University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2023
Assignment 7

Due 11:00am Wednesday, May 24, 2023

---

1. **Probability current for a charged particle.**

   Check that probability is still conserved $\partial_t \rho + \vec{\nabla} \cdot \vec{j}_A = 0$ for a charged particle, with $\rho(x,t) = |\psi(x,t)|^2$ and

   $$\vec{j}_A \equiv \frac{\hbar}{2m\mathbf{i}} \left(\psi^\star \vec{\nabla} \psi - \psi \vec{\nabla} \psi^\star \right) - \frac{e}{mc} \psi^\star \psi \vec{A}.$$

   Check that $\vec{j}_A$ is gauge invariant if the gauge transformation acts on the wavefunction by

   $$\vec{A} \to \vec{A} + \vec{\nabla}\lambda, \quad \psi \to e^{\pm \mathbf{i} \frac{e\lambda}{\hbar c}} \psi$$

   for one choice of the sign in the exponent.

   (As a check of the sign, you can check that the Schrödinger equation maps to itself under the transformation.)

2. **A charged particle in a uniform magnetic field, quantumly.** Consider an electron constrained to move in the xy plane under the influence of a uniform magnetic field of magnitude B oriented in the $+\hat{z}$ direction. The Hamiltonian for this electron is

   $$\mathbf{H} = \frac{1}{2m} \left(\left(\mathbf{p}_x - \frac{e}{c} A_x \right)^2 + \left(\mathbf{p}_y - \frac{e}{c} A_y \right)^2 \right)$$

   where m and e are the mass and charge of the electron, and c is the speed of light.

   (a) Show that a classical particle in this potential will move in circles at an angular frequency $\omega_0 = \frac{eB}{mc}$ where m is the mass.

   (b) Find a suitable expression for $\vec{A}$ so that $\mathbf{p}_x$ is a constant of motion for the above Hamiltonian.

   (c) With this choice for $\vec{A}$, show that the eigenfunctions of $\mathbf{H}$ can be written in the form

   $$\Psi(x,y) = e^{\frac{\mathbf{i}}{\hbar} p_x x} \Phi(y)$$

   where $\Phi(y)$ satisfies the Schrödinger equation for a one-dimensional harmonic oscillator whose equilibrium position is $y = y_0$. Find the effective spring constant k for this oscillator and the shift of the origin y_0 in terms of p_x, B, m, e, c.

(d) Find the energy eigenvalues for this system, and indicate degeneracies.

(e) For the remainder of the problem, suppose we further restrict the particles to live in a square of side length L. Suppose we demand periodic boundary conditions. What are the possible values of p_x?

(f) In the estimates below, use the wavefunctions found above, despite the fact that they will not satisfy the boundary conditions at $y = 0, L$. Place several electrons in the strip, and assume they only care about each other because of the Pauli exclusion principle.

Estimate the maximum number of electrons that fit in the lowest Landau level. That is: how many electrons can you add before you must put one in the first excited state of the harmonic oscillator encountered above?

3. **Landau Levels in an Electric Field.** [optional since I added it late]

In lecture I gave several arguments that a quantum Hall droplet has a linearly-dispersing edge mode. Here is a fully quantum mechanical argument. We're going to think about the physics in a neighborhood of the boundary of the sample, where the confining potential $V \simeq -Ex$ is slowly varying, and describes an electric field $E = -\partial_x V$.

The Hamiltonian in the Landau gauge (the one used in the previous problem) is

$$H = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right) - eEx. \qquad (1)$$

(a) Using the same ansatz as above, write the Hamiltonian as that of a displaced harmonic oscillator.

(b) Conclude that the eigenstates have the form

$$\psi(x, y) = \psi_{n,k}\left(x - \frac{mE}{eB^2}, y\right) \qquad (2)$$

with energies

$$E_{n,k} = \hbar\omega_c\left(n + \frac{1}{2}\right) + eE\left(k\ell_B - \frac{eE}{m\omega_c^2}\right) + \frac{m}{2}\frac{E^2}{B^2}. \qquad (3)$$

(c) Plot this spectrum, and interpret $\partial_k E_{n,k}$ as a velocity in the y direction. Compare this drift velocity with the classical behavior of a charged particle in crossed E and B fields.

4. **Topological terms, particle on a ring.**

The purpose of this problem is to demonstrate that total derivative terms in the action do affect the physics quantum mechanically.

The euclidean path integral for a particle on a ring with magnetic flux $\theta = \int \vec{B}\cdot \mathrm{d}\vec{a}$ through the ring is given by

$$Z = \int [D\phi] e^{-\int_0^\beta \mathrm{d}\tau \left(\frac{m}{2}\dot{\phi}^2 - \mathbf{i}\frac{\theta}{2\pi}\dot{\phi}\right)} \, .$$

Here

$$\phi \equiv \phi + 2\pi \tag{4}$$

is a coordinate on the ring. Because of the identification (4), ϕ need not be a single-valued function of τ – it can wind around the ring. On the other hand, $\dot{\phi}$ is single-valued and periodic and hence has an ordinary Fourier decomposition. This means that we can expand the field as

$$\phi(\tau) = \frac{2\pi}{\beta} Q\tau + \sum_{\ell \in \mathbb{Z}\backslash 0} \phi_\ell e^{\mathbf{i}\frac{2\pi}{\beta}\ell\tau}. \tag{5}$$

(a) Show that the $\dot{\phi}$ term in the action does not affect the classical equations of motion. In this sense, it is a topological term.

(b) Using the decomposition (5), write the partition function as a sum over topological sectors labelled by the *winding number* $Q \in \mathbb{Z}$ and calculate it explicitly.

[Hint: use the Poisson resummation formula

$$\sum_n f(n) = \sum_l \hat{f}(2\pi l)$$

where $\hat{f}(p) = \int dx e^{-\mathbf{i}px} f(x)$ is the fourier transform of f.]

(c) Use the result from the previous part to determine the energy spectrum as a function of θ.

(d) Derive the canonical momentum and Hamiltonian from the action above and verify the spectrum.

5. **Aharonov-Casher effect** [Bonus problem]

Consider a neutral particle with spin-$\frac{1}{2}$ (such as a neutron), described by the Lagrangian

$$L = \frac{1}{2} m v^2 + \mu \vec{\sigma} \cdot \left(\vec{v} \times \vec{E}\right)$$

where $\vec{\sigma}$ is the spin operator and $\vec{v} \equiv \dot{\vec{x}}$.

(a) Find the canonical momentum and the Hamiltonian.

(b) Consider an experiment where a cylindrically-symmetric electric field $\vec{E}$ is produced by a line charge with charge-per-unit-length λ extended in the $\check{z}$ direction. Two beams of the particles described by L are sent in paths around the line charge and allowed to interfere. Show that the phase shift between the two waves arising from the line charge is

$$\delta_{\pm} = \pm\frac{4\pi\lambda\mu}{\hbar c}$$

for spin up/down in the $\check{z}$ basis.