t\cos(p)+\mu)^2 + 4|\Delta|^2\sin^2(k)}. \tag{28}$$

Consider the critical point $p = 0$ at which the gap closes when $\mu = -2t$. Equation (27) becomes

$$\mathcal{H}_{BdG}(p) = (-2t-\mu)\hat{\sigma}_z + 2|\Delta|p\hat{\sigma}_y, \tag{29}$$

which is Dirac equation with mass $m = -2t-\mu$ and velocity $v = 2|\Delta|$. Hence, we see that at the critical point the mass sign flips defining a trivial phase ($m > 0$) and a topological ($m < 0$). There is no way to adiabatically connect both regions without closing the gap and so they represent two different topological phases, only connected by the zero-mode. Same arguments can be given for the other critical point $p = \pi$.

KITAEV CHAIN

We can understand it better by using Majorana operators $c_j = \frac{1}{2}(a_{2j-1} + ia_{2j})$ satisfying $a_j = a_j^\dagger$ and $\{a_j, a_{j'}\} = 2\delta_{jj'}$. The lattice p-wave Hamiltonian becomes the well-known Kitaev chain

$$\mathcal{H} = \frac{i}{2}\sum_j \left(-\mu a_{2j-1}a_{2j} + (t+|\Delta|)a_{2j}a_{2j+1} - (t-|\Delta|)a_{2j-1}a_{2j+2}\right).$$

The two regimes can be understood as follows. Let us set in the trivial phase ($m = -2t-\mu > 0$) the parameters as $|\Delta| = t = 0$. In this case the Hamiltonian reduces to

$$\mathcal{H} = -\frac{i\mu}{2}\sum_j a_{2j-1}a_{2j}. \tag{30}$$

Majorana operators are coupled at each physical site but decoupled between different sites. The ground state is trivial, meaning is a product state. There are no low-energy states at the edges.

Conversely, the topological phase will display edge modes when there is open boundary conditions. Setting $|\Delta| = t > 0$ and $\mu = 0$, we see that

$$\mathcal{H} = it\sum_j a_{2j}a_{2j+1}, \tag{31}$$

will have two unpaired Majorana zero-modes localized at the ends of the chain a_1 and a_{2L}. Yet, it seems that these unpaired Majoranas are artificial, only appearing when the parameters are fine-tuned. Naively, one would expect that changing $\mu \neq 0$ will make the zero-modes interact and so disappear. In fact, the zero-modes will not be destroyed until the bulk gap closes at a critical point. Isolated zero-modes at each end of the Kitaev chain are protected by particle-hole symmetry and by the absence of zero-energy excitations in the bulk. This is an example of the bulk-edge correspondence.

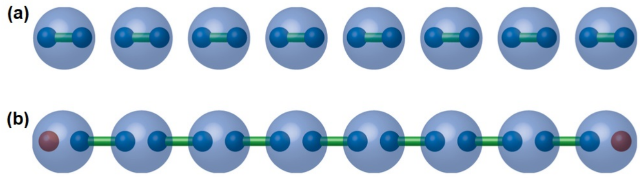

FIG. 2: Two regimes of the Kitaev chain: (a) The trivial phase where the ground state is a product state of spinless fermions. All the Majorana pairing is within the same physical site. (b) The topological phase where the ground state is not a product state and there's entanglement between Majoranas in different sites. There is no local operator that disentangles the chain.

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't Hooft-Polyakov monopole

Soliton solutions for Yang-Mills theories in which $U(1)$ is taken to be a subgroup of a larger compact covering group (as $SU(2)$) were first found by 't Hooft [3] and Polyakov [4]. Consider $SU(2)$ Yang-Mills with a Higgs potential

$$\mathcal{L} = -\frac{1}{4}\operatorname{tr} F_{\mu\nu}^2 + \frac{1}{2}\operatorname{tr}(D_\mu \Phi)^2 + \frac{\lambda}{4}(\operatorname{tr}\Phi^2 - \Phi_0^2)^2, \tag{32}$$

where $F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g\epsilon_{abc}A_b^\mu A_c^\nu$, Φ_a is a triplet spinless boson and $[T_a, T_b] = i\epsilon_{abc}T_c$.

The equations of motion are

$$D^2\Phi = \lambda(\operatorname{tr}\Phi^2 - \Phi_0^2)\Phi, \qquad D_\nu F_a^{\mu\nu} = g\epsilon_{abc}\phi_b D^\mu \Phi_c. \tag{33}$$

The vacuum of the theory is then states satisfying

$$D_\mu \Phi_a = 0, \qquad \operatorname{tr}\Phi^2 = \Phi_0^2. \tag{34}$$

However, there are more interesting configurations: finite energy configurations that should approach the vacuum solution $\operatorname{tr}\Phi^2 \to \Phi_0^2$ and $r^{d/2}D_i\Phi_a \to 0$ as $r \to \infty$. The energy of the configuration then will become

$$E = \frac{1}{2}\int d^3x \, \operatorname{tr}(D_i\Phi)^2 \sim \int d^2\Omega \, dr \, r^2 \operatorname{tr}\frac{(D_\theta \Phi)^2}{r^2}, \tag{35}$$

which if not zero would diverge linearly. Thus, we would need

$$D_\theta \Phi^a = \frac{1}{r}\partial_\theta \Phi^a + g\epsilon^{abc}A_\theta^b \Phi^c \to 0 \text{ as } r \to \infty, \tag{36}$$

which can be achieved by tuning $A_\theta \sim 1/r$. One solution is given by the ansatz

$$\Phi^a = \frac{x^a}{gr^2}\psi(r), \qquad A_0^a = 0, \qquad A_i^a = \epsilon_{aij}\frac{x^j}{gr^2}(1 - A(r)), \tag{37}$$

with boundary conditions

$$\psi(r) = \begin{cases} 0, & r \to 0 \\ g\Phi_0 r, & r \to \infty \end{cases}, \qquad A(r) = \begin{cases} 1, & r \to 0 \\ 0, & r \to \infty \end{cases}. \tag{38}$$

One can show that this ansatz satisfy (36)

$$D_i \Phi^a = \frac{x^a x^i}{gr^4}\Big(r\partial_r \psi(r) - \psi(r) - A(r)\psi(r)\Big) + \delta_{ai}\frac{A(r)\psi(r)}{gr^2} \longrightarrow 0 \text{ as } r \to \infty. \tag{39}$$

Moreover, we can look at the magnetic field

$$B_i^a = \frac{1}{2}\epsilon_{ijk}F_{jk}^a = \frac{x_i x^a}{gr^4}(1 - A^2(r) + r\partial_r A(r)) - \delta_{ai}\frac{1}{gr^3}\partial_r A(r) \to \frac{x_i x^a}{gr^4} \text{ as } r \to \infty. \tag{40}$$

Asymptotically, we have found a monopole magnetic field $\vec{B} = \frac{\hat{r}}{gr^2}$. This is a soliton solution called 't Hooft-Polyakov monopole. These solutions cannot be continuously deformed into a uniform solution, i.e. it is topologically stable.

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