University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2023
Assignment 6

Due 11:00am Wednesday, May 17, 2023

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1. **Brain-warmer.** Consider two single-particle orbitals with wavefunctions $\psi_\alpha(r), \psi_\beta(r)$. Find the position-space wavefunctions

$$\Psi(r_1\sigma, r_2\sigma') \equiv \langle r_1\sigma, r_2\sigma'|\Psi\rangle = \langle 0|\, \mathbf{a}_{r_2\sigma'}\mathbf{a}_{r_1\sigma}\, |\Psi\rangle$$

for the four states of two spinful fermions:

$$|S\rangle = \frac{1}{\sqrt{2}}\mathbf{a}^\dagger_{\alpha\sigma}\mathbf{a}^\dagger_{\beta\sigma'}\varepsilon_{\sigma\sigma'}\,|0\rangle = \frac{1}{\sqrt{2}}\left(\mathbf{a}^\dagger_{\alpha\uparrow}\mathbf{a}^\dagger_{\beta\downarrow} - \mathbf{a}^\dagger_{\alpha\downarrow}\mathbf{a}^\dagger_{\beta\uparrow}\right)|0\rangle$$

$$|A,1\rangle = \mathbf{a}^\dagger_{\alpha\uparrow}\mathbf{a}^\dagger_{\beta\uparrow}\,|0\rangle\,, \quad |A,0\rangle = \frac{1}{\sqrt{2}}\left(\mathbf{a}^\dagger_{\alpha\uparrow}\mathbf{a}^\dagger_{\beta\downarrow} + \mathbf{a}^\dagger_{\alpha\downarrow}\mathbf{a}^\dagger_{\beta\uparrow}\right)|0\rangle\,, \quad |A,-1\rangle = \mathbf{a}^\dagger_{\alpha\downarrow}\mathbf{a}^\dagger_{\beta\downarrow}\,|0\rangle\,.$$

Check that the singlet (triplet) indeed has a symmetric (antisymmetric) orbital wavefunction. Check that the labels on the A states correctly label the eigenvalues of the total spin along z, $S^z \equiv \sum_r \frac{1}{2}\mathbf{a}^\dagger_r\sigma^z\mathbf{a}_r$.

2. **Brain-warmer: a beam of particles.** Suppose the occupation numbers for a state of bosons satisfy

$$n_{\vec{p}} = ce^{-\alpha(\vec{p}-\vec{p}_0)^2/2}.$$

   (a) Determine the prefactor $c = c(n, \alpha, p_0)$ so that the average density is

$$n = \int đ^3 p n_{\vec{p}}.$$

   (b) Check that with this normalization, in the thermodynamic limit of $N \to \infty$ at fixed $n = N/V$, the pair correlation function is

$$g(x-y) = 1 + e^{-(x-y)^2/\alpha}.$$

3. **Density matrix and correlation functions.** Consider the single-particle density matrix in a mixed state $\rho = \sum_s p_s|\Psi_s\rangle\langle\Psi_s|$ of N particles, defined as

$$\rho_1(r,r') \equiv \sum_s p_s \sum_{r_2..r_N} \Psi^\star_s(r, r_2, \cdots r_N)\Psi_s(r', r_2, \cdots r_N).$$

This can be defined for either bosons or fermions.

(a) Check that ρ_1 is proportional to the two-point correlation function

$$\rho_1(r, r') \propto \mathrm{tr}\rho\psi^\dagger(r)\psi(r') \equiv \left\langle \psi^\dagger(r)\psi(r') \right\rangle$$

and find the proportionality constant. Check that it works for both bosons and fermions.

(b) [Bonus question] Prove that for a fermionic state the eigenvalues of $\rho_1(r, r')$ are between 0 and 1.

[Hint: for fermions, the expectation value of the number operator $\psi^\dagger(r)\psi(r)$ in any state is ≤ 1.]

4. **A charged particle, classically.** [If you did this problem earlier this quarter, please submit your solution again for bookkeeping purposes.] This problem is an exercise in calculus of variations, as well as an important ingredient in our discussion of particles in electromagnetic fields.

Consider the following action functional for a particle in three dimensions:

$$S[x] = \int dt \, \left(\frac{m}{2}\dot{\vec{x}}^2 - e\Phi(\vec{x}) + \frac{e}{c}\dot{\vec{x}} \cdot \vec{A}(x) \right) \ .$$

(a) Show that the extremization of this functional gives the equation of motion:

$$\frac{\delta S[x]}{\delta x^i(t)} = -m\ddot{x}^i(t) - e\partial_{x^i}\Phi(x(t)) + \frac{e}{c}\dot{x}^j F_{ij}(x(t))$$

where $F_{ij} \equiv \partial_{x^i}A_j - \partial_{x^j}A_i$. Show that this is the same as the usual Coulomb-Lorentz force law

$$\vec{F} = e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

with $B_i \equiv \frac{1}{2}\epsilon_{ijk}F_{jk}$.

(b) Show that the canonical momenta are

$$\Pi_i \equiv \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i + \frac{e}{c}A_i(x).$$

Here $S = \int dtL$. (I call them Π rather than p to emphasize the difference from the 'mechanical momentum' $m\dot{x}$.) Show that the resulting Hamiltonian is

$$H \equiv \sum_i \dot{x}^i\Pi^i - L = \frac{1}{2m}\left(\Pi_i - \frac{e}{c}A_i(x(t))\right)^2 + e\Phi.$$