University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2023
Assignment 5 – Solutions

Due 11:00am Wednesday, May 10, 2023

---

1. **Heat capacity of empty space.** What is the heat capacity of empty space?

   For the purposes of this problem you may ignore all degrees of freedom besides the electromagnetic field. We are particularly interested in the dependence on temperature.

   Compare your calculation to HW02 problem 3d.

2. **Further evidence for the clumping tendencies of bosons.**

   Consider again the model of a 1d crystalline solid that we discussed in class: It consists of N point masses, coupled to their neighbors:

$$\mathbf{H}_0 = \sum_{n=1}^{N} \left(\frac{\mathbf{p}^2}{2m} + \frac{1}{2}\kappa \left(\mathbf{q}_n - \mathbf{q}_{n-1}\right)^2 \right) = \sum_{\{k\}} \hbar\omega_k \left(\mathbf{a}_k^\dagger \mathbf{a}_k + \frac{1}{2} \right) \quad . \tag{1}$$

   Assume periodic boundary conditions $\mathbf{q}_n = \mathbf{q}_{n+N}$, so that the allowed wavenumbers are

$$\{k\} \equiv \{k_j = \frac{2\pi}{Na} j, \quad j = 1, 2...N\} \ .$$

   Consider a state with two phonons defined by

$$|k_1, k_2\rangle \equiv \mathbf{a}_{k_1}^\dagger \mathbf{a}_{k_2}^\dagger |0\rangle \ .$$

   (a) In the state $|k_1, k_2\rangle$, what is the probability of finding two phonons at the location x_1?

   Do this problem both using a first-quantized point of view and using the algebra of creation and annihilation operators. Make sure your answers agree!

   [Warning: the statement of this problem is deceptively simple.]

   (b) Make sure your probabilities add up to one.

   (c) Compare your result to the answer that would obtain if the particles were distinguishable (and occupied the same two single-particle states). Do bosons clump?

   (d) Does the story change if $k_1 = k_2$?

(e) Bonus problem: for two fermions in the state $|k_1, k_2\rangle$, what is the probability of finding one at x_1 and one at x_2? Check that your probabilities add to one. (It is possible to do this part in parallel with the others.)

If $k_1 = k_2$, then the state above is not normalized! Recall that the state of an SHO with occupation number 2 is

$$|2\rangle = \frac{\left(\mathbf{a}^\dagger\right)^2}{\sqrt{2!}} |0\rangle .$$

We need the state to be normalized so that the total probability for anything happening is

$$1 = \langle k_1 k_2 | k_1 k_2 \rangle = \sum_\alpha \langle k_1 k_2 | \alpha \rangle \langle \alpha | k_1 k_2 \rangle = \sum_\alpha P_{k_1 k_2}(\alpha)$$

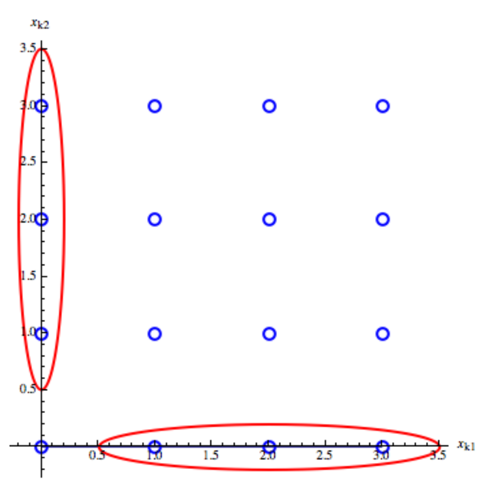
as long as $\mathbb{1} = \sum_\alpha |\alpha\rangle\langle\alpha|$ is a resolution of the identity.

Let's think about the distinguishable case first.

In a state with two particles, there are three possibilities for the number of particles at a particular location x_1: 0, 1, 2. In a plane-wave, the probability $P_k(x) = |\langle k|x\rangle|^2 = \frac{1}{N}$ is uniform in space, so we just have to count configurations. The probability of no phonons is the probability that they are both elsewhere: $p_0 = (1 - \frac{1}{N})^2$. The probability of two is $p_2 = \frac{1}{N^2}$. The probability of exactly one is $p_1 = 2\frac{(N-1)}{N^2} = \frac{2}{N} - \frac{2}{N^2}$. Note that

$$p_0 + p_1 + p_2 = (1 - \frac{2}{N} + \frac{1}{N^2}) + \frac{2}{N} - \frac{2}{N^2} + \frac{1}{N^2} = 1 \quad .$$

This counting is illustrated here for $N = 4$:



the two axes are the positions of the two (distinguishable) particles, the states with exactly one phonon at $x = x_1$ are circled.

For indistinguishable particles (bosons or fermions), with the normalization convention we have been using, the resolution of the identity on the 2-particle Hilbert space in the position basis is

$$\mathbb{1}_2 = \sum_{x_1 x_2} |x_1 x_2\rangle\langle x_1 x_2|$$

where the sum is over all x_1, x_2 and

$$|x_1 x_2\rangle \equiv \frac{1}{\sqrt{2!}} \mathbf{a}^\dagger_{x_1} \mathbf{a}^\dagger_{x_2} |0\rangle$$

no matter whether x_1 and x_2 are equal. Notice that the state $|x_1 \neq x_2\rangle$ is *not* normalized $\langle x_1 \neq x_2 | x_1 \neq x_2\rangle = \frac{1}{2}$. (!)

A good way to think about it is that $|x_1, x_2\rangle$ and $|x_2, x_1\rangle$ aren't different states, so they should only count once in the resolution of the identity, so if we normalize them to one and sum over *all* x_1, x_2, we overcount.

For $k_1 \neq k_2$, then,

$$\begin{aligned}
P(x_1, x_2) = |\langle x_1 x_2 | k_1 k_2\rangle|^2 &= \left| \langle 0 | \frac{\mathbf{a}^2_{x_1}}{\sqrt{2}} \mathbf{a}^\dagger_{k_1} \mathbf{a}^\dagger_{k_2} |0\rangle \right|^2 && (2) \\
&= \left| \frac{1}{\sqrt{2}N} \sum_{p_1 p_2} e^{\mathbf{i}p_1 x_1 + \mathbf{i}p_2 x_2} \langle 0 | \mathbf{a}_{p_1} \mathbf{a}_{p_2} \mathbf{a}^\dagger_{k_1} \mathbf{a}^\dagger_{k_2} |0\rangle \right|^2 && (3) \\
&= \frac{1}{2N} \left| \sum_{p_1 p_2} e^{\mathbf{i}p_1 x_1 + \mathbf{i}p_2 x_2} \left(\delta_{p_1 k_1} \delta_{p_2 k_2} \pm \delta_{p_2 k_1} \delta_{p_1 k_2} \right) \right|^2 && (4) \\
&= \frac{1}{2N} \left| \left(e^{\mathbf{i}k_1 x_1 + \mathbf{i}k_2 x_2} \pm e^{\mathbf{i}k_2 x_1 + \mathbf{i}k_1 x_2} \right) \right|^2 && (5) \\
&= \frac{1}{N} \left(1 \pm \cos(k_1 - k_2)(x_1 - x_2) \right).
\end{aligned}$$

For $x_1 = x_2$, this reduces to

$$P(x_1 = x_2) = \frac{1 \pm 1}{N^2}$$

which naturally is zero for fermions and $\frac{2}{N^2}$ for bosons.

So for identical bosons,

$$p_2 = \frac{2}{N^2},$$

which is bigger by a factor of two than it would be for distinguishable particles: bosons clump together!

To check the normalization, note that for $\Delta k = k_1 - k_2 = \frac{2\pi j}{N}$ with $j = 1..N-1$, we have

$$\sum_{x_1 x_2} P(x_1, x_2) = \sum_{x_1 = x_2} \frac{1 \pm 1}{N^2} + \frac{1}{N^2} \sum_{x_1 \neq x_2} \left(1 \pm \cos \Delta k (x_1 - x_2)\right) \tag{6}$$

$$= \frac{1 \pm 1}{N} + \frac{N - (1 \pm 1)}{N} = 1. \tag{7}$$

In the case where $k_1 = k_2$, this effect goes away, since

$$P(x_1 = x_2) = \left| \langle 0 | \frac{\mathbf{a}_{x_1}^2}{\sqrt{2}} \frac{\left(\mathbf{a}_k^\dagger\right)^2}{\sqrt{2}} |0\rangle \right|^2 = \left| \frac{1}{2N} \sum_{p_1 p_2} e^{\mathbf{i} p_1 x_1 + \mathbf{i} p_2 x_1} \underbrace{\langle 0 | \mathbf{a}_{p_1} \mathbf{a}_{p_2} \left(\mathbf{a}_k^\dagger\right)^2 |0\rangle}_{=2\delta_{p_1 k} \delta_{p_2 k}} \right|^2 = \frac{1}{N^2}.$$

$$P(x_1 \neq x_2) = \left| \langle 0 | \frac{\mathbf{a}_{x_1} \mathbf{a}_{x_2}}{\sqrt{2}} \frac{\left(\mathbf{a}_k^\dagger\right)^2}{\sqrt{2}} |0\rangle \right|^2 = \left| \frac{1}{2N} \sum_{p_1 p_2} e^{\mathbf{i} p_1 x_1 + \mathbf{i} p_2 x_2} \underbrace{\langle 0 | \mathbf{a}_{p_1} \mathbf{a}_{p_2} \left(\mathbf{a}_k^\dagger\right)^2 |0\rangle}_{=2\delta_{p_1 k} \delta_{p_2 k}} \right|^2 = \frac{1}{N^2}.$$

Notice that

$$\sum_{x_1, x_2} P_{k_1 = k_2}(x_1, x_2) = \sum_{x_1, x_2} \frac{1}{N^2} = 1.$$

From a first-quantized point of view, the state is

$$|k_1, k_2\rangle = \frac{|k_1\rangle \otimes |k_2\rangle \pm |k_2\rangle \otimes |k_1\rangle}{\sqrt{2}}$$

and the position eigenstates are

$$|x_1, x_2\rangle = \frac{|x_1\rangle \otimes |x_2\rangle \pm |x_2\rangle \otimes |x_1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}N} \sum_{p_1 p_2} \left(e^{-\mathbf{i} p_1 x_1 - \mathbf{i} p_2 x_2} \pm e^{-\mathbf{i} p_2 x_1 - \mathbf{i} p_1 x_2}\right) |p_1\rangle \otimes |p_2\rangle .$$

The probability is then

$$P(x_1, x_2) = \left| \langle x_1, x_2 | k_1, k_2 \rangle \right|^2 \tag{8}$$

$$= \frac{1}{2N^2} \left| \sum_{p_1 p_2} \left(e^{\mathbf{i} p_1 x_1 + \mathbf{i} p_2 x_2} \pm e^{\mathbf{i} p_2 x_1 + \mathbf{i} p_1 x_2}\right) \delta_{p_1 k_1} \delta_{p_2 k_2} \right|^2 \tag{9}$$

$$= \frac{1}{2N^2} \left| \left(e^{\mathbf{i} k_1 x_1 + \mathbf{i} k_2 x_2} \pm e^{\mathbf{i} k_2 x_1 + \mathbf{i} k_1 x_2}\right) \right|^2 \tag{10}$$

$$= \frac{1}{N} \left(1 \pm \cos(k_1 - k_2)(x_1 - x_2)\right) \tag{11}$$

as before.

3. [Bonus problem] Describe the outcome of the intensity interferometry (Hanbury-Brown and Twiss) experiment for beams of fermions.

4. [Bonus problem] Consider free fermions with single-particle Hamiltonian

$$h = t \sum_n |n\rangle\langle n+1| + h.c. + \sum_n V_n |n\rangle\langle n|.$$

   (a) For the case without an external potential, $V_n = 0$, numerically evaluate the single-particle Green's function

   $$G(n, m) \equiv \langle \Phi_0 | \, \psi_n^\dagger \psi_m \, | \Phi_0 \rangle$$

   in the groundstate. Plot it as a function of the separation between the two points.

   (b) Now add a *random* potential V_n. Choose each V_n independently from a Gaussian distribution with width v. How does this affect the Green's function? What happens if you average $G(n, m)$ over v for fixed $n - m$.