University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2023
Assignment 4

Due 11:00am Wednesday, May 3, 2023

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1. **Commutation relations of creation operators for general one-particle states.** Show that

$$\mathbf{a}(\varphi_1)\mathbf{a}^\dagger(\varphi_2) - \zeta \mathbf{a}^\dagger(\varphi_2)\mathbf{a}(\varphi_1) = \langle \varphi_2 | \varphi_1 \rangle ,$$

where these objects are as defined in the lecture notes.

2. **Fermion creation and annihilation algebra.**

Consider a single fermion mode $\mathbf{c}$. We showed in lecture that the associated Hilbert space is two-dimensional, and is spanned by

$$|0\rangle , \quad \text{with } \mathbf{c}\,|0\rangle = 0 \ \text{ and } \quad |1\rangle = \mathbf{c}^\dagger\,|0\rangle .$$

(a) Check that the two states are orthogonal.

(b) Show that acting on this Hilbert space it is indeed true that

$$\mathbf{c}^\dagger\mathbf{c} + \mathbf{c}\mathbf{c}^\dagger = \mathbb{1},$$

as long as $\langle 1|1\rangle = \langle 0|0\rangle$.

(c) Check that

$$[\mathbf{N}, \mathbf{c}] = -\mathbf{c}, \quad [\mathbf{N}, \mathbf{c}^\dagger] = \mathbf{c}^\dagger$$

where $\mathbf{N} = \mathbf{c}^\dagger\mathbf{c}$ is the number operator. Notice that this is the same algebra satisfied by bosonic modes.

3. **Majorana modes.** Given a collection of fermionic operators $\mathbf{c}_A$, satisfying the fermionic creation-annihilation algebra

$$\{\mathbf{c}_A, \mathbf{c}_B^\dagger\} = \delta_{AB}\mathbb{1} \qquad \text{and} \qquad \{\mathbf{c}_A, \mathbf{c}_B\} = 0,$$

we can decompose them into their real and imaginary parts

$$\gamma_{A1} \equiv \frac{1}{2}\left(\mathbf{c}_A + \mathbf{c}_A^\dagger\right), \quad \gamma_{A2} \equiv \frac{1}{2\mathbf{i}}\left(\mathbf{c}_A - \mathbf{c}_A^\dagger\right).$$

These are called *Majorana modes.*

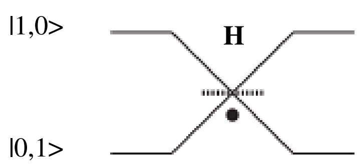
(a) Show that the Majorana modes satisfy the algebra

$$\{\gamma_a, \gamma_b\} = 2\Upsilon \delta_{ab} \mathbb{1},$$

where here a is a multi-index running over both A and $\alpha = 1, 2$. In particular, notice that $\gamma_a^2 = \Upsilon \mathbb{1}$. Find the constant Υ.

(b) Write the number operator $\mathbf{c}_A^\dagger \mathbf{c}_A$ in terms of the Majorana modes. Show that it is hermitian.

4. **Multiple photons on paths of an interferometer.**



One way to make a qubit is out of the two states of a photon moving on the upper and lower paths of an interferometer. On such a qbit, a half-silvered mirror $\mathbf{H}$ acts as a unitary gate, as indicated at left. (The dot below the mirror specifies a sign convention, to be explained below.)

On the other hand, photons are bosons. This means that if

$$\mathbf{a}^\dagger |0,0\rangle \equiv |1,0\rangle \text{ is a state with one photon on the upper path}$$

of the interferometer, then

$$\frac{(\mathbf{a}^\dagger)^n}{\sqrt{n!}} |0,0\rangle \equiv |n,0\rangle \text{ is a state with } n \text{ photons on the upper path.}$$

Similarly, define

$$\frac{(\mathbf{b}^\dagger)^n}{\sqrt{n!}} |0,0\rangle \equiv |0,n\rangle \text{ to be a state with } n \text{ photons on the lower path}$$

of the interferometer. (Note that $[\mathbf{a}, \mathbf{b}] = 0 = [\mathbf{a}, \mathbf{b}^\dagger]$ – they are independent modes.)

Now suppose we direct these two paths through a half-silvered mirror, as in the figure. A half-silvered mirror acts as a Hadamard gate

$$\mathbf{H} \equiv \frac{1}{\sqrt{2}} (\boldsymbol{\sigma}^x + \boldsymbol{\sigma}^z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

on the qubit made from the one-photon states. (The dot tells us where to put the negative entry.)

Some warm-up questions:

(a) What is the state $|0,0\rangle$? How does $\mathbf{H}$ act on $|0,0\rangle$?

(b) How does $\mathbf{H}$ act on $|2,0\rangle$ and $|0,2\rangle$?

(c) How does $\mathbf{H}$ act on the operators $\mathbf{a}^\dagger$ and $\mathbf{b}^\dagger$ (in order that the above relations are realized)?

Here's a more interesting question:

(d) A coherent state is a good cartoon of the state of light in a laser beam. What is the state which results upon sending a coherent state of photons

$$|\alpha,\beta) \equiv \mathcal{N}_\alpha \mathcal{N}_\beta \; e^{\alpha \mathbf{a}^\dagger + \beta \mathbf{b}^\dagger} \, |0,0\rangle$$

through a half-silvered mirror? ($\mathcal{N}_\alpha \equiv e^{-|\alpha|^2/2}$ is a normalization constant.) [Hint: it may be useful to insert $\mathbb{1} = \mathbf{H}^2$ in between the $e^{\alpha \mathbf{a}^\dagger + \beta \mathbf{b}^\dagger}$ and the $|0,0\rangle$.]

The half-silvered mirror is a special case of the more general notion called a beam-splitter. Suppose instead that the action on the mode operators were[1]

$$\begin{aligned} \mathbf{U}^\dagger(\theta)\mathbf{a}\mathbf{U}(\theta) &= \mathbf{a}\cos\theta + \mathbf{i}\mathbf{b}\sin\theta \\ \mathbf{U}^\dagger(\theta)\mathbf{b}\mathbf{U}(\theta) &= \mathbf{b}\cos\theta + \mathbf{i}\mathbf{a}\sin\theta \ . \end{aligned} \tag{1}$$

(e) Show that $\mathbf{U}(\theta)$ can be written as an evolution operator, in the form:

$$\mathbf{U}(\theta) = e^{\mathbf{i}\theta G}, \quad G = \mathbf{a}^\dagger \mathbf{b} + \mathbf{b}^\dagger \mathbf{a}. \tag{2}$$

(f) Show that when $\theta = \pi/4$ this beam-splitter takes the state $|1,1\rangle$ with one boson in each mode to the state

$$\frac{1}{\sqrt{2}} \left(|2,0\rangle + |0,2\rangle \right) .$$

(g) What if the operators $\mathbf{a}$ and $\mathbf{b}$ were instead fermionic operators? That is, suppose we send fermionic particles through the same beam-splitter, defined by (1). What is

$$\mathbf{U}_F(\theta = \pi/4)^\dagger \, |1,1\rangle$$

in this case? Hint: the form of the generator is different

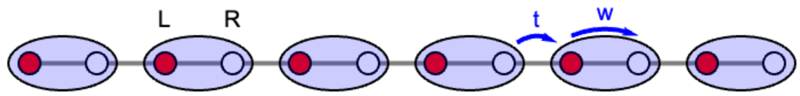
$$\mathbf{U}_F(\theta) = e^{\mathbf{i}\theta G_F}, \quad G_F = \mathbf{a}^\dagger \mathbf{b} - \mathbf{a}\mathbf{b}^\dagger.$$

(Notice that G_F is still hermitian.)

[1] The operation H in the previous parts is not $\mathbf{U}(\theta)$ for some θ; it is similar. I apologize for any confusion this caused. To get H we would have to write $\mathbf{U}'(\theta) = e^{\mathbf{i}\theta G'}$, with $G' \equiv \mathbf{i}\mathbf{a}^\dagger \mathbf{b} - \mathbf{i}\mathbf{b}^\dagger \mathbf{a}$, and set $\theta = \pi/2$.

5. **Slightly more interesting bandstructure.**

Consider a particle hopping on a chain of sites where each site involves two orbitals, one on the left and one on the right.



So the single-particle hamiltonian is

$$\mathbf{H} = \sum_n t\left(|n, R\rangle\langle n+1, L| + |n+1, L\rangle\langle n, R|\right) + \left(w|n, L\rangle\langle n, R| + h.c.\right), \quad (3)$$

where w, t are two quantities with dimensions of energy.

(a) Write down the many-body hamiltonian in terms of annihilation $\mathbf{c}_{n,\alpha}$ and creation $\mathbf{c}^\dagger_{n,\alpha}$ operators.

(b) Suppose there are N sites and N fermions and suppose $w > t$ (for simplicity take w real). Is it a metal or an insulator? Find the energy difference between the groundstate and the first excited state in the thermodynamic ($N \to \infty$) limit.

(c) What happens when $w = t$?

6. **Normalization.** Check that if $\Psi(r_1 \cdots r_n)$ is a normalized and (anti)symmetric wavefunction on n particles, then

$$|\Psi\rangle \equiv \sum_{r_1 \cdots r_n} \Psi(r_1 \cdots r_n)\, |r_1 \cdots r_n\rangle \quad (4)$$

is normalized, $\langle\Psi|\Psi\rangle = 1$.

(Interpret the sum over r as an integral if you like.)