UV Divergent Spiderwebs: Entanglement and Entropy in Quantum Field Theory

Aiden Sheckler[1]

[1]*Department of Physics, University of California at San Diego, La Jolla, CA 92093*

Following the notes of Witten [2] we introduce the construction of relative entropy and entanglement as a property of the algebra of observables on the full continuum field theory Hilbert space. We exhibit the connection to the more familiar quantum information concepts in the nonrelativistic finite-dimensional case, and discuss the bridge from this case to the continuum.

INTRODUCTION

If one is not sufficiently frustrated with the many traditional complications of quantum field theory in the continuum, let them turn their attention to the issue of pinning down the familiar notions of entanglement and entropy in this setting. Like with any interesting quantum system, we can ask questions about the information available to us when we act with local operators in different corners of the Hilbert space. Unsurprisingly, we run into nasty issues in the continuum limit: the entanglement entropy between any two regions of spacetime picks up a UV divergence. Nevertheless, we can learn valuable lessons about the underlying structure of this entropy by retreating to the case of finite-dimensional systems, and then carefully constructing a bridge to the continuum case.

I: BREAKING DOWN THE QFT HILBERT SPACE

Before we address the notion of entropy, we need some preliminary understanding of the relationship between our field theory Hilbert space $\mathcal{H}$ and the algebra of operators which act on it. If we take $\mathcal{H}$ to be the space generated by those states $|\Psi\rangle$ obtained by acting with local operators $\hat{\phi}_f \equiv \int d^D x f(x)\hat{\phi}(x)$ on the vacuum: $|\Psi\rangle = \phi_{f_1}\phi_{f_2}\dots\phi_{f_n}|\Omega\rangle$, then we can ask the question: what is the minimal amount of information needed to construct this space?

In classical mechanics we have an answer: if we establish our initial conditions on the spacetime hypersurface defined by $t = 0$, then the behavior of the system is completely defined on all of spacetime. Indeed, the same is true for field theory: defining some hypersurface Σ (take for example the surfaced defined by $t = 0$), then if we require that our field operators are localized in any neighborhood of Σ, the set of resulting states is enough to generate $\mathcal{H}$ [2].

But the surprising result summarized in the Reeh-Schlieder Theorem is that this is actually overkill: we do not need the whole $t = 0$ hypersurface. In fact we only need the fields which are localized in some arbitrarily small open subset of it. The set of states obtained by acting with these local fields on the vacuum are enough to generate all of $\mathcal{H}$! In cruder words: I can generate a state containing the Andromeda galaxy by acting with field operators localized only on the tip of my big toe, even if these regions are spacelike-separated.

The precise statement and interpretation of the Reeh-Schlieder theorem is discussed in Appendix A. But here is a core lesson of the theorem which we will carry forward: all we need to generate all of $\mathcal{H}$ is the vacuum state $|\Omega\rangle$ and the algebra of operators $\mathcal{A}_\mathcal{U}$ localized to some arbitrarily small region of spacetime $\mathcal{U}$. Importantly, this shows that every region of spacetime is highly entangled with every other region of spacetime.

Given any local algebra of operators $\mathcal{A}$, a state such as $|\Omega\rangle$ which $\mathcal{A}$ acts on to generate $\mathcal{H}$ is called a *cyclic* state for $\mathcal{A}$. In fact there are many such states, and we will use them to construct our concept of entropy.

II: SIMPLIFICATION - ENTANGLEMENT IN FINITE DIMENSIONAL SYSTEMS

Before diving into the full complexity of entanglement and entropy in the continuum field theory case, let us get a taste for what to expect in a much more comfortable setting: a finite dimensional quantum system with a factorable Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. We let $\mathcal{A}_1$ be the algebra of operators acting on $\mathcal{H}_1$, and $\mathcal{A}_2$ those acting on $\mathcal{H}_2$. If we define an orthogonal basis ψ_k for $\mathcal{H}_1$ and ϕ_k for $\mathcal{H}_2$ then a generic state Ψ of this Hilbert space can be expressed as: $\Psi = \sum_k c_k \psi_k \otimes \phi_k$.

It turns out that if these bases are orthonormal then Ψ is cyclic for $\mathcal{A}_1$ and $\mathcal{A}_2$ [2]. Moreover, in the case where the dimensions of the spaces are equal, then any generic state Ψ has an expansion of this form where all c_k coefficients are nonvanishing, and equivalently is a cyclic state.

Here we see our first clear brush with entanglement: a state for which all coefficients in this expansion are nonvanishing can be thought of as "maximally entangled". Thus, the cyclic states for these algebras are precisely those which are maximally entangled. But wait there's more!

If we represent this state with a density matrix $\rho = |\Psi\rangle\langle\Psi|$ and take traces over each of the subspaces respectively, we get reduced density matrices $\rho_1 = \mathrm{tr}_2\rho$ and $\rho_2 = \mathrm{tr}_1\rho$. Then ρ_1, ρ_2 are invertible if and only if all coefficients c_k are nonvanishing, which is the case if and only if Ψ is cyclic. Why do we care about invertibility? Well if we have two states Ψ, Φ where Ψ is cyclic, and where σ_1, σ_2 are the reduced density matrices for Φ, then let us define the *relative modular operator*:

$$\Delta_{\Psi|\Phi} = \sigma_1 \otimes \rho_2^{-1}$$

Finally this allows us to define the *relative entropy* between the states Ψ and Φ:

$$\mathcal{S}_{\Psi|\Phi} = -\langle\Psi|\log\Delta_{\Psi|\Phi}|\Psi\rangle = \mathrm{tr}\rho_1(\log\rho_1 - \log\sigma_1)$$

We recognize this as the usual definition of relative entropy for two entangled states of a quantum system! With such a tool in hand, we can tackle all sorts of questions regarding the entropy and mutual information between various states of our space, when measuring with an operator localized to only a small subspace. For those not familiar with the concept of quantum entropy and mutual information, a summary is provided in Appendix B. A demonstration of this power is the ability to prove monotonicity of relative entropy [2]: we lose information by tracing out subregions of Hilbert space: $\mathcal{S}(\rho_{AB}||\sigma_{AB}) \geq \mathcal{S}(\rho_A||\sigma_A)$ where $\rho_A = \mathrm{tr}_B\rho_{AB}$ and $\sigma_B = \mathrm{tr}_B\sigma_{AB}$

What have we learned? If we can factor the Hilbert space then we get clear criteria for cyclic states, which allows us to define the relative entropy between two states for measurements in a given subregion. We accomplish this by "tracing out" the degrees of freedom in the other factorized regions. This gives us all sorts of leverage for analyzing information and entanglement entropy amongst our states.

III: ENTROPY OF ENTANGLEMENT FOR QFT

We now take inspiration from the definition in the finite-dimensional case, and attempt to formulate a similar concept of relative entropy in our full QFT Hilbert space. The "subregion" of the Hilbert space we are interested in is that associated with some open region $\mathcal{U}$ of Minkowski space. If we let $\mathcal{A}$ be the algebra of observables localized in $\mathcal{U}$, then given any cyclic state Ψ for $\mathcal{A}$, and any other state Φ, then we can define the *relative Tomita operator* for Ψ as: $S_{\Psi|\Phi}\hat{a}|\Psi\rangle = \hat{a}^\dagger|\Phi\rangle$ for $\hat{a} \in \mathcal{A}$. The requirement for Ψ to be cyclic is necessary for this operator to be well-defined [2].

Then, in analogy with our previous construction, we define the *relative modular operator*: $\Delta_{\Psi|\Phi} \equiv S^\dagger_{\Psi|\Phi}S_{\Psi|\Phi}$. This allows us to define the *relative entropy* between our states on the spacetime region $\mathcal{U}$:

$$\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi}|\Psi\rangle$$

Is this the definition of relative entropy we are looking for? It has some encouraging properties: if $\Phi = a\Psi$ where a is a unitary element of the algebra which commutes with $\mathcal{A}$, then $\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = 0$ [2]. This shows that under such a unitary transformation then Ψ can not be distinguished from Φ by measurements in $\mathcal{U}$, which we expect on physical grounds. More promising is the monotonicity of relative entropy in this case: if $\tilde{\mathcal{U}} \subset \mathcal{U}$ then $\mathcal{S}(\tilde{\mathcal{U}}) \leq \mathcal{S}(\mathcal{U})$ [2]. But, most reassuringly: this construction agrees with the finite-dimensional case (or rather they agree in the limit), which as we saw in the previous section gives the usual definition for relative entropy.

IV: COMPARISON AND INTERPRETATIONS

Let us reconcile the lessons we have learned in the continuum and finite cases, and attempt to make some interpretations. The Reeh-Schlieder theorem tells us that the vacuum $|\Omega\rangle$ is a cyclic state for the algebra of operators $\mathcal{A}_\mathcal{U}$ over *any* open region $\mathcal{U}$ of Minkowski space, no matter how small.

In the finite-dimensional case the existence of such a cyclic state $|\Psi\rangle$ would imply a factorization of the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with the local algebra of operators over each factor acting on Ψ, which we interpret to be a "maximally entangled state". This allows us to define relative entropy by tracing out the degrees of freedom in one of the factors.

This begs us to conjecture that in the QFT case we should be able to factor our Hilbert space by spacetime region: $\mathcal{H} = \mathcal{H}_\mathcal{U} \otimes \mathcal{H}_{\bar{\mathcal{U}}}$, where $\bar{\mathcal{U}}$ is the complement of $\mathcal{U}$, so that $|\Omega\rangle$ is the "maximally entangled" cyclic state for the associated local operator algebras $\mathcal{A}_\mathcal{U}$, $\mathcal{A}_{\bar{\mathcal{U}}}$.

Sadly this is not to be. If this were achievable, then we could pick any states $\psi \in \mathcal{H}_\mathcal{U}$ and $\psi' \in \mathcal{H}_{\bar{\mathcal{U}}}$, to form the unentangled "separable" state $\psi \otimes \psi'$. This state would yield no entanglement between observables in $\mathcal{U}$ and $\bar{\mathcal{U}}$.

But this is not how quantum field theory works, in fact the reality is much worse. In field theory there exists an ultraviolet divergence in the entanglement entropy between any two adjacent regions of spacetime [2]. This UV divergence is universal in the sense that it exists regardless of the states considered: it is not a property of the states but of the algebra of observables. It is commutative with the fact that the Hilbert space can not be factored into subregions $\mathcal{H} = \mathcal{H}_\mathcal{U} \otimes \mathcal{H}_{\bar{\mathcal{U}}}$.

We contrast this with the case in finite dimensions of a factorable Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Our ability in this case to "trace out" the degrees of freedom in one of the factors allows us to define the the entanglement entropy of any state $\mathcal{S}(\rho) = \mathrm{tr}\rho_1 \log\rho_1$ where $\rho_1 = \mathrm{tr}_2\rho$. There is no such analog in continuum quantum field theory. However as we have seen we can still manage to define *relative entropy* in this case. This is a powerful enough tool to find many applications.

A caveat to the above is that: if our field theory is regularized (say on a lattice), then even if the Hilbert space does not factorize we can still form a meaningful definition of the entropy of a state [1]. However this definition will not have the usual interpretation of "tracing out" degrees of freedom.

We leave a brief discussion of the connection between the finite-dimensional and continuum case to Appendix C. The bridge between these cases is given in the form of treating the continuum Hilbert space as the limit of finite-dimensional matrix spaces in increasing dimension, so that the QFT Hilbert space has the interpretation of arising as infinitely many qubits entangled together. This is sufficient to show not only the UV divergence in the entanglement entropy, but also its universality amongst states.

APPENDIX A: THE REEH SCHLIEDER THEOREM AND IMPLICATIONS

We now give a brief statement and discussion of the Reeh-Schlieder theorem as presented in [2].

Suppose we define our Hilbert space $\mathcal{H}_0$ as the space generated by states of the form $\phi_{f_1}\phi_{f_2}\ldots\phi_{f_n}|\Omega\rangle$, where Ω is the vacuum and where $\phi_f = \int d^D x f(x)\phi(x)$ for some smooth function f. The reason for the subscript on $\mathcal{H}_0$ is because this is the so-called *vacuum sector* of our Hilbert space, and is typically only actually a subspace of the full Hilbert space $\mathcal{H}$ of our field theory that we are interested in. The other sectors are often defined by conserved quantities like charge in QED, and can not be reached by acting on $|\Omega\rangle$ with field operators which do not carry this conserved quantity. However we restrict our attention to $\mathcal{H}_0$ because the results we show can be applied to any other such sector.

Define some spacelike hypersurface Σ (the example to keep in mind is $\Sigma = \{(t,x) : t = 0\}$). Then choose any arbitrarily small open subset $\mathcal{V} \subset \Sigma$, and likewise choose any correspondingly small neighborhood $\mathcal{U}$ of $\mathcal{V}$ in the full Minkowski space.

The Reeh-Schlieder theorem states that if we restrict our functions $f_1,\ldots,f_n$ to be supported in $\mathcal{U}$, then the states of the form $|\Psi\rangle = \phi_{f_1}\ldots\phi_{f_n}|\Omega\rangle$ are sufficient to generate all of $\mathcal{H}_0$

At first this may trigger some deeply-ingrained alarms about causality: how can we effectively construct states containing complex structures, entire universes (in Witten's case, a moon) in a spacelike-separated region by acting on the vacuum only with *very* localized operators? The quick resolution is this: the Reeh-Schlieder theorem does not say that there exists *unitary* operators which acts on the vacuum to yield these states, just that *some* operator exists.

The realization that most such states are nonphysical, only mathematical elements of the Hilbert space, and not realizable by physical time evolution should be a comfort. The only way to perform any physical change to the system is by coupling our Hamiltonian and acting with unitary operators.

Therefore, it is still not possible for physical operators in one region to affect a measurement in a spacelike separated region. However, there can still be *correlations* between operators in these two regions. This is the real meat of what the Reeh-Schlider Theorem shows: that the state $|\Omega\rangle$ is cyclic for the algebra of observables $\mathcal{A}_\mathcal{U}$ of any open spacetime region $\mathcal{U}$, and therefore there is an extreme level of entanglement amongst observables in all regions of spacetime.

The same result applies in the other superselection sectors of $\mathcal{H}$, except it is not the vacuum acting as the cyclic state but some other guaranteed vector in the space.

APPENDIX B: QUANTUM RELATIVE ENTROPY AND MUTUAL INFORMATION

We now give a rough and informal review of some of the key concepts of quantum information such as von Neumann entropy and mutual information.

Shannon defined that, classically, the information content of an outcome X which occurs with probability $p(X)$ is given by $-\log p(X)$. In this way, if $p(X) = 1$ then no information is gained, and as $p(X) \to 0$ the amount of information we gain from such an outcome diverges.

Then the average information gained from a random variable X which takes values in $\mathbb{D}$, with probability distribution $p : \mathbb{D} \to [0,1]$ is:

$$\mathbb{E}(-\log p(X)) = -\sum_{x\in\mathbb{D}} p(x)\log p(x)$$

This is the *entropy* of the random variable.

If we wish for a way to measure the "distinguishability" of some proposed distribution $Q = \{q_1,\ldots,q_n\}$ from the true distribution of a system, $P = \{p_1,\ldots,p_n\}$, then we can use the *relative entropy*:

$$D(P|Q) = \sum_i \Big(p_i \log p_i - p_i \log q_i\Big)$$

All of these concepts have analogs in quantum mechanics. If we are handed some ensemble of states represented by a density matrix ρ, then the analog of the classical entropy of a distribution is the *von Neumann entropy* of ρ:

$$\mathcal{S}(\rho) = -\mathrm{tr}\rho\log\rho$$

Likewise, the quantum *relative entropy* of two states ρ,σ measures their distinguishability:

$$\mathcal{S}(\rho|\sigma) = \mathrm{tr}\rho(\log\rho - \log\sigma)$$

Relative entropy can be used to determine "how entangled" a state ρ is if we are working on a factorized Hilbert space, by measuring its optimal relative entropy to unentangled "separable" states. This is defined as the *relative entropy of entanglement*:

$$D(\rho) = \min_{\sigma} \mathcal{S}(\rho|\sigma)$$

where the minimum is taken over all separable states σ on our factorized Hilbert space.

Finally, if we are working with a single state ρ on a factorized Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, then we can ask about the degree of entanglement of ρ between the two subsystems. This is achieved first by tracing out the state in each of the subspaces: $\rho_1 = \mathrm{tr}_2\rho$, $\rho_2 = \mathrm{tr}_1\rho$. Then we define the *mutual information* as a measure of the correlation between the subspaces in this state:

$$I(\rho_1, \rho_2) = \mathcal{S}(\rho|\rho_1 \otimes \rho_2)$$

Roughly, this quantifies the amount of information learned about one subsystem by observing the other.

By constructing a concept of relative entropy in quantum field theory, as well as in finite-dimensional systems, all of these quantities are available to us.

APPENDIX C: QFT ALGEBRAS BUILT FROM FINITE-DIMENSIONAL SYSTEMS

We give a rough summary of the characterization of local algebras $\mathcal{A}_{\mathcal{U}}$ of QFT operators on spacetime regions, as presented in [2], and observe how a universal UV divergence is built directly into the resulting algebra itself. The presentation here lacks nearly all of the mathematical details and we refer the reader to the original paper to fill these in.

We say that a von Neumann algebra $\mathcal{A}$ of type I can act irreducibly on a Hilbert space $\mathcal{H}$ by bounded operators. In particular, if $\mathcal{H}$ has dimension n then the operators are clearly bounded and the algebra is said to be of type I_n. These algebras act on n-dim vector spaces V.

If we now take many copies of algebras of type I_2 and call them M_2, and pick out states:

$$K_{2,\lambda} = \frac{1}{\sqrt{1+\lambda}} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$$

for $\lambda \in (0,1)$, then these states represent entangled qubit systems which are not maximally entangled.

Given a sequence $\lambda_1, \lambda_2, \ldots$ then we can construct the space of vectors:

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes \cdots \in V^1 \otimes V^2 \otimes \ldots$$

such that $v_i = K_{2,\lambda_i}$ for all but finitely many i. This forms a countably-infinite-dimensional vector space. To see the specific details of how this is constructed into a Hilbert space is in [2]. We then take the elements of our algebra of operators to be the states:

$$a_1 \otimes a_2 \otimes \cdots \in M_2^{(1)} \otimes M_2^{(2)} \otimes \ldots$$

where $a_i = K_{2,\lambda_i}$ for all but finitely-many i. To fully make this an algebra we must also take it's closure under limits of bounded operators acting on the vector space, which is another pesky detail we refer to [2].

If the sequence $\lambda_1, \lambda_2, \ldots$ converges to some $\lambda \in (0,1)$ then the algebra $\mathcal{A}$ constructed from these operators is said to be of type III_λ. The state:

$$\Psi_\lambda = K_{2,\lambda_1} \otimes K_{2,\lambda_2} \otimes \ldots$$

is cyclic for this algebra and its complement.

The local operator algebra $\mathcal{A}_{\mathcal{U}}$ for a spacetime region $\mathcal{U}$ in quantum field theory is of type III_1 [2], so the degrees of freedom in $\mathcal{U}$ can be thought of as arising from an infinite number of entangled qubits whose levels of entanglement increase in the limit.

One can then clearly see how the entanglement entropy in the cyclic state Ψ between $\mathcal{A}_{\mathcal{U}}$ and its complement is divergent: the level of entanglement for each qubit λ_i converges to 1, a perfectly entangled system, in the limit.

Moreover, using a different cyclic operator other than Ψ will change the entanglement entropy by at most a less-divergent amount, due to the fact that the entanglement of the qubits must converge to this value in any case. Therefore the divergence of the entanglement entropy is universal amongst states as claimed.

[1] Jennifer Lin and ore Radičević. Comments on defining entanglement entropy. *Nucl. Phys. B*, 958:115118, 2020. 3

[2] Edward Witten. Aps medal for exceptional achievement in research: Invited article on entanglement properties of quantum field theory. *Rev. Mod. Phys.*, 90:045003, Oct 2018. 1, 2, 3, 4