

Physics 230: Quantum phases of matter

Spring 2024

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These lecture notes live [here](#). Please email corrections and questions to mcgreevy at physics dot ucsd dot edu.

Last updated: May 2, 2024, 14:42:17

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0.1 Introductory remarks and goals

The study of phases of matter is a topology problem. Let me explain. We wish to divide the set

{macroscopic piles of stuff, with some interactions}

into equivalence classes. The equivalence relation is roughly: two interacting piles of stuff are regarded as being in same phase if their observable properties are adiabatically connected under varying the interactions and adding in more non-interacting, gapped stuff. So phases of matter are essentially elements of $\pi_0(\text{piles of stuff})$.¹

A *topological invariant* is a quantity that does not change under such continuous variations, for example a quantity that is guaranteed to be an integer. Such invariants are wonderful because they provide labels on our equivalence classes. The simplest example of a topological invariant labelling a phase of matter is the (integer!) number of groundstates of an Ising magnet: it is 2 in the ordered phase, and 1 in the disordered phase; thus these two phases must be distinct. So you see that the use of topology in condensed matter physics is not just for ‘topological phases’.

Topological phases are those that are distinguished from others, say from the trivial state, by properties other than ordinary symmetry breaking. (A good representative of the trivial state is an *atomic insulator*, where each particle is 600 miles from its nearest neighbor and never even says ‘hello’. More generally, the trivial phase is one that has a product state representative that breaks no symmetries.) By now there is a large variety of known ways in which phases can be topological, some of which are pretty fancy mathematically. Some of them have even been found in Earth rocks. My main goal in this course will be to try to explain some of these phenomena, and the topologically-invariant labels we can attach to them, as concretely as possible.

To do this, it will occasionally be necessary to use some mathematics. A few years ago, I taught a [course](#) whose purpose was to develop some tools of algebraic topology, at least some such tools that are realized in toy models of physical systems. I will not assume this background. I am going to do my best to make our discussion here self-contained, while not making this a math course. At times I may have to ask you to do some extra background reading or to take some statements on faith. As a hopefully-useful resource, I’ve posted a [summary of the mathematical highlights](#); I don’t expect you to absorb every detail of this, but rather to use it as a resource as needed.

Likely, not everyone taking this class took the previous two quarters of the condensed matter series. If it seems like I am assuming some knowledge you don’t have, please do not hesitate to ask.

¹ π_q of this space for $q > 0$ is also interesting but [much less well-explored so far](#).

For a list of topics we might cover, see the table of contents of this document, or [this administrative handout](#). There is a lot one could say on this subject and we will have to make some selections. Your input is encouraged.

0.2 Conventions

For some of us, eyesight is a valuable and dwindling commodity. In order not to waste it, I will often denote the Pauli spin operators by

$$\mathbf{X} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{Y} \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \mathbf{Z} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(rather than $\sigma^{x,y,z}$, which hides the important information in the superscript) in the Z basis. I'll write $|0\rangle, |1\rangle$ for the Z eigenstates, $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$ and $|\pm\rangle$ for the states with $X|\pm\rangle = \pm|\pm\rangle$.

I use ijk for spatial indices, $\mu\nu\rho$ for spacetime indices. d is the number of space dimensions and $D = d + 1$ is the number of spacetime dimensions (it's bigger).

\equiv means 'equals by definition'. $A \stackrel{!}{=} B$ means we are demanding that $A = B$. $A \stackrel{?}{=} B$ means A probably doesn't equal B .

The convention that repeated indices are summed is always in effect unless otherwise indicated.

A useful generalization of the shorthand $\hbar \equiv \frac{\hbar}{2\pi}$ is

$$d^d k \equiv \frac{d^d k}{(2\pi)^d}.$$

I will also write $\delta^d(q) \equiv (2\pi)^d \delta^d(q)$.

I try to be consistent about writing Fourier transforms as

$$\int \frac{d^d k}{(2\pi)^d} e^{ikx} \tilde{f}(k) \equiv \int d^d k e^{ikx} \tilde{f}(k) \equiv f(x).$$

WLOG \equiv without loss of generality.

IFF \equiv if and only if.

RHS \equiv right-hand side. LHS \equiv left-hand side. BHS \equiv both-hand side.

IBP \equiv integration by parts.

$+\mathcal{O}(x^n)$ \equiv plus terms which go like x^n (and higher powers) when x is small.

iid \equiv independent and identically distributed.

We work in units where \hbar and k_B are equal to one unless otherwise noted.

Please tell me if you find typos or errors or violations of the rules above.

0.3 Anticipated Sources

This list will grow with the notes.

- N. Mermin, *The Topological Theory of Defects in Ordered Media*. The classic account of the use of homotopy theory to understand excitations of ordered media that, because of topology, cannot disappear.
- G. Volovik, *Exotic properties of superfluid ^3He* . A nice discussion of the Landau-Ginzburg theory of this remarkable system.
- R. Moessner, J. Moore, *Topological Phases of Matter*, Cambridge, 2021. I just got a copy of this book and it seems to have some nice things in it, with some overlap with our goals.
- S. Girvin, K. Yang, *Modern condensed matter physics*, Cambridge, 2019.
- M. Nakahara, *Geometry, Topology and Physics*. I was not a big fan of this book when I was a student because I thought it was superficial. Looking at it again now, I see its virtues more clearly. It has useful things in it and it is mostly written for physicists.
- Nash and Sen, *Geometry and Topology for Physicists*. This book has the virtue of brevity.
- D. Tong, *Lectures on the Quantum Hall Effect*.
- D. Arovas, [Lecture Notes on Quantum Hall Effect \(A Work in Progress\)](#)
- X.-G. Wen, *Quantum Field Theory of Many-Body Systems*, Oxford, 2004.
- X.-G. Wen, *Topological orders and Edge excitations in FQH states*.
- A. Zee, *Quantum Hall Fluids*.
- G. Moore, *Quantum Symmetries and Compatible Hamiltonians*.
- G. Moore, *Introduction to Chern-Simons Theories*.
- J. Harvey, *Lectures on Anomalies*.
- E. Witten, *Three Lectures On Topological Phases Of Matter*.
- E. Witten, *Fermion Path Integrals And Topological Phases*.

A. Turner and A. Vishwanath, *Beyond Band Insulators: Topology of Semi-metals and Interacting Phases*.

T. Senthil, *Symmetry Protected Topological phases of Quantum Matter*.

C. Z. Xiong, *Classification and Construction of Topological Phases of Quantum Matter*.

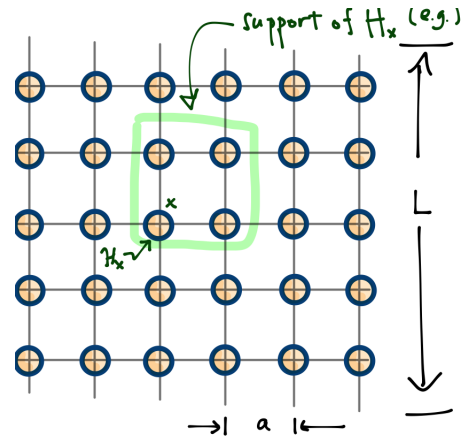
1 Introductory remarks about quantum matter

This is an introductory section on basic notions of quantum phases of matter, and how to think about them in terms of entanglement. For further reading, I suggest:

- Zeng, Chen, Zhou, Wen, *Quantum Information Meets Quantum Matter: From Quantum Entanglement to Topological Phase in Many-Body Systems*
- [These](#) TASI lectures.
- For more on applications of quantum information theory to many body physics from a similar viewpoint, see the lecture notes and problems posted [here](#).

1.1 States of matter, classified by level of desperation

In this class we are going to talk about *extensive quantum systems*. A quantum system can be specified by its Hilbert space and its Hamiltonian. By the adjective *extensive* I mean that the Hilbert space is defined by associating finite-dimensional Hilbert spaces \mathcal{H}_x to chunks of *space*, labelled by some coordinates x . Then couple them by a local Hamiltonian, $H = \sum_x H_x$, where H_x acts only on the patch at x and not-too-distant patches (and as the identity operator on the other tensor factors in \mathcal{H}).



The phenomena whose study we will find most fulfilling only happen in the *thermodynamic limit*, where the number of patches grows without bound. I will use L to

²We can allow the local Hilbert space to be infinite-dimensional (as for rotors or bosons) if we add terms to the Hamiltonian that leave a finite-dimensional set of low-energy states. In practice, when simulating such systems for example, we can always truncate the Hilbert space to some finite value.

denote the linear size of the system. For a cubic chunk of d -dimensional hypercubic lattice, there are $(\frac{L}{a})^d$ patches, where a is the size of the patches. So the thermodynamic limit is $L \rightarrow \infty$, or more precisely $L \gg a$. In the mysterious first sentence of this paragraph, I am referring to *emergent* phenomena: qualitatively new effects which can never be accomplished by small systems, such as spontaneous symmetry breaking (magnetism, superconductivity, the rigidity of solids), phase transitions, topological order, and all the other things we have not thought of yet because we are not very smart.^{3 4}

Perhaps the most basic question we can ask about such a system is: how many degrees of freedom are there at the lowest energies (lower than any interesting scale in the problem, in particular in the Hamiltonian)? By degrees of freedom (dofs) I mean excitations that can be created by a local operator, as in an experiment where we scatter particles (neutrons, photons...) off the material. There are essentially three possibilities:

1. **None.**
2. **Some.**
3. **A lot.**

As we proceed down this classification, our level of understanding rapidly decreases.

A more informative tour through that list goes like this. To get started let me make the assumption that the system has (at least discrete) translation invariance, so we can label the excitations by momentum.

1. **None:** Such a system has an energy gap ('is gapped'): the energy difference $\Delta = E_1 - E_0$ between the first excited state and the groundstate is nonzero, even in the thermodynamic limit. Note that Δ is almost always nonzero in finite volume. (Recall, for example, the spectrum of the electromagnetic field in a box

³In case you doubt that characterization, ask yourself this: How many of the items on this list were discovered theoretically before they were found to occur in Earth rocks by our friends who engage in experiments? The answer is **none**. Not one of them! Let us be humble. On the other hand: this is a source of hope for more interesting physics, in that the set of Earth rocks which have been studied carefully so far is likely to represent a very small sample of the possible emergent quantum systems.

⁴Can you think of other elements I should add to this list? One possibility (thanks to Ibou Bah for reminding me) can be called *gravitational order* – the emergence of dynamical space (or spacetime) (and hence gravity) from such ordinary-seeming quantum systems. The best-understood example of this is AdS/CFT, and was discovered using string theory. I was tempted to claim this as a victory for theorists, but then I remembered that we discovered gravity experimentally quite a while ago.

of linear size L : $E_n \sim \frac{n}{L}$.) The crucial thing here (in contrast to the case of photons) is that this energy stays finite even as $L \rightarrow \infty$.

The excitations of such a system are generally massive particles⁵.

Actually, it is useful to allow a finite number of states below the gap (which are split by something of order $e^{-\alpha L}$ and therefore rapidly become degenerate in the thermodynamic limit) in our definition of a gapped system.

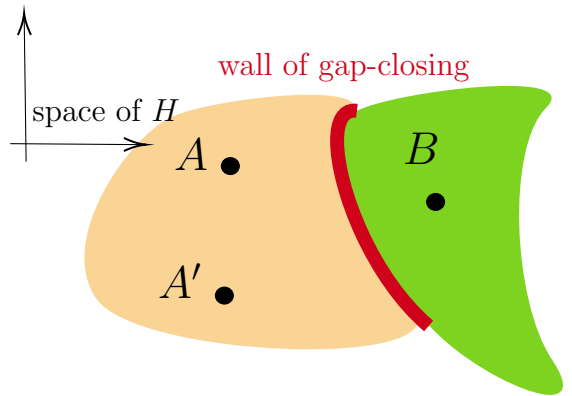
2. **Some:** An example of what I mean by ‘some’ is that the system can have excitations which are massless particles, like the photon.

The lowest energy degrees of freedom occur at isolated points in momentum space: The dispersion relation of the photon $\omega(k) = c\sqrt{\vec{k} \cdot \vec{k}}$ vanishes at $\vec{k} = 0$.

In this category I also put the gapless fluctuations at a critical point. It’s not necessarily true that $\omega \sim k^{\text{integer}}$ and those excitations are not necessarily *particles*. But they are still at $k = 0$ ⁶.

3. **A lot:** What I mean by this is things like Fermi surfaces, where there are excitations on a whole codimension-one locus in momentum space. This includes not just free fermions or adiabatic continuations of free fermions (Landau Fermi liquid theory). Such systems exist, for example in the half-filled Landau level and in the strange metal regime of cuprate superconductors.

Let’s reconsider the case of gapped systems and define the notion of a quantum phase. Different gapped states are in different phases if we can’t deform the Hamiltonian to get from one to the other without closing the gap. So a gapped phase is an equivalence class of Hamiltonians. In the figure at right, $[A] = [A'] \neq [B]$.



⁵Verstraete et al proves a version of this statement. I think it is worth looking for loopholes here.

⁶or some other isolated points in momentum space.

⁷⁸You might be bothered by the following: it is hard to imagine checking that there is no way around the wall of gaplessness. It is therefore important to find sharp characterizations of such states, like integer labels, which cannot change smoothly. This is the very definition of topology. An important goal in condensed matter physics is to figure out labels that can be put on states which can distinguish them in this way as distinct phases of matter.

Here are two classes of examples (even in the absence of symmetry) of topological labels. These are the two main ways in which a state of matter can be topological: protected edge states and topological order. Fractional quantum Hall systems exhibit both of these phenomena, while integer quantum Hall states only exhibit the former.

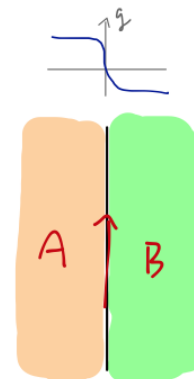
1) Edge modes. There exist nontrivial topological phases without topological order. Another (distinct!) possibility is that even if the system in infinite space has an energy gap, if we cut the space open, new stuff can happen; for example there may be gapless edge modes. There is a class of topological states which are characterized by their edge modes, generally called *invertible states* (special cases include Symmetry-Protected Topological states (SPTs) and Topological Insulators). The edge modes of such a state must carry some property which cannot be recreated locally on the surface; such a property is called an *anomaly*⁹.

⁷Actually, there is an important extra equivalence relation that we must include: We don't care if on top of some nontrivial phase of matter someone sprinkles a dust of decoupled qubits which are totally inert and do nothing at all. This modification represents the same phase of matter. (A good example to keep in mind is the K -shell electrons that sit passively next to the nucleus while the conduction electrons do the hard work of forming some strongly-correlated phase.) Then, further, we are allowed to adiabatically deform the hamiltonian, including these decoupled bits, so that they can interact with the original degrees of freedom. So: in addition to allowing adiabatic variation of couplings, we also allow the addition of decoupled qubits.

⁸Note that the closing of the gap does not by itself mean a quantum critical point: at a first order transition, just the lowest two levels cross each other.

⁹This definition of SPT as a state characterized by its anomalous edge modes may be imperfect. There are examples of distinct states protected by lattice symmetries which seem not to have interesting edge modes. See appendix A of [this paper](#) or [this paper](#). Thanks to Mike Hermele for bringing this phenomenon to my attention. A related exception would seem to be “higher-order topological insulators,” which have no edge states, but have excitations localized to sharp corners of the boundary.

A reason to think that an interface between the vacuum and a gapped state of matter which is distinct from the trivial one might carry gapless modes is that the couplings in the hamiltonian are forced to pass through the wall where the gap closes. In fact there are important exceptions to this conclusion. For example, the toric code admits gapped boundaries (interface with vacuum). We'll come back to the possibility of distinguishing phases of matter by their edge states in §4.



2) Topological Order. Even the lowest-energy (even below the gap) physics of gapped systems can be deeply fascinating. For example, it may be that the *number* of groundstates depends on the topology of the space on which we put the system. Since this is an integer, it cannot vary continuously and can only jump when the gap closes. This is a symptom of the phenomenon is called *topological order*¹⁰.

As an example of a state with topological order, consider the toric code, aka \mathbb{Z}_2 gauge theory. A representative groundstate wavefunction (the fixed point one) locally has the form

$$|\text{gs}, 0\rangle = \sum_{\text{closed loops}, C} |C\rangle = \left| \text{cylinder} \right\rangle + \left| \text{cylinder with one loop} \right\rangle + \left| \text{cylinder with two loops} \right\rangle + \dots$$

Here it is useful to think of the degrees of freedom as qubits living on the links of a lattice; a basis of the Hilbert space of each link is labelled ‘yes string’ or ‘no string’, and accordingly we do or do not draw a little red line segment over the link. The (local) Hamiltonian picks out only configurations where these links form closed loops, and picks out a uniform superposition of such loops. (We can talk about a specific form of such a Hamiltonian later.)

I say ‘locally’ because on a space with non-contractable loops (like a cylinder), we get orthogonal groundstates by including (or not) the loops that wind around the non-contractible cycles. In the picture above, I tried to indicate that space is a cylinder,

¹⁰In order for this degeneracy to be stable, it must be that no local operator maps one of these groundstates to another. Suppose our hamiltonian H_0 happens to annihilate two states $|\psi_{1,2}\rangle$ (i.e. WLOG, set the groundstate energy to zero). If we perturb H with any ΔH such that $\langle \psi_1 | \Delta H | \psi_2 \rangle$, the degeneracy will be split – the levels repel each other.

This property makes the groundstate subspace of a system with topological order into a quantum error-correcting code, with a *code distance* (the number of errors that can be corrected) of order the system size L .

and I included only contractable loops in the sum. An orthogonal groundstate is

$$|\text{gs}, 1\rangle = \left| \text{cylinder with one red loop} \right\rangle + \left| \text{cylinder with two red loops} \right\rangle + \left| \text{cylinder with three red loops} \right\rangle + \dots$$

So the number of groundstates depends on the topology of space. States in these different sectors are only related by an operator that creates a whole large loop winding around the nontrivial cycle, hence not by any local operator:

$$|\text{gs}, 1\rangle = \mathcal{O} \left(\text{cylinder with one red loop} \right) |\text{gs}, 0\rangle$$

The anyonic excitations arise by allowing the loops to end,

$$|\text{anyons at } x \text{ and } y\rangle = \mathcal{O} \left(\text{cylinder with red line from } x \text{ to } y \right) |\text{gs}\rangle$$

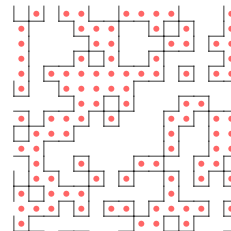
(where we can act on any of the groundstates). Such a state can arise as the groundstate of a spin system – it is a (gapped) *spin liquid* state.¹¹

The low-energy physics of gapped phases is often described by a (unitary) topological field theory; this is a theory of groundstates, and it can provide a way to distinguish states of matter. When this is case, the phase is called a *liquid*. Exceptions include fracton topological phases, where the lattice is not forgotten by the topological groundstates.

So topological order means a robust spacetime-topology-dependent groundstate degeneracy. Associated with this phenomenon is also a fractionalization of the quantum numbers of the microscopic constituents. That is, the emergent quasiparticle excitations carry quantum numbers (statistics, spin, charge) which are rational fractions of those of the constituents. Particles of fractional spin are called *anyons*. In three space dimensions, there are also string excitations with robust and interesting fractional properties. In the example of the toric code above, although the microscopic degrees of freedom are all bosonic (just spins), there is an excitation that is a fermion,

¹¹To see that topological order is a subtle thing that one might miss if asking the wrong questions, consider the following model of spins at the sites of the square lattice: $\mathbf{H} = \sum_i X_i$, where X_i is the Pauli σ^x operator.

Its groundstate, written in the σ^z eigenbasis, is (the product state) $\otimes_i |\rightarrow\rangle_i \propto \prod_i \sum_{s_i=\uparrow,\downarrow} |\{s_i\}\rangle$. We can also visualize this as a sum over closed loops: draw a loop around each region of $s_i = \downarrow$ (the red dots in the figure at right). They are closed loops because they are the boundaries of a region. Isn't this a toric code groundstate? No! One way to see the difference is that since these loops are defined as the boundaries of regions, they are always contractable.



and there are excitations that are mutual semions, meaning that they get a minus sign when braided around each other.

These two symptoms of topological order are not independent. A better way to define anyons (or more generally topological excitations) is: localized excitations that cannot be created by any local operator. But then these extended operators that create the anyons can do other things, too. That is, the fractional statistics of the quasiparticles implies a groundstate degeneracy on *e.g.* the torus: Pair-create a quasiparticle-antiquasiparticle pair, move them around a spatial cycle, and then re-annihilate them. This process is accomplished by an operator \mathcal{F}_x maps one groundstate to another. But \mathcal{F}_x does not commute with some \mathcal{F}_y (an analogous operator around the other direction), by the anyonic statistics. The space of groundstates must represent the algebra of these operators. Conversely, a *robust* groundstate degeneracy requires that the different groundstates are related by the action of non-local operators. In cases where the groundstate degeneracy is independent of geometry¹², these non-local operators describe the transport of fractionalized excitations around the cycles of the space.

[End of Lecture 1]

1.2 Toric code

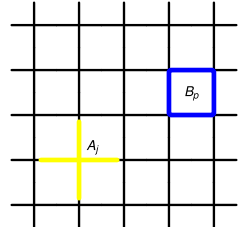
Here's the [toric code](#). It emerges \mathbb{Z}_2 gauge theory from a local Hilbert space. There is a sense in which it exists in certain forms of artificial condensed matter (cold atoms in optical lattices, trapped ions).

¹²In contrast, in type-2 fracton models, the operators taking one topological groundstate to another are supported on fractals, and so are hard to interpret as transporting anyons. In such models the groundstate degeneracy depends (in a complicated way) on the system size, and not just on the topology.

Consider a 2d cell complex. This means a graph (a set of vertices who know with whom they share an edge) with further information about plaquettes, who know which edges bound them):

$$\text{plaquettes} \xrightarrow{\partial} \text{edges} \xrightarrow{\partial} \text{vertices} \quad (1.1)$$

where ∂ is the boundary map, and I really mean formal linear combinations of these objects (we'll see a natural quantum mechanical realization of these linear combinations). For example, consider the square lattice at right. Now place a qubit on each *edge*. Now let's make the terms in the Hamiltonian. Associate to each plaquette a plaquette operator or 'flux operator', $B_p = \prod_{\ell \in \partial p} Z_\ell$, and to each vertex a star operator or 'gauss law operator', $A_v = \prod_{\ell \in \partial^{-1}v} X_\ell$. (The former names just describe the support of the operators on the graph. The latter names are natural if we consider Z to be related to a gauge field by $Z \sim e^{iA}$, and X is its electric flux. For more on the translation to gauge theory see §5.2 [here](#).) These definitions are not special to the square lattice and work for any cell complex, in any dimension.

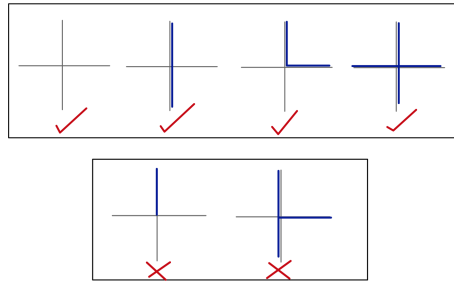


[Fig by D.Ben-Zion, after

[Kitaev](#)]

The hamiltonian is $\mathbf{H}_{\text{TC}} = -\Gamma_m \sum_p B_p - \Gamma_e \sum_v A_v$. These terms all commute with each other (since each vertex and plaquette share zero or two links), and they each square to one, so the Hamiltonian is easy to diagonalize. Let's find the groundstate(s).

Which states satisfy the 'gauss law condition' $A_v = 1$? In the X basis there is an extremely useful visualization: we say a link l of $\hat{\Gamma}$ is covered with a segment of string (an electric flux line) if $\mathbf{e}_l = 1$ (so $X_l = -1$) and is not covered if $\mathbf{e}_l = 0$ (so $X_l = +1$): $\overline{\ell} \equiv X = -1$. In the figure at right, we enumerate the possibilities for a 4-valent vertex. $A_v = -1$ if a flux line ends at v .



So the subspace of \mathcal{H} satisfying the gauss law condition is spanned by closed-string states (lines of electric flux which have no charge to end on), of the form $\sum_{\{C\}} \Psi(C) |C\rangle$.

Now we look at the action of B_p on this subspace of states:

$B_p = \prod_{\ell \in \partial p} Z_\ell$ creates and destroys strings around the boundary of the plaquette p :

$$\begin{aligned}
 B_\square | \rangle &= | \square \rangle \\
 B_\square | \square \rangle &= | \rangle \\
 B_\square | \text{loop} \rangle &= | \text{loop} \rangle
 \end{aligned}$$

$$B_p |C\rangle = |C + \partial p\rangle .$$

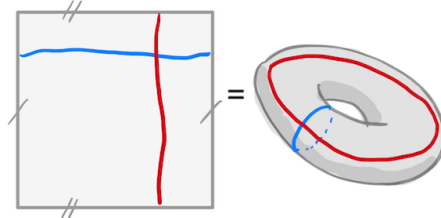
The argument of the ket is to be understood mod two. The condition that $B_p |gs\rangle = |gs\rangle$ is a homological equivalence. In words, the eigenvalue equation $\mathbf{B}_\square = 1$ says $\Psi(C) = \Psi(C')$ if C' and C can be continuously deformed into each other by attaching or removing plaquettes.

If the space is simply connected (like a sphere) – if all curves are the boundary of some region contained in the lattice – then this means the groundstate

$$|gs\rangle = \sum_C |C\rangle \tag{1.2}$$

is a uniform superposition of all loops.

Topological order. If the space has non-contractible loops, however, then the eigenvalue equation does not determine the relative coefficients of loops of different topology! The two-dimensional torus obtained by considering periodic boundary conditions in x and y is an example of such a space:



$$\tag{1.3}$$

On a space with $2g$ independent non-contractible loops, there are 2^{2g} independent groundstates. (In fact, the above is the very definition of the simplicial homology of the space, with \mathbb{Z}_2 coefficients; more generally the number of independent groundstates is 2^{b_1} where $b_1 \equiv \dim H^1(M, \mathbb{Z}_2)$. For more on the connection with homology and algebraic topology in general, see [these notes](#).)

No local operator mixes these groundstates. This makes the topological degeneracy stable to local perturbations of the Hamiltonian. The degenerate groundstates are instead connected by the action of (Wegner-Wilson) loop operators:

$$W_C = \prod_{\ell \in C} Z_\ell \quad V_{\tilde{C}} = \prod_{\ell \perp \tilde{C}} X_\ell .$$

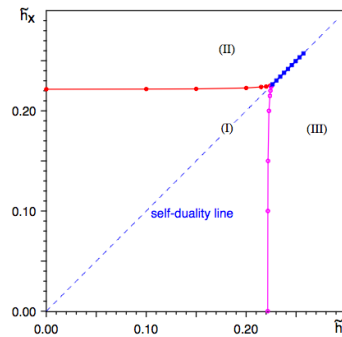
The second object is supported on a loop \tilde{C} that lives in the *dual lattice*. In 2d, this is the lattice whose vertices are faces of the original lattice and vice versa. For the

square lattice this is another square lattice shifted by half a lattice spacing in each direction. (Notice that the loop operator for a single plaquette $W_{\partial\Box} = B_p$ is the plaquette operator.) V, W commute with \mathbf{H}_{TC} and don't commute with each other (specifically W_C anticommutes with $V_{\check{C}}$ if C and \check{C} intersect an odd number of times). This algebra must be represented on the groundstates, and it doesn't have any one-dimensional representations. In terms of our picture of strings, W_C creates a loop on C , and $V_{\check{C}}$ detects a loop intersecting \check{C} .

The deconfined phase. So far everything I've said works on any graph (actually: cell complex, since we need to know where the plaquettes are). And so far I've described the solvable limit, where $H = H_{\text{TC}}$.

But the fact that the code distance goes like L (no local operator mixes the topological groundstates) is also the reason that the topological degeneracy is *robust*: adding local operators to the Hamiltonian will never split the degeneracy in perturbation theory. Therefore, this physics is characteristic of a phase of matter, and not just the special solvable Hamiltonian H_{TC} . The toric code is a (special, RG fixed point, with zero correlation-length) representative of a phase of matter.

Perturbations such as $\Delta H = \sum_l (h_X X_l + h_Z Z_l)$ produce a nonzero correlation length. Let's focus on $D = 2 + 1$ for what follows. These couplings h_X and h_Z are respectively a string tension and a fugacity for the electric flux string endpoints: charges. Make these too big and the model is higgsed or confined, respectively. These fancy-sounding phenomena are actually adiabatically connected [Fradkin-Shenker]: Both are connected to the trivial state where e.g. $H = \sum_l X_l$ whose groundstate is a product $\otimes_l |-\rangle_l$.



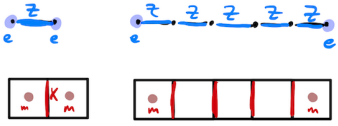
The lower left corner is the toric code. The region on the lower left is the phase with topological order. Everything else is adiabatically connected to a product state.
[from [Tupitsyn-Kitaev-Prokof'ev-Stamp](#)]

Anyons. There are two kinds of elementary excited states of the toric code: violations of $A_s = 1$ and violations of $B_p = 1$.¹³

¹³Cultural note: The limit where the coefficient of the star term A_s goes to infinity is called 'pure \mathbb{Z}_2 gauge theory', where the condition $A_s = 1$, the Gauss' law constraint, is imposed exactly. The e particle defects cost infinite energy and hence are strictly forbidden in this theory.

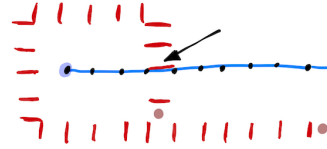
Here is how to make them. The defects are created by the endpoints of open Wilson lines. Again there are two kinds:

$$W(C) = \prod_{\ell \in C} Z_\ell, \quad V(\check{C}) = \prod_{\ell \perp \check{C}} X_\ell. \quad (1.4)$$



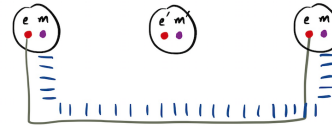
Here C is an open curve in the lattice, and \check{C} is an open curve in the dual lattice. Endpoints of $W(C)$ violate A_s and endpoints of $V(\check{C})$ violate B_p .

These two kinds of particles have nontrivial mutual statistics, as you can see by moving one of them around the other and keep track of the strings trailing away from them. The process results in a net factor of (-1) on the state.



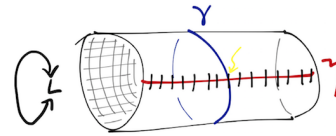
This has the further consequence that their bound state is a fermion, despite the fact that the model is entirely made from local, bosonic degrees of freedom. Hence, fractionalization.

To see this, observe that exchanging two particles can be accomplished by first rotating one around the other by a π rotation, and then translating both of them by their separation. As you can see in the figure, the first step requires the string creating the e particle to cross that creating the m particle on an odd number of links. (The second step is innocuous.)



Consider the cylinder. There is one nontrivial class of loops; call a representative γ . Let η be a line running along the cylinder. The two groundstates are generated by the action of the Wilson loop operator

$$V(\eta) \equiv \prod_{\ell \text{ crossed by } \eta} X_\ell$$



in the sense that

$$|\text{gs}_2\rangle = V(\eta) |\text{gs}_1\rangle .$$

This is also a groundstate (at $h_X, h_Z = 0$) since there is no plaquette with $\mathbf{B}_p = -1$ (more simply: $[\mathbf{H}_{h_X=h_Z=0}, V_x(\eta)] = 0$). They are *distinguished* by $W(\gamma) \equiv \prod_{\ell \in \gamma} X_\ell$ in the sense that the two groundstates are eigenstates of this operator with distinct eigenvalues:

$$W(\gamma) |\text{gs}_\alpha\rangle = (-1)^\alpha |\text{gs}_\alpha\rangle, \quad \alpha = 1, 2.$$

This follows since $W(\eta)V(\gamma) = -V(\gamma)W(\eta)$ – the two curves share a single link (the

one pointed to by the yellow arrow in the figure).

At finite h_X, h_Z (and in finite volume), there is tunneling between the topologically degenerate groundstates, since in that case

$$[\mathbf{H}, \prod_{l \in \gamma} X_l] \neq 0.$$

This means that for some n

$$\langle \text{gs}_2 | \mathbf{H}^n | \text{gs}_1 \rangle \neq 0.$$

The process that mixes the groundstates requires the creation of magnetic flux on some plaquette (*i.e.* a plaquette P with $B_P = -1$, which costs energy $2\Gamma_m$), which then must hop (using the h_X term in \mathbf{H}) all the way along the path η , of length L , to cancel the action of $V(\eta)$. The amplitude for this process goes like

$$\Gamma \sim \frac{\langle \text{gs}_2 | (hX_1)(hX_2) \cdots (hX_L) | \text{gs}_1 \rangle}{2\Gamma_m \cdot 2\Gamma_m \cdots 2\Gamma_m} \sim \left(\frac{h}{2\Gamma_m} \right)^L = e^{-L|\ln 2\Gamma_m/h|}$$

which is *extremely tiny* in the thermodynamic limit. The way to think about this is that the Hamiltonian is itself a local operator, and cannot distinguish the groundstates from each other. It takes a non-perturbative process, exponentially suppressed in system size, to create the splitting.

- I've focussed on the case of two spatial dimensions, but the toric code is well-defined on an arbitrary cell complex, in particular on a lattice in any number of dimensions. It has various generalizations:
- For example instead of the putting the dofs on the links, we can put them on the p -cells. Instead of using qubits, we can use \mathbb{Z}_N clock and shift variables.
- The version on the p -cells with \mathbb{Z}_N variables computes $H_p(C, \mathbb{Z}_N)$, the p th homology of the cell complex, as its groundstate subspace.
- With the dofs on the links, the model can be generalized to any finite group G (in fact this step was already taken in Kitaev's original paper). This is usually called the quantum double model.
- I haven't emphasized the connection to gauge theory above. The toric code is (a limit of) G lattice gauge theory with the gauss law condition imposed energetically, meaning that the low energy states satisfy the gauss law condition. At low energies it is governed by the TQFT described by Greg Moore and Meng Cheng in their lectures.

- By attaching various phases to the plaquette operators, we can make twisted gauge theory, as introduced by Dijkgraaf and Witten. A better framework for making explicit solvable lattice models for such states is the string net models developed by Levin and Wen [here](#) (see [here](#) for more).

[End of Lecture 2]

The toric code.

To summarize, the energy eigenstates of the toric code look like:

$$|gs\rangle = | \rangle + | \circ \rangle + | \circ \circ \rangle + | \heartsuit \rangle + | \text{B} \rangle + \dots$$

$$|\text{anyons}\rangle = | \text{J} \rangle + | \text{OJ} \rangle + | \text{JO} \rangle + | \text{JO} \heartsuit \rangle + | \text{S} \rangle + \dots$$

The groundstates on the torus can be labelled as:

$$| \square \rangle, | \square \text{---} \rangle, | \square \text{---} \rangle, | \square \text{---} \rangle \simeq | \square \text{---} \rangle$$

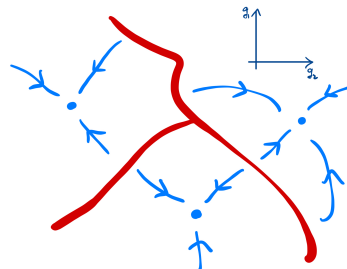
These groundstates are *locally indistinguishable*:

$$\langle \square \text{---} | \mathcal{O}_x | \square \rangle = 0 \quad \forall \text{ local operators } \mathcal{O}_x.$$

1.3 Back to the big overview

A word about the virtues of model Hamiltonians. In the previous subsection, we spent some time talking about a particular special Hamiltonian. It is pretty artificial-looking, in the sense that the terms in the Hamiltonian involve four or more spins at a time, and in that all the terms commute with each other, a very non-generic situation. Why do we think we can learn universal lessons from such a special system? The answer comes from the renormalization group.

Each gapped phase contains a single special point which is an attractive fixed point of the renormalization group, with zero correlation length. The fact that there can only be one such point in each gapped phase can be proved by contradiction: if we suppose there were more than one, we would not be able to draw the flow lines without discovering a repulsive fixed point separating them, but such a point must lie on the wall of gap-closing. So every gapped phase contains a special representative with zero correlation length. This special representative is easier to understand because all the irrelevant microscopic details have already been coarse-grained away. So we might as well use it to learn about the phase.



A third essential symptom of topological order is long-range entanglement in the groundstate wavefunction. I claim that the groundstate of a system with topological order cannot be made from a product state by a finite-depth local unitary circuit. I will explain this point in §1.4. Actually the converse of this is not true: there are some exceptional states with this property but which nevertheless do not have topological order, namely integer quantum Hall states and a few other examples we'll discuss below. A sharper diagnostic is the topological entanglement entropy, defined in terms of the entanglement entropy of a subsystem, which vanishes if and only if there are no anyons. This means that a state with topological order is (in a certain sense) far from a product state.

Let's enshrine these symptoms of topological order in a list:

1. Fractionalization of quantum numbers.
2. Groundstate degeneracy that depends on the topology of space.
3. Long-ranged entanglement.

I emphasize that the quantum numbers of the anyons (their statistics and (if there are global symmetries) charges) characterize the phase of matter. Especially in $D = 2 + 1$, the theory of anyons (their statistics and fusion rules) is a highly-developed mathematical edifice called topological field theory (TQFT), and more specifically in the case of 2+1 dimensions, unitary modular tensor category (UMTC) theory. Perhaps now is a good time to mention the most elementary distinction, between abelian and non-abelian topological order. By fusion of anyons, I mean the following. An anyon is a particle whose presence can be detected from a distance, by circling some other excitation around it and measuring the change of the resulting state. Given two anyon types a and b , I can consider circling other excitations around both of them. If I have

a complete basis of all the anyon types in the topological order under study, the result must look like one of them, but which one we get need not be uniquely determined:

$$a \times b = c_1 + c_2 + \dots . \tag{1.5}$$

If the fusion rules look like the special case

$$a \times b = c$$

(a group law) for all the anyons, we say the topological order is abelian. Braiding such particles merely acts by a phase on the resulting unique state. In contrast, fusion rules like (1.5) require that the lowest-energy state in the presence of a and b is degenerate; in this case, braiding the two particles involves not only a phase, but a whole unitary matrix acting on this degenerate subspace.

If you have studied conformal field theory (CFT), you will notice a formal similarity between (1.5) and the operator product expansion. This is not a coincidence – a 2d CFT also defines a UMTC. In fact, the structure was defined first in that context, by [Moore and Seiberg](#).

It was believed for a long time that all gapped phases are described at the lowest energies by a TQFT. However, this is not true. An interesting special case of topologically ordered states is *fracton phases*. A fracton phase has excitations (called fractons) that cannot be *moved* by any local operator (perhaps only in some directions of space). This is a strictly stronger condition than topological order, since an excitation can effectively be moved by annihilating it and creating it again elsewhere. This means that a fracton phase has a number of anyon types that grows with the system size – fractons in different places are really distinct anyons, since they are not related by any local operator. A consequence of this defining property is a groundstate degeneracy whose logarithm grows linearly with system size, and a subleading linear term in the scaling of the entanglement entropy of a region with the size of the region. Thus, the system knows about the geometry of space, and not just the topology, and hence cannot be described by an ordinary TQFT.

Gaplessness is something special that needs to be explained. An energy gap (and no topological order or special edge modes) should probably be the generic expectation for what happens if you pile together a bunch of degrees of freedom and couple them in some haphazard (even translation invariant) way. At the very least this follows on general grounds of pessimism: if you generically got something interesting by doing this, physics would be a lot easier (or more likely: we wouldn't find it interesting anymore). More seriously, gaplessness is an extreme case of a finite degeneracy: if allowed local operators could mix the the low-lying states, the levels would repel and create a gap under generic perturbations of the Hamiltonian.

Here is a list of some possible reasons for gaplessness (if you find another, you should tell me):

1. tuning to a critical point – notice that this requires some agent to do the tuning, and will only occur on some subspace of the space of couplings of nonzero codimension.
2. spontaneously-broken continuous symmetry (Goldstone bosons).
3. continuous unbroken gauge invariance (*e.g.* photons). Actually, this is a special case of item 2 for continuous one-form symmetries. If you are eager to learn more about this point of view, take a look [here](#).
4. Fermi surface (basically only in this case do we get gapless degrees of freedom at some locus of dimension greater than one in momentum space)
5. edge of an invertible topological phase. Here the gaplessness or degeneracy is protected by an anomaly.
6. a symmetry that forbids mass terms in some other way. This is called ‘technical naturalness’. An example is unbroken chiral symmetry, which forbids fermion masses. This is an explanation of the lightness of quarks and leptons in the Standard Model compared to the Planck scale. Supersymmetry (where chiral symmetry prevents fermion masses, and supersymmetry relates boson masses to fermion masses) goes in this category.
7. CFT with no relevant operators. I am not sure if there are examples of this which are not examples of item 3. Sometimes this is called self-organized criticality. See [here](#) for a recent search.

Each entry in this list is something to be understood¹⁴. If you encounter a gapless model and it does not fit into this list then I will bet you \$5 that it is fine tuned, meaning that its creator simply didn’t add enough terms to the Hamiltonian.

We can no longer define the boundary of a gapless phase by a wall of gap-closing. Instead, a useful definition that works for both gapped and gapless cases is that perturbation theory (in the difference of Hamiltonians) works within the phase. A phase

¹⁴Note that the masslessness of the graviton is a mystery not obviously solved by an element of this list.

is thus an equivalence class of hamiltonians such that within the phase, physics (thermodynamics, local operator expectations) varies smoothly¹⁵.

Refinement by symmetry. Another important axis along which we may organize states of matter is by symmetry. Specifically, we can label states according to the symmetry group G that acts on their Hilbert space, and restrict our discussion to the space of Hamiltonians to those which are preserved by G . Here I am speaking about what are called *global symmetries*, that is, symmetries (not redundancies of our labelling, like gauge transformations).

Now, in the presence of a symmetry, for each H , we can ask whether the symmetry is preserved by the groundstate? If not, this is called spontaneous symmetry breaking, and provides a label that distinguishes that state from the trivial state. This is the old story of classifying phases with symmetries.

But there is something else that can happen. There can be phase boundaries (walls of gaplessness) that cannot be circumvented within the space G -symmetric Hamiltonians, but which we can go around if we allow G -breaking terms in H . The example of SSB is already of this form: if we explicitly break the symmetry, there is no longer a sharp distinction between the paramagnetic phase and the broken phase, and we can go around the phase transition. (Consider the phase diagram of the Ising model including a longitudinal field.) If a G -symmetric state is connected to the trivial phase in the space of all H but not in the space of G -symmetric Hamiltonians, and does *not* spontaneously break G , it is called an SPT (symmetry-protected topological) state (protected by G). Such a state is very nearly trivial and we may wonder how it can be distinguished from the trivial state. The answer is that the edge theory realizes the symmetry in an interesting way, more precisely in an *anomalous* way. More on this later.

We can also have both topological order and anomalous edge modes. Various such G -symmetric states with topological order are called SET (symmetry-enriched topological) states.

Entanglement. An important perspective for organizing our understanding of quantum phases of matter – on which we focus for the rest of this chapter – is the amount and structure of entanglement in the groundstate.

A pure state is completely unentangled if it is a product state, $\otimes_x |s_x\rangle$. In a phase

¹⁵An annoying fact is that sometimes within a phase there are observables which vary non-analytically across a point where the thermodynamics and all local observables are perfectly smooth. A classic example is the roughening transition of Wilson loops in lattice gauge theory. So when I say ‘physics varies smoothly’ I really mean local observables. For non-local Hamiltonians, we do not even know how to define a notion of phase.

with a product-state representative, mean field theory applies. (And when it applies it is very useful: it predicts phase transitions and the associated critical theory of the order parameter, and even the excitation spectrum (*e.g.* spin waves).) Such states can be distinguished only by symmetries acting independently on each site. This problem is approximately solved (from the point of view of the experiment-free discussion of condensed matter physics we are having – of course the questions about energetic competition, which we are completely ignoring, can be very interesting from other points of view) by the representation theory of G . Elsewhere in such a phase, the effects of entanglement are a perturbative correction to non-universal quantities.

We’ve just defined what is ‘unentangled’. We will want to be more quantitative about entanglement between A and its complement \bar{A} ; here $\mathcal{H} = A \otimes \bar{A}$ is any bipartition of the Hilbert space, such as a region of space. Since we are talking about pure states (as opposed to *e.g.* thermal density matrices), we can do this using the entanglement entropy:

$$\rho_A \equiv \text{tr}_{\bar{A}} |\psi\rangle\langle\psi| \quad S_A \equiv -\text{tr} \rho_A \log \rho_A \quad .$$

This vanishes for product states. It equals $\log 2$ for $|\psi\rangle_{AB} = (|00\rangle + |11\rangle) / \sqrt{2}$, a single Bell pair. The dependence of S_A on the size and shape of A provides a lot of information about a state, some of which is universal, meaning a property of the phase, independent of the representative state.

So: highly-entangled and mean-field are antonyms. The description in terms of weakly-interacting waves above an ordered groundstate breaks down when the entanglement matters. The frontier of our understanding is states of matter where quantum mechanics is essential, not just a correction that can be included perturbatively. This is now a big industry (some interesting reviews are [1210.1281](#), [1302.0899](#)) and I will try to give some flavor of it. The states of interest here are distinguished instead by their patterns of quantum entanglement. Furthermore, since such new states of matter are distinguished by different new kinds of orders, the phase transitions which separate them go beyond those described by fluctuations of local symmetry-breaking order parameters. This leads to new renormalization-group fixed points and new conformal field theories (CFTs).

1.4 Adiabatic continuation and local unitary circuits

[Zeng, Chen, Zhou, Wen, chapter 7] A useful alternative characterization of a gapped phase motivates this entanglement-based point of view.

First, a quantum phase is actually a property of the groundstate. This statement is a version of the Principle of Entanglement Bootstrap: all the universal data about

a phase of matter can be extracted from a single wavefunction, even just the density matrix of a ball. For example, the groundstate seems to know whether it is gapped or gapless, because in the former case the equal-time correlation functions will fall off exponentially. In the case of topological order, all of the data about the characteristic anyon excitations are encoded in the groundstate wavefunctions on a torus (see [here](#)) or indeed even a single wavefunction (see [here](#) or, most elegantly, [here](#)). We'll see some evidence below.

Claim: Two groundstates are representatives of the same phase¹⁶ iff there is a **quasi-local** unitary circuit U of finite depth (the depth of a circuit is the (maximum) number of elementary gates acting on each site, 'finite' means independent of L , and I will explain quasi-local below) which maps one state to the other¹⁷. In symbols, $[\mathbf{H}_0] = [\mathbf{H}_1] \Leftrightarrow |\psi(1)\rangle = U |\psi(0)\rangle$.

[End of Lecture 3]

\Rightarrow Suppose there is a path $\mathbf{H}(s)$ in the space of Hamiltonians starting at \mathbf{H}_0 (whose groundstate is $|\psi(0)\rangle$) and ending at \mathbf{H}_1 (whose groundstate is $|\psi(1)\rangle$), with a gap for every s in between. In finite volume, the adiabatic theorem says we can construct a unitary which *probably* maps $|\psi(0)\rangle$ to $|\psi(1)\rangle$, namely slow-enough time-evolution along the path $\mathbf{H}(s)$,

$$\mathcal{T} e^{i \int_0^1 dt \mathbf{H}(t)} |\psi(0)\rangle \propto |\psi(0)\rangle + \dots \quad (1.6)$$

Slow enough means compared to the timescale set by the gap, $1/\Delta$. Since the gap is independent of L , the required duration is too. The failure rate (the amplitude for the \dots in (1.6), however, is extensive. This problem can be fixed by a procedure called *quasi-adiabatic filtering* introduced by Hastings (a review is [here](#)) – one can construct a modified family of Hamiltonians $\tilde{\mathbf{H}}(s)$ which are almost as local¹⁸ but precisely map groundstates to groundstates (the idea is to filter out the contributions from the excited states to which non-adiabatic transitions can happen)

$$|\psi(1)\rangle = \mathcal{T} e^{i \int_0^1 dt \tilde{\mathbf{H}}(t)} |\psi(0)\rangle.$$

(So really the title of this section should have been 'quasi-adiabatic continuation...')

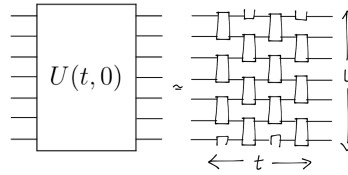
¹⁶In this discussion, we assume that the hamiltonians have a unique groundstate. So if we are talking about a phase with TO, we study it on a simply-connected space. The notion of phase is a local property.

¹⁷Actually, if you look back at our definition of gapped phase, we should also allow ourselves to tensor in ancillas in a product state before acting with the unitary.

¹⁸I've oversimplified the discussion here. Actually, there is a trade-off between locality of the filtered $\tilde{\mathbf{H}}$ and the precision with which the groundstates are mapped to each other. In fact, in order to precisely map the groundstates to each other, the operators $\tilde{\mathbf{H}}_x$ must have some tails, that is they have a profile which behaves like $e^{-r^{1-\delta}}$ where r is the distance from the point x – not quite exponential decay. This is the meaning of the modifier 'quasi-local'. Approximations to the exact map which are just as good for practical purposes can be made with \tilde{H}_x which are strictly local.

Now this continuous time evolution can be *Trotterized*. That is, we can approximate it by a circuit, by breaking the time evolution into tiny steps. The crucial ingredient is the Baker-Campbell-Hausdorff formula, in the form

$$e^{i\sum_x H_x \Delta t} = \prod_x e^{iH_x \Delta t} + \mathcal{O}(\Delta t^2). \quad (1.7)$$



The range of the terms in the Hamiltonian then determines the range of the individual unitary gates. The crucial point is that finite time evolution (independent of L) means a finite number of layers of elementary gates – this is a finite-depth circuit.

Thus we can regard circuits and unitaries from continuous-time evolution by local Hamiltonians as equivalent. A key point which is visible from the circuit picture is that there is a useful notion of lightcone, even in non-relativistic systems. Consider the domain of influence of a given input qubit in the circuit. It cannot effect output qubits that are arbitrarily far away, because the information about it only propagates by the local gates. The rigorous version of this statement is called the Lieb-Robinson bound.

Lieb-Robinson bound. Even non-relativistic theories have lightcones. Given a local Hamiltonian $\mathbf{H} = \sum_Z H_Z$ where the terms H_Z are supported on a subset Z and $\|H_Z\|$ shrinks rapidly with the diameter of Z (exponentially is good), then we can bound the correlations of local operators (A_X is supported on a set X and $A_X(t) = e^{-i\mathbf{H}t} A_X e^{i\mathbf{H}t}$ is its time evolution by \mathbf{H}):

$$\|[A_X(t), B_Y]\| \leq c_3 e^{-c_1 d_{XY}} (e^{2c_2 t} - 1)$$

where $d_{XY} = \min_{i \in X, j \in Y} |i - j|$ is the distance between the sets X, Y and $c_3 = 2\|A_X\| \|B_Y\| |X|$, c_1, c_2 are constants. The quantity $2c_2/c_1$ is the *Lieb-Robinson velocity*.

You can find a relatively accessible proof (and many important applications) [here](#).

⊞ Given a circuit $U = \mathcal{T} e^{i\int_0^1 dt \tilde{\mathbf{H}}(t)}$ that accomplishes $|\psi(1)\rangle = U |\psi(0)\rangle$, we can define $U(s) \equiv \mathcal{T} e^{i\int_0^s dt \tilde{\mathbf{H}}(t)}$ (just truncate the circuit at time s) and a family of states $|\psi(s)\rangle = U(s) |\psi(0)\rangle$. These states are the gapped groundstates of

$$\tilde{\mathbf{H}}(s) = \sum_x U(s) \tilde{\mathbf{H}}_x U(s)^\dagger$$

(gapped because the spectrum is independent of s) where $\mathbf{H}(0) = \sum_x \mathbf{H}_x$ is local, meaning that each \mathbf{H}_x has finite range ξ (independent of L); the range ξ of the terms in the filtered Hamiltonian $\tilde{\mathbf{H}} = \sum_x \tilde{\mathbf{H}}_x$ is still effectively finite (in the sense that there is still a Lieb-Robinson bound on its growth of correlations). But then the range of $U(s)\tilde{\mathbf{H}}_x U(s)^\dagger$ is bounded by $\tilde{\xi} + sv_{\max}$, where v_{\max} is the maximum speed of propagation of correlations via $\tilde{\mathbf{H}}(t \leq s)$, which is again (according to the Lieb-Robinson bound) independent of L . ■

Notice that there is a lot of freedom in defining the unitary U that relates the two groundstates – we’re actually only specifying its action on a single vector. What it does to the excited states (for example, the fact that it preserves the spectrum) is largely meaningless.

Here is an important consequence of this result. Recall that by a trivial phase we’ll mean one with a representative groundstate which is a product state. This result implies that any groundstate in a nontrivial phase *cannot* be made from a product state by a finite-depth circuit. An example is a toric code ground state $\sum_{\text{loops}, C} |C\rangle$.

1.5 Entanglement, short and long

Mean field theory is product states, which means there is no entanglement between regions of space at all. The next level of complication and interest to consider for possible groundstates of quantum many body systems is the case of states obtained by acting with a short-ranged quantum circuit of small depth on a product state. Let us consider such states, which are called short-range-entangled. What does their entanglement entropy of subregions look like and how do we distinguish which bits might be properties of a phase?

Let us focus on $d = 2$ space dimensions for definiteness. If the entanglement is short-ranged, we can construct a local ‘entanglement entropy density’ which is supported along the boundary of the region A [[Grover-Turner-Vishwanath](#)], and follow the logic of Landau theory to determine its form:

$$S_A = \oint_{\partial A} s dl = \oint (\Lambda + bK + cK^2 + \dots) dl = \Lambda \ell(\partial A) + \tilde{b} + \frac{\tilde{c}}{\ell(\partial A)} + \dots$$

In the first step, we use the fact that the entanglement is localized at the boundary between the region and its complement. In the second step we parametrize the local entropy density functional in a derivative expansion; K is the extrinsic curvature of the boundary. Since the total system is in a pure state, $S(A) = S(\bar{A})$, which implies $b = 0$: since interchanging A and \bar{A} reverses the orientation of the boundary, the extrinsic curvature cannot contribute. This means that the subsystem-size-independent term

cannot come from terms local on the boundary; it is universal in the sense that it cannot be changed by changing the UV regulator (*e.g.* by rearranging lattice details). Where can such a term come from? For the example of the groundstate of \mathbb{Z}_2 gauge theory (the toric code), a closed string that enters the region A must leave again. This is one missing bit of freedom for the reduced density matrix of A , which means a contribution to the EE that is independent of the size of A :

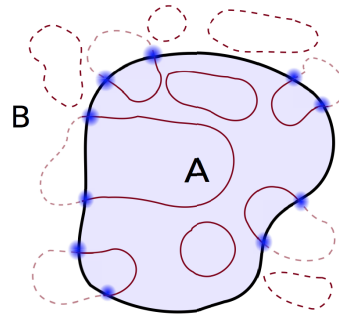
$$S_A = |\partial A| \Lambda - \log 2 \equiv |\partial A| \Lambda - \gamma \quad (1.8)$$

where the area-law coefficient Λ is some short-distance-dependent junk and γ is a universal characterization of the nature of the topological order.

This is true for each component of the boundary of A individually, so the generalization of (1.8) to regions with $b_0(\partial A)$ boundary components is $S(A) = |\partial A| \Lambda - \gamma b_0(\partial A)$.

The universal constant term γ is called the topological entanglement entropy (TEE)¹⁹. For more general topological orders, γ can be related to the spectrum of anyons; for Abelian states γ is $\frac{1}{2} \log(\#\text{torus groundstates})$. A beautiful argument for this is the [Kitaev-Preskill wormhole construction](#) (see their Fig. 2).

It is instructive to try to combine entropies of different regions to isolate the TEE from the area-law junk.



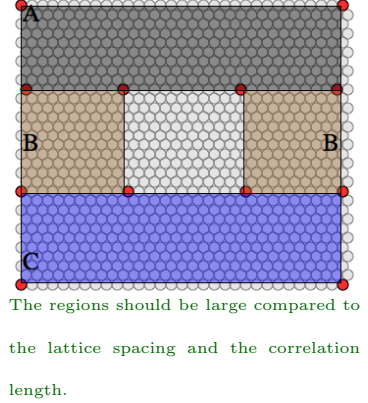
[fig: Tarun Grover]

¹⁹It was introduced for $d = 2$ by [Hamma-Ionicioiu-Zanardi](#), [Kitaev-Preskill](#), [Levin-Wen](#); the higher-dimensional generalizations are explained in the Grover et al paper linked above.

If the entanglement is indeed all short-ranged, then for collections of regions where the boundaries cancel out, $\partial(AB) + \partial(BC) = \partial(B) + \partial(ABC)$, (such as in the figure at right) nothing will be left. Let $S(x)$ be the EE of the subregion x in the state in question.

$$I(A : C|B) := S(AB) + S(BC) - S(B) - S(ABC)$$

is the conditional mutual information – correlations between variables A and C if we knew B . In general this combination of entropies satisfies a deep inequality called Strong Subadditivity (SSA), $I(A : C|B) \geq 0$. In general gapped phases in 2d, for the arrangement of regions at right, $I(A : C|B) = 2\gamma$, where γ is the subleading term to the area law defined in (1.8). The area-law contributions cancel out pairwise (notice that the corners cancel too).



When $\gamma = 0$, SSA is saturated. $I(A : C|B) = 0$ means ρ_{ABC} is a ‘quantum Markov chain,’ a state which can be reconstructed from its marginals ρ_A, ρ_B, ρ_C (by a formula due to Petz). So the quantity γ is an obstruction to this automatic reconstruction of the global state from local data.

The above argument shows that the TEE is not a short-distance artifact, but is it a property of a phase for any choice of A, B, C ? And is it only nonzero for states with topological order? Almost. The papers linked above argue – assuming that the system is a liquid – that the TEE is independent of small changes in the regions (using $S_A = S_{\bar{A}}$ for pure states) and therefore insensitive to changes in the Hamiltonian that keep the correlation length short. There is, however, [an important exception](#) if the phase is not a liquid, whereby small changes of the regions lead the TEE to jump, and to give nonzero answers in states without TO. A consolation is that the correct value [minimizes](#) the answers you can get for γ .

In $d = 3$, ∂A is characterized by its number of components b_0 and its number of noncontractable loops b_1 ; these are related by $\chi = 2b_0 - b_1 = V - E + F = \frac{1}{2\pi} \int_{\partial A} R$ (the Gauss-Bonnet theorem) to the integral of a local density. The EE of A is linear in b_0 and b_1 (see Appendix E of the Grover-Turner-Vishwanath paper) but only one combination of them is a signature of long-range entanglement. Again this 3d TEE can be extracted by combining regions whose boundaries and corners cancel.

The TEE is only one number characterizing the nature of the topological order, and by no means uniquely characterizes it. For example, the double semion state is a distinct topological order from the toric code in $d = 2$, whose representative wavefunction is $\sum_{\text{closed loops}, C} (-1)^{b_0(C)} |C\rangle$ (where $b_0(C)$ is the number of components

of the loops). As you can see from the form of the wavefunction it also has four groundstates on the torus and hence the same TEE. However, by now humans have learned to extract a great deal of the data specifying a given topological order from the entanglement properties of a single wavefunction, the most advanced incarnation of which is the *entanglement bootstrap*.

2 Some quantum Hall physics

By popular request, and because it still provides the best experimental examples of all the most interesting quantum topology phenomena, let's spend some time talking about effective descriptions of quantum Hall physics, both fractional and integer. I warn you that we will start from a distinctly macroscopic perspective, and will not say as much as others do about the important microscopic questions of energetic competition. For a perspective on those questions, I recommend chapters 12-14 of the textbook by Girvin and Yang (or [these lectures](#) by Leggett). I learned about this aspect of the subject from [these notes by Girvin](#). For a great account of the whole subject I recommend [David Tong's lectures](#).

2.1 Electromagnetic response of gapped states in $D = 2 + 1$

Let's think about a gapped state of matter made of some stuff in $D = 2 + 1$, out of which we can construct a conserved $U(1)$ current j_μ (if you like, think of it as the current that keeps track of electron number). This means we can couple this current to an external, background, non-dynamical gauge field \mathcal{A}_μ , by adding to the action functional like so:

$$S_{\text{microscopic}}[\text{the stuff}, \mathcal{A}] = S_{\text{microscopic}}[\text{the stuff}] + \int j^\mu \mathcal{A}_\mu + \dots$$

where \dots is whatever other terms are needed to make this action fully gauge invariant. Here we'll treat \mathcal{A} as a background field that we control²⁰. The theory with this deformation is gauge invariant because the current is conserved $\partial_\mu j^\mu = 0$ (integrate by parts).

Integrate out the stuff to see the electromagnetic response:

$$e^{iS_{\text{eff}}[\mathcal{A}]} \equiv \int [D\text{stuff}] e^{iS[\text{stuff}, \mathcal{A}]}.$$

²⁰Notice that what we've done here is *not* gauging the $U(1)$ symmetry. We are not changing the Hilbert space of the system. The background gauge field here just describes a particular collection of coupling constants.

The fact that the term linear in \mathcal{A} gives the current density:

$$\langle j^\mu(x) \rangle = \frac{\delta}{\delta \mathcal{A}_\mu(x)} S_{\text{eff}} \quad (2.1)$$

where the RHS is evaluated on the configuration of background fields of interest, which could be $\mathcal{A} = 0$. Terms quadratic in \mathcal{A} encode linear response:

$$\langle j^\mu(x) j^\nu(y) \rangle = \frac{\delta^2}{\delta \mathcal{A}_\mu(x) \delta \mathcal{A}_\nu(y)} S_{\text{eff}}.$$

Recall that $\langle jj \rangle$ is the main ingredient in Kubo's formula for the conductivity.

Because the stuff is gapped, S_{eff} is local. By the Landau-Ginzburg-Wilson logic, we can then determine $S_{\text{eff}}[\mathcal{A}]$ in a derivative expansion, as follows. To figure out the power counting, note that \mathcal{A} is a gauge field, which is something that we can add to a derivative to make it a covariant derivative; therefore \mathcal{A} has dimension 1, it counts the same as a derivative.

$$S_{\text{eff}}[\mathcal{A}] = \int \left(\underbrace{0 \cdot \mathcal{A}^2}_{\text{no symmetry breaking}} + \frac{\nu}{4\pi} \mathcal{A} \wedge \mathcal{F} + \frac{1}{g^2} \mathcal{F} \cdot \mathcal{F} \right) + \dots \quad (2.2)$$

($\mathcal{F} = d\mathcal{A}$.) The \mathcal{A}^2 term is forbidden by gauge invariance²¹. With time-reversal symmetry (and only one gauge field), $\nu = 0$. If $\nu \neq 0$, Maxwell is less important than the term with ν , the Chern-Simons (CS) term. (Actually, without Lorentz invariance we can have non-vacuum dielectric constant and magnetic permittivity ϵ, μ , but this won't affect our story.) g is an energy scale that we can take to be large compared to our energies of interest. [End of Lecture 4]

The Kubo formula then says that the Hall conductivity is:

$$\sigma^{xy} = \lim_{\omega \rightarrow 0} \frac{1}{i\omega} \underbrace{\langle j^x j^y \rangle}_{= \frac{\delta}{\delta \mathcal{A}_x(k)} \frac{\delta}{\delta \mathcal{A}_y(k)} S_{\text{eff}}[\mathcal{A}]} \Big|_{k=0} = \frac{\nu}{2\pi} \stackrel{\text{in experimenter's units}}{=} \nu \frac{e^2}{h}.$$

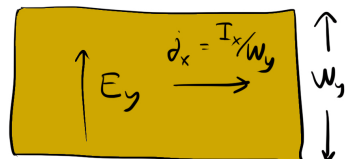
The analogous Kubo formula for the longitudinal conductivity σ^{xx} says that it is zero. This is one sense in which the system is an insulator. Note that one could argue with this characterization, since $\rho_{ij} = (\sigma^{-1})_{ij}$ has $\rho_{xx} = 0$, like a perfect conductor. While it's true that there is no dissipation (since the current is perpendicular to the voltage drop), no charge moves in the direction of the electric field, so I think it's safe to call it an insulator. Also, there's an energy gap (by assumption).

²¹To make it gauge invariant, we would have to add more (gapless) degrees of freedom, in particular the Goldstone mode ϕ for the broken U(1) symmetry, which would appear in the gauge-invariant combination $(\partial_\mu \phi + \mathcal{A}_\mu)^2$.

Next we'll show that, if there is no fractionalization, ν is quantized to be an integer. As a result, different values of ν label distinct phases of matter, since an integer can't change continuously. (Note that there could be further distinctions – states with the same ν could be distinct.)

Notice that 2d is special here because the conductivity is essentially dimensionless, and moreover the resistance is independent of the width of the sample:

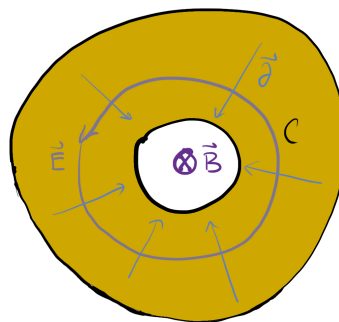
$$R_{xy} = V_y/I_x = E_y W_y/I_x = E_y/j_x = \frac{1}{\sigma_{xy}}$$



(Here W_y is the width of the sample in the y direction, perpendicular to the current, along the direction of voltage drop.) So it makes sense to say that σ^{xy} is quantized (in units of the quantum of conductivity $\frac{e^2}{h}$).

Flux-threading. I must emphasize that the following argument involves an important special case of studying the response of the system to background fields, called *flux-threading*. Rather than thinking about static background EM fields, we consider (still externally-fixed) EM fields that slowly vary in time – in a loop.

Consider the system on an annulus (sometimes called, in this context, the ‘Corbino geometry’). Adiabatically thread 2π worth of magnetic flux through (a solenoid in) the hole in the annulus. This means we slowly vary the magnetic field in the hole, so that the change in flux is the flux quantum $\Phi_0 \equiv \frac{hc}{e} = 2\pi$.



In the following equation only I restore un-natural units:

$$\Phi_0 = \Delta\Phi = \int dt \partial_t \left(\int_{\text{hole}} d\vec{a} \cdot \vec{B} \right) \stackrel{\text{Faraday}}{=} -c \int dt \oint_C \vec{E} \cdot d\vec{\ell} \quad j_r = \sigma_{xy} E_\varphi \quad -\frac{c}{\sigma_{xy}} \underbrace{\int dt I_r}_{=\Delta Q}$$

where C is a curve going around the hole, and $I_r = \oint_C j_r$ is the total radial current passing through the curve C . Note that Faraday's law is the spatial components of the equation $dF = 0$ stating the absence of magnetic monopoles, so does not depend on the form of the effective action for the EM field.

We conclude that an amount of charge


$$\Delta Q = \frac{\Phi_0}{c} \sigma_{xy} = \nu e \tag{2.3}$$

(e is the charge of the electron) is transferred from one edge of the cylinder to the other. Because of the energy gap, we can do the flux-threading adiabatically. Moreover, the initial and final Hamiltonians are related by a gauge transformation:

$$H(\Phi = 0) \cong H(\Phi = 2\pi).$$

(We detect a magnetic field by moving a charged particle around a loop and acquiring a phase $e^{iq \oint_C A}$; since the charge is quantized to be an integer, 2π flux is the same as no flux.) They have the same spectrum. Moreover, the work done on the system is $\int Id\Phi \propto \int dt \left(\frac{d\Phi}{dt}\right)^2$ which goes to zero in the thermodynamic limit, if our process is adiabatic. Therefore the initial and final states must be degenerate in the thermodynamic limit. But the adiabatic deformation can take one state to another. The states differ in that an amount of charge ν has been moved from one boundary to the other. Since, in the absence of fractionalization, charge is carried only by electrons, localized objects with integer charge, we conclude $\mathbb{Z} \ni \nu = \Delta Q/e = \sigma_{xy}$.

If we make the further assumption that the states can be labelled with the same labels as free-electron states, *i.e.* electron occupation numbers, we can say more. We've identified two different states related by the flux threading. The single-particle states whose occupation numbers have changed must lie near the Fermi level. Since we've assumed the bulk is gapped, we conclude that there must be gapless edge states.

The following argument implies further that a gapped system with $\sigma^{xy} = \nu \frac{e^2}{h}$  hosts a particle excitation with charge νe and exchange statistics $\pi\nu$.

Now consider the system on the plane. Adiabatically thread 2π worth of localized magnetic flux through some localized region R of the sample (as in the \otimes at right). To do this, we have to stick a really thin solenoid through the 2d surface on which the system lives. This means as above that (I now return to units with $\hbar = c = e = 1$)

$$2\pi = \Delta\Phi = \int dt \partial_t \left(\int_{R|\partial R=C} d\vec{a} \cdot \vec{B} \right) \stackrel{\text{Faraday}}{=} - \int dt \oint_C \vec{E} \cdot d\vec{\ell} \stackrel{j_r = \sigma_{xy} E_\varphi}{=} - \frac{1}{\sigma_{xy}} \underbrace{\int dt j_r}_{=\Delta Q}.$$

We conclude that the inserted flux sucks in an amount of charge

$$\Delta Q = \nu e.$$

Because of the energy gap, we can do this adiabatically. And because the flux is a multiple of 2π we end up with another state of the same system – the inserted flux is not an extrinsic defect²². This object is a localized excitation of the system – it can

²²This is slightly less obvious than in the case where the flux was in a hole in the system. Put the

move around, it's a particle²³. And if $\nu \notin \mathbb{Z}$, it has fractional charge νe .

But now consider two of them. From the Bohm-Aharonov effect, each has statistics angle $\pi\sigma_{xy}$ (for $\sigma_{xy} = \frac{e^2}{h}$, this is a fermion). Therefore, no topological order (hence no fractional statistics) implies (again) that $\sigma_{xy} \in \mathbb{Z}\frac{e^2}{h}$.

This argument has a stronger consequence for a system made only of bosons: In a gapped state with no fractionalization, all particles, including this one, must have the same statistics as the microscopic constituents. For a non-fractionalized state made from fermions, we can conclude that $\nu \in \mathbb{Z}$, since odd ν will produce a fermionic particle. But for a system made only of bosons, without topological order, σ_{xy} must be an *even* multiple of $\frac{e^2}{h}$. For a careful recent discussion of these arguments, see [here](#).

Roles of topology. Quantum Hall insulators provide examples that are topological in two distinct ways. The Hall conductivity (apply small electric field in x direction, measure current in y direction, take ratio)

$$\sigma^{xy} = \frac{p e^2}{q h} \tag{2.4}$$

is a rational number – $p, q \in \mathbb{Z}$ – despite (in fact, with the help of)²⁴ disorder.

1. The integer quantum Hall effect (IQHE) is governed by (2.4) with $q = 1$. We know such a phase with $p \neq 0$ is not adiabatically connected to the trivial phase

solenoid at the origin and choose the gauge $A = \Phi_0 \frac{d\varphi}{2\pi}$, where φ is the azimuthal coordinate in the plane (this is the field involved with the flux-threading, in addition to any magnetic field responsible for supporting the quantum Hall state). The gauge transformation that removes A is $g = e^{i\varphi}$, which is singular at the origin of polar coordinates. This singularity has no effect, since it just changes the overall phase of the wavefunction, $\Psi \rightarrow \Psi \prod_j e^{i\varphi_j}$.

²³Here's something I'm confused about at the moment. Unlike in the case of the 'Corbino geometry' above, where the flux was inserted in a hole in the sample, in this case we do expect that the final state has a different energy than the initial state – the quasiparticle we've created has some rest mass larger than zero, and (by charge conservation) the gap should be of order twice this value. Where does the argument about the work done break down?

²⁴If the system is translation invariant, one can show that the Hall conductivity must be strictly linear in the continuously-variable filling fraction $\nu = \rho\Phi_0/B$ (ρ is the electron density, $\Phi_0 = hc/e$ is the flux quantum), so there can be no quantized plateaux. One possibility for breaking the symmetry leading to this conclusion is disorder; this is the sense in which it helps. (It is then less obvious that it doesn't just make the Hall conductivity zero by Anderson localizing all the states.) Another possibility is that the symmetry could be broken spontaneously, as discussed [here](#). Another possibility is the presence of a lattice – the effective description of QHE in terms of Chern-Simons gauge theory works just as well for Chern insulators, *i.e.* a tight-binding model of fermions hopping a lattice where the filled bands have a nonzero Chern number. We will talk about Chern insulators more later on. It seems to me that another possibility could be boundaries of the sample.

because this integer cannot vary continuously – it’s definitely a distinct phase of matter. This happens for free electrons filling Landau levels or Chern bands. The quantization $p \in \mathbb{Z}$ arises because of topology of single-particle orbits; p is the Chern number. This does *not* exhibit topological order. This is an example of a ‘topological insulator’. It is a band insulator – the electrons completely fill some bands, and therefore there is an energy gap, measured by the energy difference to the next band (or to the next Landau level, *i.e.* the cyclotron frequency eB/m). What’s special about such a phase? We’ll see below that it has gapless edge modes.

Chern bands. To understand the possibility of IQHE without a magnetic field, first consider the possibility that our electrons in a 2d electron gas (2DEG) in a big magnetic field may be constrained to live at the sites of a lattice. (In fact this could be true of all the degrees of freedom of the Standard Model for all we know, if the lattice spacing is small enough.) That is, it could be described by a tight-binding model, like

$$H_{B=0} = - \sum_{ij} t c_i^\dagger c_j + h.c. \quad (2.5)$$

where i, j label sites of a lattice (say the square lattice). This hamiltonian is solved by going to Fourier space

$$c_k \propto \sum_i e^{i\vec{k} \cdot \vec{r}_i} c_i \quad (2.6)$$

so that $H = \int_{\text{BZ}} c_k^\dagger c_k \epsilon(k)$. If we expand $\epsilon(k)$ about the minimum, it will generically be quadratic and we get back a continuum description of non-relativistic electrons.

How do we include the magnetic field in such a description? The answer is that we replace the hopping parameter $t \rightarrow t e^{i a_{ij}}$ so that

$$H_B = - \sum_{ij} t e^{i a_{ij}} c_i^\dagger c_j + h.c. \quad (2.7)$$

with

$$a_{ij} = \frac{2\pi}{\Phi_0} \int_{\vec{r}_i}^{\vec{r}_j} \vec{A}(r) \cdot d\vec{r}. \quad (2.8)$$

(This is called Peierls’ substitution.)

Consider the case with $\vec{B} = \vec{\nabla} \times \vec{A} = B \hat{z}$ uniform. The description depends a bit on whether $\Phi_\square \equiv \oint_{\partial \square} \vec{A} \cdot d\vec{l}$, the flux through a single plaquette, is a rational

multiple of Φ_0 . For simplicity assume that it is, say $\Phi_\square = \Phi_0/k, k \in \mathbb{Z}$. Then we can choose the phases so that there is a finite unit cell (with k sites) as follows: We'll choose an analog of $\vec{A} = -Bx\hat{y}$ gauge. This is accomplished by choosing

$$t_{(x,y),(x,y+1)} = \omega^x, \quad \omega \equiv e^{\frac{2\pi i}{k}}. \quad (2.9)$$

With this choice of phases, the holonomy around each square $\prod_{\ell \in \partial \square} t_\ell$ is equal to ω . (Note that $t_{ij} = t_{ji}^*$.)

Again we can diagonalize the resulting Hamiltonian in momentum space, but now the unit cell has k sites, so there will be k bands:

$$c_{k\alpha} \propto \sum_i e^{i\vec{k} \cdot \vec{r}_i} c_{i+\alpha\hat{x}} \quad (2.10)$$

so that $H = \int_{\text{BZ}} c_{k\alpha}^\dagger c_{k\alpha} \epsilon_\alpha(k)$. As k grows, these bands get flatter and flatter and it becomes a better approximation to the continuum Landau levels.

But what's special about these bands that if I fill some of them I get a nonzero quantized Hall response? The answer is given by the following formula, called the TKNN formula. For any free fermion system, with some number of fully-filled bands, the Hall conductivity is

$$\sigma_{xy} = -\frac{e^2}{h} \sum_{\text{occupied bands}, \alpha} C_\alpha \quad (2.11)$$

where C_α is the *Chern number* of band α , defined as follows:

$$C_\alpha \equiv \frac{1}{2\pi} \int_{\text{BZ}} \mathcal{F}_\alpha, \quad \mathcal{F}_\alpha \equiv \left(\vec{\nabla}_k \times \mathcal{A}_\alpha \right)_z, \quad (2.12)$$

where \mathcal{F} is the Berry curvature, and

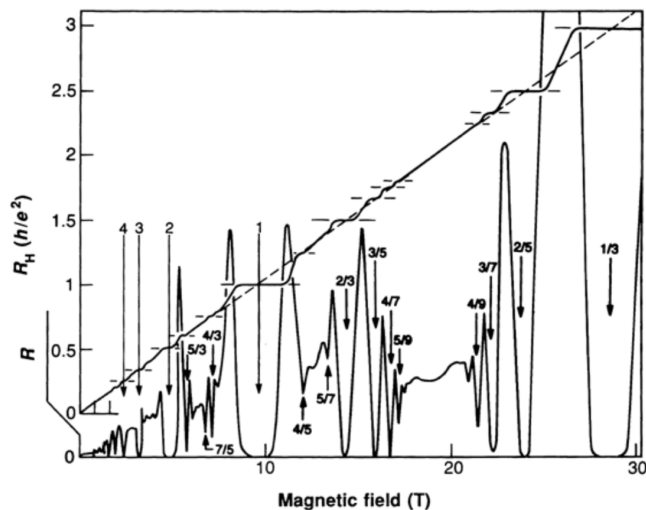
$$\mathcal{A}_\mu^\alpha \equiv i \langle u^\alpha(k) | \partial_{k\mu} | u^\alpha(k) \rangle \quad (2.13)$$

is the Berry connection. Here $|u^\alpha(k)\rangle$ is the wavefunction at momentum k of the α th band. The numbers C_α are always integers if the bands don't touch each other. We'll talk later about why this formula is true, and how to think about it so that it is robust to including interactions.

2. The fractional quantum Hall effect (FQHE) is described by $q \geq 2$, and requires interactions. There is necessarily topological order. $q \in \mathbb{Z}$ because of the topology of *many-body* wave function. The electron *fractionalizes*: as we'll see excitations

have charge $1/q$, fractional statistics.

Perhaps I should pause to emphasize that so far we have *assumed* that a bunch of stuff in $D = 2 + 1$ with a conserved charge but without time-reversal symmetry (such as a 2d electron gas in a magnetic field) can form a state with an energy gap. What we've shown, using the Landau-Ginzburg-Wilson logic, is that if this does happen, the system exhibits a quantized Hall conductivity. It is a remarkable fact, not at all obvious from anything we've said here, that this actually happens. As evidence, I include the [classic plot](#):



In this plot, the electron density is fixed, and the horizontal axis varies B . The key point is that around the value of B where the actual filling fraction (number of electrons per flux quantum) hits certain microscopically-preferred rational values, the measured σ_{xy} shows a plateau (along which $\sigma_{xx} = 0$), in striking contrast with the prediction from translation symmetry²⁵. When deforming away from the middle of the plateau,

²⁵**Theorem:** translation symmetry implies $\sigma^{xy} = \frac{\rho ec}{B}$, where ρ is the electron density. Consider the Hamiltonian H for a 2d collection of electrons in a uniform magnetic field $\vec{B} = B\hat{z}$, with $\vec{E} = E\hat{y}$ uniform as well. Choose the gauge $\vec{A}(r, t) = (-cEt + Bx)\hat{y}$. Gauge invariance means that the momentum of the i th electron appears in H only in the combination $\vec{\Pi} = \vec{p}_i + \frac{e}{c}\vec{A}(r_i, t)$. Remove the Et term by changing variables to $\vec{r}'_i = \vec{r}_i - c\frac{E}{B}t\hat{x}$, $\vec{p}'_i = \vec{p}_i$, $t' = t$. The current in the new frame is $j'_x = -\rho e \langle \dot{X}' \rangle$ where X' is the center-of-mass position in the \hat{x} direction. This is time-independent. The center-of-mass momentum in the y direction $P'_y = P_y \equiv \sum_j (p_j)_y$ is conserved. Therefore if $\langle \dot{X}' \rangle \neq 0$, $\Pi'_y = \sum_i ((p'_i)_y + \frac{e}{c}BX')$ would blow up, and therefore so would the kinetic energy. We conclude that $0 = \langle \dot{X}' \rangle \propto j'_x = 0$. But $\vec{j}'(r') = \vec{j}(r) + ec\frac{E}{B}\rho\hat{x}$ and therefore

$$j_x = -\rho ec \frac{E}{B} = \sigma^{xy} E^y$$

from which we conclude $\sigma^{xy} = -\frac{\rho ec}{B}$ without any further assumptions. A closely-related argument

the extra electrons must somehow be prevented from participating in the transport – they are somehow localized, either by disorder or by forming a (Wigner) crystal.

2.2 Abelian Chern-Simons theory

[Wen’s book or [this review](#); Zee, [Quantum Hall Fluids](#); Zee’s QFT book §VI.2] I want to explain an example of how properties 1 and 2 can be realized in a simple physical system, using the EFT (effective field theory) that describes the canonical examples of topologically-ordered states: (abelian) *fractional* quantum Hall states in $D = 2 + 1$.

The low-energy effective field theory is Chern-Simons-Witten gauge theory, whose basic action is:

$$S_0[a_I] = \sum_{IJ}^n \frac{K_{IJ}}{4\pi} \int a_I \wedge da_J \quad (2.14)$$

a^I are a collection of abelian gauge fields.

Where did these gauge fields come from? We’ll discuss some perhaps-more-informative possibilities below, but one very simple way to motivate their introduction is as follows. By assumption, our system has a conserved $U(1)$ current, J^μ , satisfying $\partial_\mu J^\mu = 0$. In $D = 2 + 1$, we can *solve* this equation by introducing a field a and writing

$$J^\mu \propto \epsilon^{\mu\nu\rho} \partial_\nu a_\rho. \quad (2.15)$$

The continuity equation is automatic if J can be written this way (for nonsingular a) by symmetry of the mixed partials. (The equation could also be solved by a sum of such terms, as we write below. This ambiguity reflects some of the enormous multiplicity of different quantum Hall states.) Then we must guess what dynamics should govern a . Here we just add all terms allowed by the symmetries, as usual. When it’s not forbidden by time-reversal symmetry or parity, the Chern-Simons term is the most important term at low energies.

Notice that we wrote this action in a coordinate-invariant way without needing to mention a metric. This is a topological field theory. In the absence of charges, the equations of motion say simply that $0 = \frac{\delta S_0}{\delta a} \propto f = da$. Unlike Maxwell theory, there are no local, gauge invariant degrees of freedom. And, by Legendre transformation, the Hamiltonian is just zero. It is a theory of groundstates.

[\[End of Lecture 5\]](#)

Consider the simplest case of (2.14) with a single such field a , $S_0[a] = \int \frac{k}{4\pi} a \wedge da$. As we’ll see, this describes *e.g.* the Laughlin state of electrons at $\nu = 1/k$ for k an

assuming boost invariance appears in many places, in particular in the Girvin lectures.

odd integer. (More general K describe the so-called hierarchy states, and give some understanding of the pattern of plateaux that appear.)

When I say there are no local dofs, I am thinking of the limit where we totally ignore the Maxwell term. The Maxwell term is irrelevant: its effects go away at low energies. Let's add it back in and look at the spectrum of fluctuations with the action:

$$L = \frac{k}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \frac{1}{4M} f_{\mu\nu} f^{\mu\nu}$$

where M is some microscopic energy scale above which the Maxwell term matters. The equation of motion is

$$0 = \frac{\delta S}{\delta a_\lambda} = \frac{k}{2\pi} \epsilon^{\lambda\rho\nu} f_{\rho\nu} + \frac{\partial_\mu f^{\mu\lambda}}{M}. \quad (2.16)$$

In terms of $f^\lambda \equiv \epsilon^{\lambda\rho\sigma} f_{\rho\sigma}$ this is

$$\epsilon^{\mu\nu\rho} \partial_\nu f_\rho + \frac{Mk}{2\pi} f^\mu = 0. \quad (2.17)$$

Taking curl of the BHS ($\epsilon_{\mu\alpha\beta} \partial^\alpha(\text{BHS})$) gives

$$\left(\partial_\mu \partial^\mu - \left(M \frac{k}{2\pi} \right)^2 \right) f_\rho = 0. \quad (2.18)$$

This is the dispersion relation for an excitation of mass $\frac{Mk}{2\pi}$. As $M \rightarrow \infty$, the excitation goes off to infinite energy.

If we demand that (2.14) is invariant (or rather e^{iS_0} is invariant) under $U(1)^n$ gauge transformations, including large gauge transformations, then k must be an integer²⁶.

²⁶Note that when k is odd, our flux-threading argument shows the existence of gauge-invariant fermionic excitations, and so can only arise from a theory with microscopic fermions.

To see the mysterious factor of two, here is my advice:

1. Recall that $\int a_1 \wedge da_2 = \int a_2 \wedge da_1$ by IBP. This is why the K -matrix is symmetric.
2. Consider the system on a spacetime of the form $S^1 \times S_g$, where S_g is a compact Riemann surface. An arbitrary connection on such a spacetime is of the form $a = a_1 + a_2$ where $a_{1,2}$ are polarized along the first and second factor of the spacetime respectively. Then

$$\int a \wedge da = \int (a_1 \wedge da_2 + a_2 \wedge da_1). \quad (2.19)$$

Note that the second term need not be zero because a_1 can still depend on the coordinates of S_g . Using the first item, this gives

$$\frac{1}{4\pi} \int a \wedge da = \frac{1}{2\pi} \int a_1 \wedge da_2. \quad (2.20)$$

3. Now consider the variation $a \rightarrow a + \mathbf{i}g^{-1}dg$, $g = e^{i\varphi}$, where φ is the coordinate along the circle.

Thanks to Aidan Sheckler for help with this.

From the point of view of (2.15), the demand that the gauge group is really $U(1)$, and the concomitant quantization of flux of da , comes from demanding that the charge of the current J_μ is quantized (in units of the charge of the electron). It's pretty interesting that this seemingly-metaphysical microscopic information that all charges come in integer multiples of the electron charge has such strong consequences for the low-energy description of macroscopic quantum phases.

Flux quantization. A crucial ingredient in the above argument for the quantization of the level is the fact of *flux quantization*. This is a consequence of the fact that the gauge group is compact, in this case $U(1)$ (as opposed to \mathbb{R}). Flux quantization is also the reason that the TKNN invariant (aka Chern number of a band) (2.12) is an integer, so I really should have explained it earlier. Let me stop to say a few words about it. The basic statement is that the integral of the flux F over a closed submanifold is a topological property.

I'm just going to explain the basic example of the phenomenon. To begin, let's think about a configuration of the electromagnetic field in 3-space with a magnetic monopole at the origin, *i.e.* satisfying $\vec{\nabla} \cdot B = 4\pi g \delta^3(x)$. The LHS of this equation is the time component of the 1-form $\star dF = \star d^2 A$, so A must not be globally well-defined or else this would vanish. A way out is to cover space with patches. Actually, all the action happens on the unit sphere surrounding the monopole (that is, $\mathbb{R}^3 \setminus \{0\} \simeq S^2$), so let's just think about that. We cover this two-sphere with two patches U_N and U_S consisting of everything but the south and north poles respectively. The overlap deformation retracts to the equator. On U_N and U_S respectively we take the gauge potential to be^a

$$A^N = g(1 - \cos \theta)d\varphi, \quad A^S = g(-1 - \cos \theta)d\varphi = A^N - 2gd\varphi = A^N + \mathbf{i}g_{NS}^{-1}dg_{NS}$$

where

$$g_{NS}(\theta, \varphi) \equiv e^{i2g\varphi}$$

is a function on $U_N \cap U_S$. Notice that $F = dA^N = dA^S = g \sin \theta d\theta \wedge d\varphi$ are both proportional to the volume form on the 2-sphere, consistent with the demand that magnetic flux is coming out from the origin in a spherically-symmetric way. g_{NS} has two names here: Mathematically, it is the transition function for a complex vector bundle of rank one between our two patches on the 2-sphere. Physically it is a function parameterizing a gauge transformation

$$A \rightarrow A^g = g^{-1}(A - \mathbf{id})g. \tag{2.21}$$

between two choices of gauge for the vector potential.

What is the structure group G of the vector bundle in question? g_{NS} is single valued under $\varphi \cong \varphi + 2\pi$ and therefore $G = \text{U}(1)$ iff

$$4\pi g \in 2\pi\mathbb{Z}. \quad (2.22)$$

This is the Dirac quantization condition. Notice that we didn't actually say anything about quantum mechanics. We just demanded that there was some $\text{U}(1)$ -valued transition function connecting the gauge potentials on the two patches. Quantum mechanics comes in for example if we put a charged particle in this background field, because the phase of the wavefunction of a charged particle transforms under a gauge transformation by multiplication by the transition function; if this isn't single-valued, the wavefunction is not well-defined. In that context, this is the physical reason we want the structure group to be $\text{U}(1)$ and not \mathbb{R} .

This conclusion leads to *flux quantization*: A vector bundle with a compact structure group has quantized fluxes, $\oint_S \frac{F}{2\pi} \in \mathbb{Z}$, where S is any compact 2d submanifold of B . In this example, we have

$$\begin{aligned} \oint_{S^2} \frac{F}{2\pi} &= \frac{1}{2\pi} \left(\int_{H_N} dA^N + \int_{H_S} dA^S \right) \stackrel{\text{Stokes}}{=} \frac{1}{2\pi} \oint_{\text{equator}} (A^N - A^S) \quad (2.23) \\ &= -\frac{1}{2\pi} \oint_{\text{equator}} \mathbf{i}g_{NS}^{-1} dg_{NS} = \frac{1}{2\pi} \int_0^{2\pi} 2gd\varphi = 2g \stackrel{(2.22)}{\in} \mathbb{Z} \end{aligned}$$

where H_N and H_S are the north and south hemispheres, which lie respectively in U_N and U_S and which have $\partial H_{N/S} = \pm \text{equator}$.

Although I explained it in the context of a $\text{U}(1)$ bundle on S^2 this phenomenon is general for compact gauge groups. In the case of the TKNN invariant, the Bloch wavefunctions define a $\text{U}(1)$ bundle on the Brillouin zone, which is a 2-torus. In this case the structure group is $\text{U}(1)$ because the gauge transformations are associated with rephasing the (normalized) wavefunction $\psi(k) \rightarrow g(k)\psi(k)$, so $g(k) = e^{i\phi(k)}$ is a phase.

Let me be more explicit about this bundle over the BZ defined by some Bloch bands in 2d. The eigenfunctions are defined by $h(k)u_\alpha(k) = \epsilon_\alpha u_\alpha(k)$. Because space is a lattice, k takes values in a 2-torus. Let's think about the special case when there are two bands, $\alpha = \pm 1$, so the hamiltonian takes the form

$$h(k) = a_0(k)\mathbb{1} + \sum_{i=x,y,z} \sigma^i a^i(k). \quad (2.24)$$

The $a_0(k)$ bit doesn't affect the eigenfunctions so forget it. Assuming the gap is nonzero for all k requires $X \equiv \sum_i (a^i(k))^2 > 0$. Without changing the topology

we can divide h by X so that (the new) $a^i(k)$ defines a unit vector, *i.e.* a point on S^2 . So $a^i(k)$ is a map from T^2 to S^2 . The a_i determine the wavefunctions by

$$u_-(k) = \begin{pmatrix} \cos \frac{\theta(k)}{2} \\ \sin \frac{\theta(k)}{2} e^{i\varphi(k)} \end{pmatrix} \quad (2.25)$$

where $\theta(k), \varphi(k)$ are the polar coordinates on S^2 labelling $a^i(k)$. From this information you can check that the Chern numbers of the two bands are just plus or minus the number of times the T^2 winds around the S^2 .

^aI'm using polar coordinates on the unit sphere where $x = \sin \theta \cos \varphi, y = \sin \theta \sin \varphi, z = \cos \theta$.

More generally, K must be a symmetric matrix (don't forget the sign from integration by parts) of integers.

Two more ingredients are required for this abelian CS theory to describe the low-energy EFT of a quantum Hall state:

(1) We must say how the stuff is coupled to the EM field. Notice that these gauge fields imply conserved currents $j_\mu^I = \frac{1}{2\pi} \epsilon_{\mu\nu\rho} \partial_\nu a_\rho^I$. This is automatically conserved by antisymmetry of $\epsilon_{\mu\nu\rho}$, as long as a is single-valued. In its realization as the EFT for a quantum Hall state, a linear combination of these currents is coupled to the external EM field \mathcal{A}_μ :

$$S_{EM}[a_I, \mathcal{A}] = \int \mathcal{A}^\mu t_I j_\mu^I,$$

i.e. the actual EM current is $J_\mu = \sum_I t_I j_\mu^I$. The normalization is determined so that flux quantization implies quantization of charge.

(2) Finally, we must include information about the (gapped) quasiparticle excitations of the system. Creating a quasiparticle excitation costs some energy of order the energy gap, and their dynamics is not included in this ultra-low-energy description. As I described above, however, the quantum numbers of these excitations is a crucial part of the data specifying the topological order. This is encoded by adding (conserved) currents minimally coupled to the CS gauge fields:

$$S_{qp} = \int a_I j_{qp}^I.$$

Alternatively, we can think of this as inserting Wilson lines $e^{i\oint_W a^I q_I}$ along the trajectories W of a (probe) anyon of charge q^I .

Now let's show item 1, fractional statistics, in the simplest case with a 1×1 K -matrix. In this case, the quasiparticles are anyons of charge e/k . The idea of how this

is accomplished is called flux attachment. The CS equation of motion is $0 = \frac{\delta S}{\delta a_\mu} \sim -\epsilon^{\mu\nu\rho} f_{\nu\rho} \frac{k}{2\pi} + j_{qp}^\mu$, where j_{qp} is a quasiparticle current, coupling minimally to the CS gauge field. The time component of this equation $\mu = t$ says $b = \frac{2\pi}{k} \rho$ – a charge gets $2\pi/k$ worth of magnetic flux attached to it. Then if we bring another quasiparticle in a loop C around it, the phase of its wavefunction changes by (the ordinary Bohm-Aharonov effect)

$$\Delta\varphi_{12} = q_1 \oint_C a = q_1 \int_{R, \partial R=C} b = q_1 \frac{2\pi}{k} q_2.$$

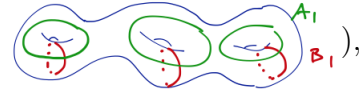
Hence, the quasiparticles have fractional braiding statistics²⁷.

Now topological order property 2: # of groundstates = $|\det(K)|^{\text{genus}}$. Consider the simplest case, where $K = k$, and put the system on a torus $T^2 = S^1 \times S^1$. The gauge-invariant operators acting on the Hilbert space of the CS theory on a torus are of the form $\mathcal{F}_x \equiv e^{i\oint_{C_x} a}$, $\mathcal{F}_y \equiv e^{i\oint_{C_y} a}$ and *integer* powers of these operators. These are the operators that transport the anyons around the cycles of the torus. The restriction to integers comes from the demand that they are invariant under large gauge transformations, which take $\oint_C a \rightarrow \oint_C a + 2\pi\mathbb{Z}$. According to the CS action, a_x is the canonical momentum of a_y . Canonical quantization then implies that

$$[a_x(r), a_y(r')] = \frac{2\pi\mathbf{i}}{k} \delta^2(r - r')$$

and hence (by the BCH formula) that these flux-insertion operators satisfy a Heisenberg algebra: $\mathcal{F}_x \mathcal{F}_y = \mathcal{F}_y \mathcal{F}_x e^{2\pi i/k}$. The smallest irrep of this algebra is k dimensional, where \mathcal{F}_x and \mathcal{F}_y look like clock and shift matrices.

If space is a Riemann surface with g handles (like this:



then there are g pairs of such operators, so g independent Heisenberg algebras, all of which commute with the Hamiltonian, and hence k^g groundstates.

It is also possible to show that CS theory also exhibits the third property of long-range entanglement. See [here](#).

This description shows a quasiparticle with charge e/k : If we stick in a quasiparticle at the origin, the equations of motion become

$$0 = \frac{\delta S}{\delta a_0(x)} = \frac{k}{2\pi} f_{xy} - \delta^2(x). \quad (2.26)$$

²⁷The fractional statistics of the charge- $\frac{1}{3}$ quasiparticles of the $\nu = 1/3$ Laughlin state were finally observed experimentally [just recently](#). Their charge had been measured using shot-noise measurements long ago.

From the relation $J^\mu = \frac{e}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu a_\rho$, the actual electric charge is then

$$\rho = e \frac{1}{2\pi} f_{xy} = \frac{e}{k} \delta^2(x). \quad (2.27)$$

Finally, we can do the (gaussian!) path integral over a to produce an effective action for \mathcal{A} of the form (2.2). (Complete the square.) We find a rational Hall conductivity

$$\sigma^{xy} = t_I (K^{-1})^{IJ} t_J \frac{e^2}{h}. \quad (2.28)$$

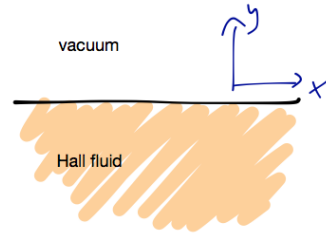
In the simplest case of $K = k, t = 1$, this is $\sigma^{xy} = \frac{1}{k} \frac{e^2}{h}$. The fact that the Hall conductivity is not an integer is not a problem – $e^{iS_{\text{eff}}[A]}$ does not need to be invariant under large gauge transformations, since there are k groundstates on the torus, which are permuted by flux-threading.

[End of Lecture 6]

So far, we've shown that abelian CS theory reproduces the bulk phenomenology of some fractional quantum Hall states. Now here is a bonus: we can see what it does when the sample has a boundary in space (which actual samples in the laboratory tend to have).

Edge physics. Consider U(1) CS theory living on the lower-half plane.

$$S = \frac{k}{4\pi} \int_{\mathbb{R} \times \text{LHP}} a \wedge da$$



Let's work in $a_0 = 0$ gauge. We must still impose the equations of motion for a_0 , which say $0 = f_{ij} = \epsilon_{ij} \partial_i a_j$. This is solved by $a = \mathbf{i}g^{-1}dg = d\phi$ ($g = e^{-i\phi}, \phi \simeq \phi + 2\pi$), where d is the exterior derivative in just the spatial directions. This looks like a gauge transformation.

Only gauge transformations that approach 1 at the boundary preserve S_{CS} . This implies that the would-be-gauge-parameter ϕ is dynamical on the boundary. (Or equivalently, we must add a degree of freedom identical to ϕ to cancel the gauge variation of the action.)

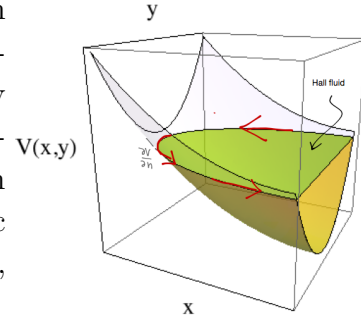
A good choice of boundary condition is: $0 = a - v(\star_2 a)$ i.e. $a_t = va_x$. The velocity v is some non-universal UV data; it arises from a gauge invariant local boundary term, $\Delta S = \int_{\partial \text{LHP}} \frac{kv}{4\pi} a_x^2$. Plugging back into the CS action and adding the boundary term,

we find²⁸

$$S_{CS}[a = d\phi] = \frac{k}{4\pi} \int dt dx (\partial_t \phi \partial_x \phi + v (\partial_x \phi)^2). \quad (2.32)$$

Conclusion: ϕ is a chiral boson. $kv > 0$ is required for stability. The sign of k determines the chirality.

For the case of IQHE ($k = 1$), the microscopic picture in terms of free fermions is at right. For free fermions in a magnetic field, the velocity of the edge states is determined by the slope of the potential which is holding the electrons together. (This can be understood by considering the motion of a classical charged particle in a large enough magnetic field that the inertial term can be ignored: $q\vec{v} \times \vec{B} = -\vec{\nabla}V$, solve for v .) It is clearly not universal information.



The Hamiltonian H depends on the boundary conditions; the Hilbert space \mathcal{H} does not.

I have to emphasize that a chiral theory like this cannot be realized from a local lattice model in $D = 1 + 1$ dimensions. There are more powerful arguments for this statement, but a viscerally appealing argument is simply to draw the bandstructure arising from any lattice Hamiltonian of free fermions. Each band is periodic in momentum space. This means that an even number of bands cross the Fermi level, and moreover that each band that crosses with positive slope must cross again with negative slope to return to its starting point. This is the essence of the *Nielsen-Ninomiya fermion doubling theorem*. An analogous argument applies in any number of dimensions. In fact, interactions provide a [real loophole](#) in the case of $D = 3 + 1$. But in $D = 1+1$, a nonzero chiral central charge (which in the simple examples we've discussed is just the number of right-movers minus the number of left-movers) is associated with a *gravitational anomaly*. A lattice model has zero gravitational anomaly, and this is a scale-independent quantity that must agree between the microscopic description and the EFT. The real obstruction to making a local lattice model is the anomaly.²⁹

²⁸In more detail, let \tilde{d} denote the exterior derivative in just the spatial directions.

$$S_0[a = \tilde{d}\phi] = \frac{k}{4\pi} \int_{\mathbb{R} \times LHP} a \wedge (dt \partial_t + \tilde{d}) a = \frac{k}{4\pi} \int_{\mathbb{R} \times LHP} \tilde{d}\phi \wedge dt \partial_t \tilde{d}\phi \quad (2.29)$$

$$= \frac{k}{4\pi} \int_{\mathbb{R} \times LHP} \tilde{d}(\phi \wedge dt \partial_t \tilde{d}\phi) \stackrel{\text{Stokes}}{=} \frac{k}{4\pi} \int_{\mathbb{R} \times \partial LHP} \phi dt \partial_t \tilde{d}\phi \quad (2.30)$$

$$= \frac{k}{4\pi} \int_{\mathbb{R} \times \partial LHP} dx dt \phi \partial_t \partial_x \phi \stackrel{\text{IBP}}{=} - \int_{\mathbb{R} \times \partial LHP} dx dt \partial_x \phi \partial_t \phi. \quad (2.31)$$

²⁹If, however, we break time translation symmetry, we can evade this outcome even in $D = 1 + 1$: for example, in a floquet system, where $H(t+T) = H(t)$, the set of energy eigenvalues is also periodic,

In the case with general K matrix,

$$S = \frac{K^{IJ}}{4\pi} \int_{\mathbb{R} \times \text{LHP}} a_I \wedge da_J$$

$$S_{CS}[a^I = d\phi^I] = \frac{1}{4\pi} \int dt dx (K^{IJ} \partial_t \phi^I \partial_x \phi^J + v_{IJ} \partial_x \phi^I \partial_x \phi^J).$$

(v is a positive matrix, non-universal.) This is a collection of chiral bosons. The number of left-/right-movers is the number of positive/negative eigenvalues of K .

Abelian Chern-Simons theory of the toric code. Consider now the following theory of two gauge fields with a *mutual* Chern-Simons term:

$$S[a, b] = \frac{k}{4\pi} \int d^3x (a\partial b + b\partial a) .$$

So the K -matrix is $\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$. The argument above suggests that a boundary of this model should have one left-mover and one right-mover, altogether an ordinary boson in $1 + 1d$. In this case, we can add local, single-valued, gauge-invariant terms to the boundary (such as $\cos \phi$) to kill the edge mode. Notice that unlike the generic abelian CS theory, this system has a time-reversal symmetry acting by $a \leftrightarrow b$.

So the TO described by this K matrix allows a gapped boundary. In fact it is an effective field theory of a familiar system. To see this, consider the anyon types: they can be labelled by their electric charges under the two gauge fields (a, b) . Because of the CS term, the electric charge of a gets k units of magnetic flux of b attached to it, and vice versa. The well-defined operators (ferrying these anyons around) are

$$W_C = e^{i\oint_C a}, \quad V_{\tilde{C}} = e^{i\oint_{\tilde{C}} b}.$$

Because of the Aharonov-Bohm phase, if we place the curves in a fixed-time slice, they satisfy

$$W_C V_{\tilde{C}} = \omega^{\#\tilde{C} \cap C} V_{\tilde{C}} W_C.$$

These are the operators that ferry the e and m particles of the \mathbb{Z}_k toric code.

As with the CS theories above, the anyons themselves are not dynamical degrees of freedom here, they are static external objects. However, we can include their dynamics, for example, by adding massive scalar fields that are charged under the CS gauge group. By varying the parameters of the potential for such scalar fields, we can move around the phase diagram. When their mass-squared passes through zero, they can condense

so we can have a band that starts below the Fermi level and ends above it, separated by $2\pi/T$ from its starting energy.

and take us to a new phase. In the toric code example, condensing a field charged under a higgses a and confines b . This takes us to the trivial phase, as condensing e should do.

Given a K -matrix theory with equal numbers of left-movers and right-movers, when can we gap out the boundary? The question is whether we can add local operators that give them a mass. For a chiral mode, $e^{i\phi_R} + h.c. = \cos \phi_R$ is not a local operator because of the commutation relations of ϕ_L determined from (2.32). But $\cos(\phi_R + \phi_L)/2$ is local. A keyword for the answer is ‘Lagrangian subalgebra’. Roughly, this is a subset of the anyons with the property that they are all (self and mutual) bosons, and that they braid nontrivially with everyone else. The idea is that we can *condense* these anyons (since they are bosons), and the result, because of the nontrivial braiding, is to confine everyone else, producing a trivial phase. If we do this condensation in the upper half plane, the real axis is a gapped boundary between the original TO and the trivial phase. In the case of the toric code, $a + b = \partial\phi_R$, $a - b = \partial\phi_L$, and both $\cos(\frac{1}{2}(\phi_R + \phi_L)(x)) = e^{iJ^x a} + h.c.$ and $\cos(\frac{1}{2}(\phi_R - \phi_L)(x)) = e^{iJ^x b} + h.c.$ are local. These two choices correspond to boundaries on the toric code where e and m are condensed, respectively.

Thermal Hall conductivity. Above I mentioned the notion of ‘central charge’ at some point. This is a concept from 1+1d CFT, which also appears in another universal observable. This observable is important for example when we are studying a system that does not couple directly to electromagnetism (such as spins) so doesn’t have a Hall conductivity. So even if there is no $U(1)$ symmetry, if the system is time-translation invariant, there will be energy conservation, and we can measure an energy current. One way to make an energy current is by applying a temperature gradient. The thermal Hall conductivity is defined as κ_{xy} in the linear response equation

$$I_y = \kappa_{xy} \vec{\nabla}_x T . \quad (2.33)$$

In a system with a free boson chiral edge mode like the one we found above, we can compute this. For each edge, any excitation moves along to the right with velocity v , so the energy current is $I = v\epsilon$, where ϵ is the energy density. In thermal equilibrium at temperature T , for a real chiral scalar field, this is

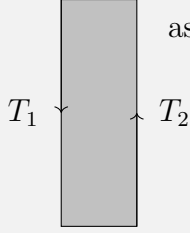
$$\epsilon(T) = \int_0^\infty \mathfrak{d}q v q b_T(vq) \quad (2.34)$$

where $b_T(\varepsilon) \equiv \frac{1}{e^{\varepsilon/T} - 1}$ is the Bose distribution, we used $\omega = vq$ is the dispersion relation, and because the field is real $\phi_{-q} = \phi_q^*$, the independent modes are

labelled only by positive momenta. Therefore each edge gives

$$I(T) = v\epsilon(T) = \int_0^\infty dE E \frac{1}{e^{E/T} - 1} = T^2 \frac{1}{2\pi} \int_0^\infty \frac{x dx}{e^x - 1} = \frac{\pi}{12} T^2 \quad (2.35)$$

Now place the left and right edges at slightly different temperatures as in the figure at left, so the total current is



$$I_y = \frac{\pi}{12} (T_1^2 - T_2^2) \simeq \frac{\pi}{6} T \Delta_x T. \quad (2.36)$$

$$\kappa_{xy}/T = c_- \frac{\pi}{12} \frac{k_B^2}{\hbar}. \quad (2.37)$$

In the last step I restored human-centric units (the \hbar comes from the density of states), and placed c_- (which is 1 for this example) where it belongs in general.

To give some evidence that this is where c_- belongs, let's consider an example of a different edge theory. Suppose instead the edge is inhabited by a chiral majorana fermion field η , with action

$$S[\eta] = \int dx dt \eta (\partial_t - v \partial_x) \eta. \quad (2.38)$$

The critical Ising model in $D = 1+1$ is described by two copies of this theory, one right-moving and one left-moving. This chiral version arises as the edge theory of several interesting gapped 2+1d phases, including the $p + ip$ superconductor (which has a free-fermion description and therefore is completely understood), and the non-abelian phase of the Kitaev honeycomb model. I will talk about both of those more later.

Again all the stuff just moves to the right with velocity v , and again the dispersion relation is $\omega = vq$, so each edge now gives

$$I(T) = v \int_0^\infty dq v q f_T(vq) \quad (2.39)$$

where now $f_T(\varepsilon) \equiv \frac{1}{e^{\varepsilon/T} + 1}$ is the Fermi distribution. Following the same steps,

$$I(T) = T^2 \frac{1}{2\pi} \int_0^\infty \frac{x dx}{e^x + 1} = \frac{\pi}{24} T^2. \quad (2.40)$$

Therefore, we again find (2.37), but with $c_- = 1/2$, which is indeed the chiral central charge of a single right-moving majorana mode.

More, generally the central charge is a measure of the number of degrees of freedom. In fact, two chiral majoranas (with $c_- = \frac{1}{2} + \frac{1}{2} = 1$) can be related to a single chiral boson by bosonization.

Non-abelian CS theory. So far we've talked about CS theory with gauge group $U(1)^n$. CS theory with more general gauge groups G , such as a non-abelian Lie group, can also arise as an EFT for states of matter. The non-abelian CS action looks like³⁰

$$S_{CS}[a] = \frac{k}{4\pi} \int_M \text{tr} \left(a \wedge da + \frac{2}{3} a \wedge a \wedge a \right)$$

where now a is a Lie-algebra-valued one-form, *i.e.* $a = \sum_{A=1}^{\dim G} a^A T^A$ where T^A are generators of the Lie algebra, say in the fundamental representation.

Again invariance under large gauge transformations, $g : M \rightarrow G$, requires that k is quantized. The variation of the CS Lagrangian

$$\mathcal{L}_{CS} = \frac{k}{4\pi} \text{tr} \left(a \wedge da + \frac{2}{3} a \wedge a \wedge a \right)$$

under $a \rightarrow gag^{-1} - \partial gg^{-1}$ is

$$\mathcal{L}_{CS} \rightarrow \mathcal{L}_{CS} + \frac{k}{4\pi} d(\text{tr} dg g^{-1} \wedge a) + \frac{k}{12\pi} \text{tr}(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg).$$

The first term is a total derivative and integrates to zero on a closed manifold. Over any closed surface, the second term integrates to the winding number of the map $g : M \rightarrow G$, and therefore the integral of the second term is an integer. We conclude that $e^{iS_{CS}}$ is gauge invariant if $k \in \mathbb{Z}$.

A similar story holds for the edge modes on $M = \mathbb{R} \times \Sigma$ with $\partial\Sigma \neq \emptyset$. Again we work in $a_0 = 0$ gauge, and the constraint $0 = \frac{\delta S}{\delta a_0} \propto f = da + a \wedge a$ is solved by $a = g^{-1} \tilde{d}g$, where \tilde{d} is the spatial exterior derivative. Only g that approach $\mathbb{1}$ at the boundary of Σ are gauge redundancies, and so the boundary value of g is a physical degree of freedom. Plugging into the action, and adding a local boundary term because

³⁰Full disclosure: in treating a as a Lie-algebra-valued one-form I am assuming that it is a connection on a trivial G -bundle on M . More generally, M must be covered by patches between which a is related by a gauge transformation. One way to robustly define the CS action is to realize $M = \partial N$ as the boundary of some 4-manifold N and use the fact that $\frac{1}{4\pi^2} \text{tr} f \wedge f = d\omega_{CS}$. Therefore the integral $\int_N \frac{1}{8\pi^2} \text{tr} f \wedge f = \int_M \omega_{CS} = S_{CS}[a]$ is perfectly well-defined. One shortcoming of this method is that not every M is the boundary of some N . For example, if M has a boundary, then it cannot be the boundary of something. From this point of view, the quantization of the level comes from demanding that the result for S_{CS} is independent of *which* 4-manifold we choose: the exp of the difference between the result for M and M' is

$$e^{ik(\int_M p_1 - \int_{M'} p_1)} = e^{ik \int_W p_1} \tag{2.41}$$

where $W = M \cap M'$ is now a 4-manifold without boundary, and $p_1 \equiv \frac{1}{4\pi^2} \text{tr} f \wedge f$. A generalization of flux quantization says that this integral is an integer multiple of 2π .

you can't stop me,

$$S_{\text{CS}}[a = g^{-1}\tilde{d}g] + \int_{\partial\Sigma \times \mathbb{R}} kv\text{tr}a_x^2 = k\text{tr} \left(\int_{\partial\Sigma \times \mathbb{R}} (g^{-1}\partial_t g g^{-1}\partial_x g + v g^{-1}\partial_x g g^{-1}\partial_x g) + \int_{\Sigma \times \mathbb{R}} \frac{1}{12\pi} g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg \right).$$

The first two terms are just like in the abelian case. The third term is still written as a 3d integral, but it only depends on the boundary value of g . It is called a WZW term. The resulting 1 + 1d field theory is a conformal field theory (CFT) called a chiral G_k WZW model. The central charge for $G = \text{SU}(N)$ at level k is

$$c = \frac{k \dim G}{k + N}.$$

[End of Lecture 7]

For non-abelian G , G_k CS theory (at least for $k > 1$) realizes non-abelian topological order. For example, $\text{SU}(2)_2$ is a description of the (non-abelian) Moore-Read state that seems to occur at filling $\nu = 5/2$ in GaAs (see *e.g.* p. 45 of [this useful review](#)). In case you missed it, [the class I mentioned in the introduction](#) concludes with a discussion of some bulk observables in non-abelian CS theory.

2.3 Representative wavefunctions

You’ll notice that I haven’t said very much about microscopic energetic questions. Quantum Hall states have been realized by now in many very different materials (in semiconductor heterojunctions, in other sandwiches made from semiconductors and insulators and metals, in graphene), and these energetic questions are pretty different in each case. Part of the reason to avoid that discussion is that it is different in each of these platforms. Quantum Hall states can even be realized in lattice models without any external magnetic field, namely Chern insulators. This just means that the hopping matrix elements are such that the bands have nonzero Chern number. A filled band with Chern number one has the same effect on the EM response as a filled Landau level.

But there is a valuable perspective more microscopic than CS theory, but still more universal than lattice details, namely representative wavefunctions. By this I mean a groundstate wavefunction *somewhere* in the same phase. There is [mounting evidence](#) that the groundstate wavefunction of a gapped phase contains all the universal data of the phase. This is particularly valuable in the case of quantum Hall states where there is a [topological obstruction](#) to exactly solvable models with exactly zero correlation length (like the toric code).

How to write down a wavefunction in the right phase? One way is to make an educated guess, which is what Laughlin did. For times when we are feeling less inspired, here is a reliable method.

Parton construction. [I recommend [Sung-Sik Lee’s TASI 2010 lectures](#)] Here is a strategy for writing down wavefunctions that represent a phase with topological order. It also produces a candidate effective field theory, and has many other virtues. It is widely regarded with suspicion.

A practical point of view on what I’m going to describe here is a way to guess variational wavefunctions for fractionalized groundstates. A more ambitious interpretation is to think of the parton construction as a low-energy duality between a model of interacting electrons ([or spins or bosons or ...](#)) and a gauge theory of (candidate) ‘partons’ or ‘slave particles’. Like any low-energy duality, it is a guess for useful low-energy degrees of freedom. The goal is to describe states in roughly the same Hilbert space³¹ as the original model, in terms of other (hopefully better!) variables. The appearance of gauge fields (perhaps only discrete ones) is an inevitable side effect when there is frac-

³¹I don’t mean exactly the same Hilbert space. The construction takes advantage of our ability to add in ancillary, decoupled, inert bits in changing our representative of a phase. Sometimes condensed matter physicists use the phrase “the same Hilbert space” to mean up to this equivalence, and it is in this sense that we mean it here.

tionalization of quantum numbers (spin-charge separation, fractional charge ...) in $D > 1 + 1$.

I will describe the construction in two steps. For definiteness, let's focus on the following example. Suppose c to be the annihilation operator for a (spinless) electron. Suppose we are interested in the (difficult) model with

$$H = \sum_{\langle ij \rangle} \left(t_{ij} c_i^\dagger c_j + h.c. \right) + \sum_{\langle ij \rangle} V n_i n_j \quad (2.42)$$

Comments:

1. We're going to talk about spinless electrons. This can be reasonable in a big magnetic field, which implies a big Zeeman splitting, so that the wrong-pointing spin states are high energy states we can ignore. (However, see the section of Girvin's review about QH ferromagnetism – the Zeeman splitting in GaAs is not that big.)
2. We can suppose that the hopping terms t_{ij} include some lattice version of the magnetic field, so $t_{ij} = t e^{iA_{ij}}$. If you like, you could think of my lattice model here as just a discretization of electrons in the continuum in a magnetic field.
3. This kind of 'Hubbard-V interaction' is the shortest range interaction we can have for spinless fermions (since the density $n_i = c_i^\dagger c_i$ is zero or one and so satisfies $n_i^2 = n_i$).
4. To fully specify the system (2.42), we need to specify the filling – how many electrons are there per site. If the electrons fully fill some bands and $V = 0$, the system is an insulator; since there's a gap we expect this fact to persist even for nonzero V . If the filled bands have nonzero Chern number, this is a Chern insulator, and there is a quantized Hall response. It is just a lattice version of the IQHE.
5. If we partially fill some Chern bands, without V the system would be a metal. Interactions have a chance to change that. Indeed such a model can produce fractional quantum Hall groundstates. On the lattice, such a thing is called a fractional Chern insulator; as of just recently, these exist in various twisted bilayer systems.

Parton construction: step 1 of 2 (Kinematics)

Relabel states of the many-body \mathcal{H} with new, auxiliary variables.

For example, a parton ansatz appropriate to the $\nu = \frac{1}{3}$ Laughlin FQH state is

$$e.g. \quad c = f_1 f_2 f_3 = \frac{1}{3!} \epsilon_{\alpha\beta\gamma} f_\alpha f_\beta f_\gamma$$

f s are complex fermion annihilation operators (they must be fermionic in order that three of them make up a grassmann operator).

Not all states made by f s are in \mathcal{H} . There is a redundancy: if we change

$$f_1 \rightarrow e^{i\varphi(x)} f_1, f_2 \rightarrow e^{-i\varphi(x)} f_2, f_3 \rightarrow f_3, \text{ or } f_1 \rightarrow f_1, f_2 \rightarrow e^{i\varphi(x)} f_2, f_3 \rightarrow e^{-i\varphi(x)} f_3, \quad (2.43)$$

then the physical variable c is unchanged. In fact, there is an $SU(3)$ redundancy $f_\alpha \rightarrow U_\alpha^\beta f_\beta$, $c \rightarrow \det U c$ (of which (2.43) is a Cartan (maximal abelian) subgroup). We are making the ansatz that c is a baryon.

In any state in \mathcal{H} , the number of actual electrons is equal to the number of partons of each color, since c^\dagger creates one of each. The Lagrange multipliers imposing

$$f_1^\dagger f_1 = f_2^\dagger f_2 = c^\dagger c = \text{number of } e^-; f_2^\dagger f_2 = f_3^\dagger f_3 \quad (2.44)$$

are the time components a_0 of a gauge field.

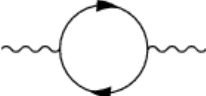
To write an action for the f s that is covariant under this redundancy, introduce the spatial components of the gauge field, a_i . Perhaps you don't like this idea since it seems like we added degrees of freedom. Alternatively, we can think of it as arising from e^- bilinears, in decoupling the $c_x^\dagger c_x c_{x+i}^\dagger c_{x+i}$ interaction by the Hubbard-Stratonovich trick. What I mean by this is:

$$\begin{aligned} e^{iV \int dt n_i(t) n_j(t)} &= e^{iV \int dt c_i^\dagger(t) c_i(t) c_j^\dagger(t) c_j(t)} \\ &\stackrel{(2.44)}{=} e^{i \int dt \frac{V}{9} \sum_\alpha f_{i\alpha}^\dagger(t) f_{i\alpha}(t) \sum_\beta f_{j\beta}^\dagger(t) f_{j\beta}(t)} \\ &= \int \left[\prod_{\alpha, \beta} D\eta_{ij}^{\alpha\beta} \right] e^{i \int dt \sum_{\alpha\beta} \left(\frac{9|\eta_{ij}^{\alpha\beta}|^2}{V} + f_{i\alpha}^\dagger(t) f_{j\beta}(t) \eta_{ij}^{\alpha\beta} + h.c. \right)} \end{aligned} \quad (2.45)$$

where η is a new complex (auxiliary) bosonic field on each link. Now let $\eta_{ij} = |\eta_{ij}| e^{ia_{ij}}$ (for each $\alpha\beta$) and ignore the (massive) fluctuations of the magnitude $|\eta_{ij}| = t_{ij}$. *Voilà* the gauge field, and the parton kinetic term.

How does the practical viewpoint of constructing possible wavefunctions arise? Guess weakly interacting partons: $H_{\text{partons}} = - \sum_{ij} t_{ij} f_i^\dagger e^{ia_{ij}} f_j + h.c.$ Then fill bands of f and project onto the gauge invariant subspace.

But what about the fluctuations of a (*i.e.* we still have to do the a integral)? Microscopically, a has no kinetic term; in that sense the partons are surely strongly coupled and confined at *short* distances (of course they are – the system is made of electrons if you look closely enough). a only gets a kinetic term from the parton

fluctuations, by processes like this:  . The hope is that with enough

other partons around, they can be shared and juggled amongst the electrons, so that which parton is in which electron fluctuates.

Parton construction: step 2 of 2 (Dynamics)

Such a rewrite is always possible, and there are many possibilities. The default result of such a rewriting is that the gauge theory also confines the partons at low energies. By a confining state, I mean one in which the energy cost to separate colorful partons is much larger than other scales in the problem, namely, the gap, or the inverse lattice spacing, or energies associated with chemistry (gasp). A picture of a confined state might be something like this:



In this picture, each of the sites has one of each color of parton localized to it – this is just an electron. This means there is no fractionalization and no topological order and usually leads us back to the microscopic description in terms of the microscopic degrees of freedom. (It doesn't mean the parton description is useless however; see §4).

Pure 2+1d gauge theory (without a CS term) likes to do this. Recall that the Maxwell or Yang-Mills kinetic term is an irrelevant operator according to naive dimensional analysis, if we treat the gauge field as a connection (*i.e.* something we can add to a spatial derivative). This is [true](#) even of (compact) $U(1)$ gauge theory: In terms of the dual photon σ , defined by $\partial_\mu \sigma \equiv \frac{1}{2\pi} \epsilon_{\mu\nu\rho} \partial_\nu a_\rho$, the gas of monopole instantons produces an effective potential of the form

$$V_{\text{eff}} = \Lambda^3 e^{i\sigma} + h.c. = \Lambda^3 \cos \sigma.$$

Expanding around the minimum of this potential, we find a mass for σ , and hence for the photon a_μ . The statement that abelian gauge theory with compact gauge group in $D = 2 + 1$ likes to confine is due to Polyakov³².

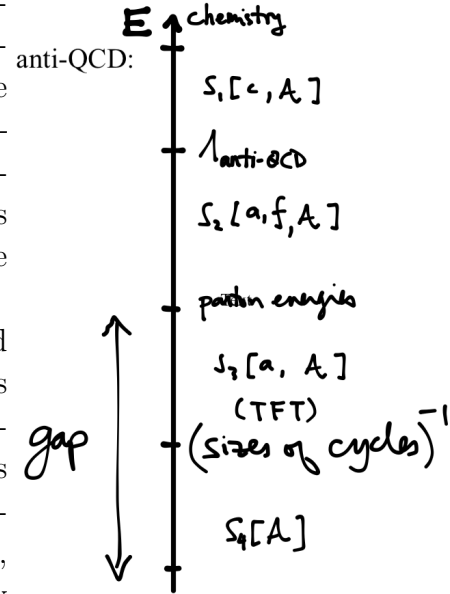
³²See section 7.2 of [these notes](#) for more details about this from the point of view of a regularization on the lattice.

energy scales in QCD:



Let me emphasize again that it's *deconfined* states of parton gauge theories that are most interesting here. So we are looking for gauge theories that behave oppositely to QCD, really like *anti-QCD*, where the partons are deconfined *below* the confinement scale $\Lambda_{\text{anti-QCD}}$, as in the figure at right. Interesting states we can make this way correspond to interesting phases of gauge theory, a worthy subject.

Our discussion in this section has followed this diagram starting from the highest energies (chemistry!) and guessing the lower-energy degrees of freedom that result from the interactions of the constituents. (This dialectic between high-energy physics and condensed matter physics, of GUT and anti-GUT, is described vividly by [Volovik.](#))



Like gaplessness, deconfinement requires an explanation. Known exceptions which allow for this:

- enough dimensions that the Maxwell term becomes marginal or relevant and we can have a Coulomb phase.
- partial Higgsing to \mathbb{Z}_n . Condensing electric charge makes monopoles heavy.
- lots of charged degrees of freedom at low energy. One way to describe their effects is that they produce zeromodes on the monopole configuration, and the monopoles only contribute to higher-dimension operators involving insertions of the light fields. (Interesting constraints on how many modes is enough, from strong-subadditivity of the entanglement entropy, were derived [here.](#)) Partons that are gapless at points in k -space inhabit phases called algebraic (something) liquids; the ‘something’ is whatever visible quantum numbers they carry, *e.g.* spin; if they happened in the model (2.42), it would be charge. If the partons form a Fermi surface, that is certainly enough (Sung-Sik Lee reviews his proof of this in the notes linked above). This is a kind of spin liquid which may have been observed in various materials in the past decade or so.
- in $D = 2 + 1$: the Chern-Simons term $a \wedge da$ is marginal, and can gap out gauge dynamics, as we saw in §2.2, producing a stable, deconfined, topological phase. Mutual CS terms can accomplish the same goal.

If I've forgotten some please tell me.

Parton construction of Laughlin state. Let's pursue the Laughlin example a bit further, but let's retreat to the continuum. So consider a pile of electrons in 2+1 dimensions, on a space of area A with periodic boundary conditions, in large uniform B . And let's put only enough electrons to fill a third of the lowest Landau level. That is, the number of electrons per lowest Landau level state is

$$\frac{1}{3} = \nu_e \equiv \frac{N_e}{N_\Phi(e)} = \frac{N_e}{eBA/(hc)}.$$

The fact that this band is partially filled means that if the electrons are free, the system is gapless. But this degeneracy is fragile. Any interaction between the electrons will split the degeneracy somehow.

If, on the other hand, the electron fractionalizes as $c = f_1 f_2 f_3$, then f_α carries charge $1/3$ ³³. Consider then each f_α in the same external field B , and suppose the partons are free (as a first approximation). Their filling fraction is:

$$\nu_f = \frac{N_f}{N_\Phi(e/3)} = \frac{N_e}{N_\Phi(e/3)} = 3\nu_e = 1 \quad .$$

The wonderful thing about this guess is that the partons can now form a gapped state: that is, we can pretend they are free and fill their bands, so that they make a band insulator. However, because they are filling a Landau level, this band insulator is an integer quantum Hall (IQH) state. Then, integrating out the gapped partons produces a (nonsingular but nontrivial) contribution to the effective action for the gauge field: the IQH nature of the bands means that there is a Hall response for any gauge fields to which they are coupled, just as we've discussed above. This is encapsulated precisely by the CS term!³⁴

³³Actually, it is completely arbitrary how we divide up the electron charge amongst the partons; different choices differ by relabelings of the gauge group which cannot affect the physics.

³⁴We showed that QHE means a CS term earlier. The massive Dirac fermion in 2+1 dimensions also has a Hall response. The mass term $m\bar{\psi}\psi$ breaks parity in $D = 2 + 1$. This slightly-more-microscopic calculation can be done in just the same manner as the path integral calculation of the chiral anomaly, and the $\epsilon_{\mu\nu\rho}$ arises for the same reason:

$$\begin{aligned} \log \int D\psi D\psi^\dagger e^{-\int d^3x \bar{\psi}(\mathbf{i}\not{D}-m)\psi} &= \log \det(\mathbf{i}\not{D}-m) = \text{Tr} \log(\mathbf{i}\not{D}-m) \\ &\equiv \text{Tr} \log(1 - \mathbf{i}\not{D}/m) e^{-\square/M^2} + \text{cst} \quad \square \equiv (\mathbf{i}\not{D})^2 = -(\partial+a)^2 - \frac{1}{2}\Sigma_{\mu\nu}f^{\mu\nu} \\ &= -\text{Tr} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\mathbf{i}\not{D}}{m}\right)^n e^{-\square/M^2} . \end{aligned}$$

where $\Sigma_{\mu\nu} \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu]$ is the rotation generator. Now expand the regulator exponential as well and

Here is the simplest route to the low-energy theory³⁵. We saw above that a useful description of the IQHE is in terms of a dynamical U(1) gauge field in terms of which the current describing the dofs forming the IQH state takes the form $j = \frac{1}{2\pi}\epsilon\partial a$, and the Lagrangian has a term $\frac{ada}{4\pi}$. So let's introduce such a gauge field for each of the three species of parton:

$$j^{(\alpha)\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu b_\rho^{(\alpha)}.$$

We also have gauge fields associated with the parton gauge redundancy (2.43)³⁶, that I'll call a^1 and a^2 . The full effective action is then

$$4\pi L = \sum_\alpha b^\alpha db^\alpha + 2A \sum_\alpha q_\alpha db^\alpha + 2a^1(db^1 - db^2) + 2a^2(db^2 - db^3) \quad (2.46)$$

where q_α are the electric charges of the partons, which satisfy $\sum_\alpha q_\alpha = 1$. $a^{1,2}$ are just Lagrange multipliers setting $b^1 = b^2$ and $b^2 = b^3$. Setting $a = b^1 = b^2 = b^3$, then, the action becomes

$$4\pi L = 3ada + 2Ada \sum_\alpha q_\alpha = 3ada + 2Ada$$

which is the effective action we advertised above.

Hence we arrive at a CS theory, like (2.14), for some particular choice of K , determined by the QH response of the partons, *i.e.* by their charges-squared times the Chern numbers of their bands.

The Hall conductivity is just sums of the contributions of the partons:

$$\sigma^{xy} = \frac{(e/3)^2}{h} \times 3 = \frac{1}{3}e^2/h.$$

The parton groundstate is $|\Phi_{mf}\rangle = \mathbb{P}|\text{free parton state}\rangle$, where \mathbb{P} is the projection onto the gauge invariant subspace, and the free parton state is obtained just by filling the lowest Landau levels of the partons. The electron wavefunction is

$$\Psi(r) = \langle 0 | \prod_i c(r_i) | \Phi_{mf} \rangle = \left(\underbrace{\prod_{i<j}^N z_{ij} e^{-\sum_i^N |z_i|^2 / (4\ell_B^2 (e/3))}}_{\nu=1 \text{ Slater det of charge } 1/3 \text{ fermions}} \right)^3 = \prod_{i<j} z_{ij}^3 e^{-\sum_i^N |z_i|^2 / (4\ell_B^2)}$$

in $D = 3$, the term that survives the trace over Dirac matrices is

$$\text{Tr} \frac{\mathbf{i}a_\rho \gamma^\rho}{m} \frac{1}{M^2} \left((\partial + a)^2 + \frac{1}{2} \Sigma_{\mu\nu} f^{\mu\nu} \right) = \text{sign}(m) \underbrace{\text{tr} \gamma_\rho \Sigma_{\mu\nu}}_{=\epsilon_{\rho\mu\nu}} \int d^3x \frac{1}{2} \mathbf{i}a^\rho f^{\mu\nu}.$$

Please see my QFT notes from s22 if that was too fast.

³⁵One place in the literature where it appears is section IV of [this paper](#).

³⁶You can ask: what happened to the rest of the SU(3)? One possible answer is that it is spontaneously broken down to this U(1)² subgroup in the state we are describing.

where $z_{ij} \equiv z_i - z_j$. This is the Laughlin wavefunction. (Note that $\ell_B^2(e) = \frac{\hbar}{eB}$, so $\ell_B^2(e/3) = 3\ell_B^2$.)

Another route to the low-energy theory is as follows. Just couple the partons to an $SU(3)$ gauge field a and

$$\text{integrate out the gapped partons: } \int [Df] e^{i \int L(f,a)} = e^{ik\text{CS}(a)+\dots}$$

The resulting low-energy effective field theory of a is $SU(3)_1$ CS theory (with gapped fermionic quasiparticles). It's a non-trivial fact that $SU(3)_1$ CS theory with gapped fermionic quasiparticles is dual to the $U(1)_3$ CS theory that we found earlier – they have the same groundstate degeneracy and anyon types and bulk response theory. I hope to explain more about this duality after we speak about invertible phases, since the derivation essentially involves subtracting an invertible phase from the BHS.

The Laughlin quasiparticle is the parton f with a Wilson line to make it gauge invariant.

$D = 2 + 1$ is kind of cheating from the point of view of emergent gauge fields. This is because the Chern-Simons term is a self-coupling of gauge fields that gives the photon a mass without the addition of degrees of freedom. We have seen above (in the toric code example) that this does not necessarily require breaking parity symmetry.

For this reason partons work extremely well to describe QH physics, but they are also useful for other kinds of quantum matter with strong correlations. For more about parton gauge theory I heartily recommend [Sung-Sik Lee's TASI 2010 lectures](#). In his lectures 2 and 3, he applies this method to bosons and to spins and provides a great deal of insight.

For a long time I thought that gauge fields were only interesting for condensed matter physics when deconfinement could be somehow achieved, *i.e.*, when there is topological order. We'll see examples in §4 where even confined emergent gauge fields can do something interesting!

Attempted parable. The parton construction is a method for ‘solving’ non-holonomic constraints, like inequalities. In what sense were we solving such a constraint above? Suppose that the nearest-neighbor repulsion V is the biggest scale in the problem. Then we want the number of electrons on each pair of neighboring sites to be ≤ 1 .

Here is a much simpler example: I can solve the condition $y > 0$ by writing $y = x^2$. So we can do a 0-dimensional path integral (integral) over $y > 0$ in terms of an

unconstrained variable x by writing

$$\int_0^\infty dy e^{-S(y)} = \frac{1}{2} \int_{-\infty}^\infty dx e^{\log|2x| - S(x^2)}.$$

In this model, the operation $x \mapsto -x$ is a gauge redundancy. In this case, it is a finite dimensional gauge group and we account for it by the factor of $\frac{1}{2}$ out front.

The extra $\log|2x|$ term in the action from the Jacobian is like a contribution from the gauge fluctuations. If I were clever enough I would illustrate deconfinement here, but I guess that isn't going to happen in zero dimensions.

[End of Lecture 8]

The parton construction makes possible

- new mean field ansatz,
- candidate many-body groundstate wavefunctions,
- good guesses for low-energy effective theory,
- accounting of topological ground-state degeneracy and edge states,
- an understanding of transitions to nearby states. (I'll give an example below.)

It has the following difficulties:

- making contact with microscopic description,
- its use sometimes requires deciding the IR fate of strongly coupled gauge theories.

Plasma analogy. [For more on this subject and the next please see [Dan Arovas' QHE notes](#).] So the Laughlin wavefunction at filling $\nu = \frac{1}{k}$ is

$$\Psi_k(z) = \prod_{i < j} z_{ij}^k e^{-\sum_i^N |z_i|^2 / (4\ell_B^2)}. \quad (2.47)$$

For even k this is a fermionic wavefunction, and for odd k this is a bosonic wavefunction. It has a number of very interesting properties. One is that correlation functions of operators diagonal in position space can be computed by a certain auxiliary classical system of N particles in 2d. The N -particle probability density, which determines all such correlations is:

$$|\Psi_k(z)|^2 = e^{-\beta h(\vec{r}_1 \cdots \vec{r}_N)}, \quad \text{with } \beta = 1/k \quad (2.48)$$

and

$$h(\vec{r}_1 \cdots \vec{r}_N) = -2k^2 \sum_{i < j} \log|r_i - r_j| + \frac{k}{2\ell_B^2} \sum_i r_i^2. \quad (2.49)$$

This is the Boltzmann distribution for a two-dimensional one-component plasma: N point charges of charge $\sqrt{2}k$ interacting via the 2d Coulomb potential ϕ that satisfies

$$\vec{\nabla}^2 \phi(r) = 2\pi\delta^2(r) \implies \phi(r) = -\log r \quad (2.50)$$

with a uniform background charge

$$\rho = -\frac{1}{\sqrt{2}\pi\ell_B^2} \implies \phi_{\text{BG}} = \frac{kr^2}{2\ell_B^2}. \quad (2.51)$$

To minimize energy the charges form a uniform density (at least at large enough β), to cancel out the background charge $nk\sqrt{2} + \rho = 0$, so that $n = \frac{1}{2\pi k\ell_B^2}$. With N particles, they fit in a disk of radius R with $\pi R^2 n = N$, so $R = \sqrt{2kN}\ell_B$.

An important such observable is the *pair density distribution*:

$$ng(r) \equiv \frac{1}{N} \left\langle \sum_{i \neq j}^N \delta^2(\vec{r} + \vec{r}_j - \vec{r}_i) \right\rangle. \quad (2.52)$$

It is proportional to the probability of finding two particles separated by the vector \vec{r} . $g(r)$ determines the static structure factor (measurable in scattering experiments) by

$$\hat{s}(k) = 1 + n \int d^2r (g(r) - 1) e^{-i\vec{k}\cdot\vec{r}}. \quad (2.53)$$

One funny thing is that at some large value of $\beta = 1/k$, this plasma crystallizes! That is, a delta-function peak forms in $\hat{s}(k)$. This is a result of Monte Carlo simulations (and the critical value of $k \sim 70$). This is a completely different phenomenon from the energetic competition between the Laughlin state and a possible Wigner crystal state of the electrons – it says that the Laughlin wavefunction itself exhibits translation-symmetry breaking. Note that this shows that topological order and ordinary symmetry breaking are not necessarily inimical.

Fermi statistics demands that $g(r)$ vanish as $r \rightarrow 0$. But the Laughlin wavefunction does even better:

$$g_{\text{Laughlin}}(r) \stackrel{r \rightarrow 0}{\equiv} c_k \left(\frac{r}{\ell_B}\right)^{2k} = c_{k+1} \left(\frac{r}{\ell_B}\right)^{2(k+1)} + \dots \quad (2.54)$$

Parent Hamiltonians. The behavior of the pair correlation function (2.54) in the Laughlin states can be used to identify a family of exact parent Hamiltonians for it, that is, nice local Hamiltonians with (very) short-ranged interactions whose groundstate

is the Laughlin state. They take the uncontroversial form $H = H_{\text{kin}} + H_{\text{int}}$, where $H_{\text{kin}} = \frac{1}{2m} \int c^\dagger (\vec{\nabla} + A)^2 c$ is the usual kinetic term and

$$H_{\text{int}} = \int d^2 r_1 d^2 r_2 n(r_1) n(r_2) v(|\vec{r}_1 - \vec{r}_2|). \quad (2.55)$$

In this expression I emphasize that the interaction potential is central, *i.e.* $v(r) = v(|r|)$. This means that its Fourier transform is of the form

$$\hat{v}(k) = \int d^2 r v(r) e^{-i\vec{k}\cdot\vec{r}} = 2\pi \int_0^\infty dr r v(r) J_0(k) = \sum_{j=0}^\infty A_j (-k^2 \ell_B^2)^j. \quad (2.56)$$

Therefore back in real space

$$v(r) = \sum_{j=0}^\infty A_j (\ell_B^2 \vec{\nabla}^2)^j \delta^2(r). \quad (2.57)$$

For a state made only from lowest Landau-level orbitals, the kinetic energy doesn't contribute. The interaction energy of any state is

$$E_{\text{int}}/N = \langle \Psi | H_{\text{int}} | \Psi \rangle = \frac{n}{2} \int d^2 r v(r) g(r) = \frac{n}{2} \sum_{j=0}^\infty A_j (\ell_B^2 \vec{\nabla}^2)^j g(r)|_{r=0}. \quad (2.58)$$

For fermions, A_j doesn't contribute. For the Laughlin state at filling $\nu = 1/k$, the A_j with $j < k$ don't contribute. Thus, for any Hamiltonian with $A_1, \dots, A_{k-1} \neq 0$, but $A_k = A_{k+1} = \dots = 0$, Ψ_k has $E_{\text{int}} = 0$ exactly. But we can choose the nonzero A_j so that $H_{\text{int}} > 0$, and thus Ψ_k is a groundstate.

Furthermore, we can argue that it's the unique groundstate (on the plane or sphere), the only homogeneous wavefunction at $\nu = 1/k$ with $E_{\text{int}} = 0$. The idea is that any such state must vanish at least as fast as z_{ij}^k as any two particles approach, and therefore must have a factor of $V(z)^k$, where $V(z) \equiv \prod_{i < j} z_{ij}$ is the vandermonde determinant. But including any higher-degree polynomial $\Psi = e^{-\frac{|z|^2}{4\ell_B^2}} V(z)^k \tilde{P}(z)$ will dilute the particles below $\nu = 1/k$. (The filling is $\nu = \frac{N(N-1)}{J}$, where J is the total angular momentum, which for a holomorphic wavefunction is the degree in λ under $z_i \rightarrow \lambda z_i$. A single power of $V(z)$ has degree $\frac{1}{2}N(N-1)/2$.)

One can interpolate at fixed ν between the H_{exact} described above and the actual Coulomb interaction Hamiltonian governing particles in a 2DEG made from an insulating heterojunction. The claim is that there is no phase transition.

Lowest-Landau-Level Projection. To actually do this it is best to project into the LLL. Let me remind you what this means. Recall that in first-quantized notation,

in a uniform magnetic field $\vec{B} = -B\hat{z}$,

$$H_{\text{kin}} = \frac{1}{2m}\Pi^2 = \hbar\omega_c(a^\dagger a + \frac{1}{2}). \quad (2.59)$$

where

$$\begin{aligned} \vec{\Pi} &\equiv \vec{p} + \frac{e}{c}\vec{A}, & \Pi &\equiv \Pi_x + \mathbf{i}\Pi_y = -\frac{\mathbf{i}\sqrt{2}\hbar}{\ell_B}a \\ \vec{\kappa} &\equiv \vec{p} + \frac{e}{c}\vec{A} - \frac{e}{c}\vec{B} \times \vec{r} & \kappa &\equiv \kappa_x + \mathbf{i}\kappa_y = \mathbf{i}\frac{\sqrt{2}\hbar}{\ell_B}b^\dagger. \end{aligned} \quad (2.60)$$

Here $[b, b^\dagger] = 1 = [a, a^\dagger]$, $[a, b] = 0 = [a, b^\dagger]$. The kinetic energy depends only on the Landau level quantum number, and is independent of the so-called ‘guiding center’ operators b . For future reference, the complex coordinates are related to these operators by

$$z = \frac{\mathbf{i}\ell_B^2}{\hbar}(\Pi - \kappa) = 2\ell_B(a + b^\dagger), z^\dagger = 2\ell_B(a^\dagger + b). \quad (2.61)$$

The derivatives are

$$\partial = \frac{1}{\sqrt{8}\ell_B}(b - a^\dagger)\bar{\partial} = \frac{1}{\sqrt{8}\ell_B}(a - b^\dagger). \quad (2.62)$$

The LLL projection $\mathcal{O} \rightarrow \mathbb{P}\mathcal{O}\mathbb{P} \equiv \mathcal{O}_L$, $\mathbb{P} \equiv |0\rangle\langle 0|$, $a|0\rangle = 0$. The resulting operator \mathcal{L}_L is still an operator acting on the guiding center dofs. Although $[\hat{x}, \hat{y}] = 0$ in the full Hilbert space,

$$[\hat{x}_L, \hat{y}_L] = \mathbf{i}\ell_B^2. \quad (2.63)$$

A better way to think about this projection is just to expand the full electron annihilation operator

$$c_x = \sum_{nk} \Phi_{nk}(x)c_{nk}, \quad \mathbb{P}c_x\mathbb{P} = \sum_k \Phi_{0k}(x)c_{0k}. \quad (2.64)$$

Recall that the LLL wavefunctions are holomorphic functions times a gaussian factor. In the LLL, the operator \bar{z}_i can be replaced by $2\ell_B^2\partial_{z_i}$.

2.4 Composite fermions and hierarchy states

The following line of thought, which allows us to understand other abelian FQH that actually occur (at fractions besides $\nu = \frac{1}{m}$) can be regarded as an important special case of the parton construction.

Consider what happens as we move away from the center of the plateau where the filling fraction is exactly $\frac{1}{m}$, say by varying the external magnetic field as in the famous

plot. The lowest-energy way to add charge is to create some charge- e/m quasiparticles, so when we change the filling by a finite amount, we produce some nonzero density of these objects, still in a large magnetic field. What do they do? They interact with each other somehow. If they are localized by disorder (*i.e.* form an Anderson insulator) or form a Wigner crystal, the Hall conductivity stays at the plateau value. Indeed at the lowest densities, charged particles with Coulomb interactions do form a Wigner crystal. But at higher densities, what's to stop them from forming their own FQH state?

There are several nice ways to describe this. One is called composite fermions, where the idea is to think of the factors in the Laughlin wavefunction z_{ij}^m as $z_{ij}z_{ij}^{m-1}$, one IQH wavefunction, and one boson $\nu = \frac{1}{m-1}$ (bosonic) Laughlin wavefunction – of the same variables. Fermi statistics of the electron require $\Psi \propto z_{ij}$, but the extra $m-1$ powers are something else. Regard the particle whose wavefunction this is as a fermion (the one in the IQH state) with $m-1$ units of some kind of flux attached (we'll see precisely what flux in a moment). Such a particle experiences a reduced magnetic field:

$$B^* = B - (m-1)\rho\Phi_0. \quad (2.65)$$

Since the number of such ‘composite fermions’ is the same as the number of electrons, we have (if $B^* > 0$)

$$\rho = \frac{\nu B}{\Phi_0} = \frac{\nu^* B^*}{\Phi_0} \quad (2.66)$$

and hence the actual filling ν is related to the filling of composite fermions ν^* by

$$\nu = \frac{\nu^*}{(m-1)\nu^* + 1}. \quad (2.67)$$

If we let the composite fermions fill $\nu^* \in \mathbb{Z}$ Landau levels (or fill bands with total Chern number ν^*), we get a gapped state with Hall conductivity given by (2.67). For $m=3$, and $\nu^* = 1, 2, 3 \dots$ this is

$$\nu = \frac{\nu^*}{2\nu^* + 1} = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9} \dots \quad (2.68)$$

You can see these plateaux in the famous plot. This picture suggests a very successful trial wavefunction for these fillings³⁷, namely:

$$\tilde{\Psi}_\nu(z) = \mathcal{P}_{LLL} \prod_{i < j} z_{ij}^2 \tilde{\Psi}_{\nu^*}(z, \bar{z}). \quad (2.69)$$

Here \mathcal{P}_{LLL} is the projector to the lowest Landau level: If we are filling multiple Landau levels, $\Psi_{\nu^* > 1}$ is no longer holomorphic; to make wavefunctions for electrons at $\nu < 1$,

³⁷Define $\Psi \equiv \tilde{\Psi} e^{-\sum_i \frac{|z_i|^2}{4t_B^2}}$ so we don't have to write the annoying gaussian factor.

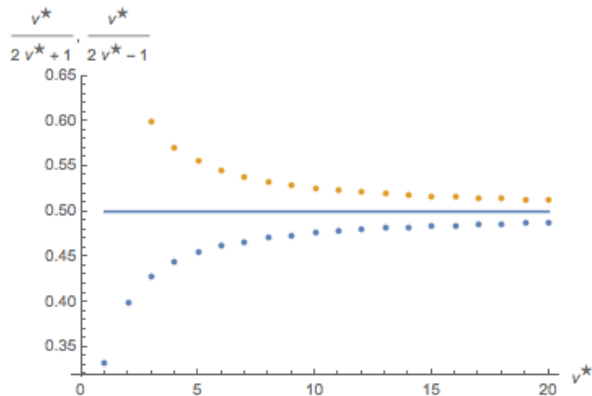
which we expect to be made just from LLL orbitals, we project this thing into the LLL by the replacement $\bar{z}_i \mapsto 2\ell_B^2 \partial_{z_i}$.

Note that there are also such ‘hierarchy states’ with fillings larger than $\frac{1}{m}$, which means $B^* < 0$. In that case (2.66) is replaced by

$$\rho = \frac{\nu B}{\Phi_0} = -\frac{\nu^* B^*}{\Phi_0} \quad (2.70)$$

so

$$\nu = \frac{\nu^*}{2\nu^* - 1} = 1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \dots \quad (2.71)$$



If we plot the fillings we achieve by this construction (2.68) and (2.71) as a function of ν^* , we get the following plot. This will be useful later. (Notice that if instead we took $m - 1$ to be some other even number as our starting point, the asymptote would be $\nu = \frac{1}{m-1}$ instead.)

Now here is an explanation of the above numerology in terms of partons. We make the following parton ansatz

$$c = fb. \quad (2.72)$$

c is the electron destruction operator, and f and b are fermionic and bosonic partons respectively. This fractionalization leads to a $U(1)$ gauge field which let's call a_1 , under which f and b have charges 1 and -1 . What are the charges of f and b under the external A ? I claim that it does not matter, as long as they add up to one. Now (here we make a choice), let's put b into a $\nu = 1/2$ Laughlin state. One way to do that is to write $b = d_1 d_2$ in terms of two more fermionic partons, introducing a second gauge field a_2 under which $d_{1,2}$ have opposite charge (note that it again doesn't matter how we distribute the a_1 charge between d_1 and d_2), and let each of d_1 and d_2 fill a Chern band. The full table of charges is

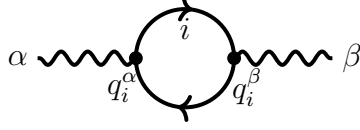
	A	a_1	a_2
d_1	q_1	0	1
d_2	q_2	1	-1
f	q_f	-1	0

Now let's integrate out d_1 and d_2 . In general, integrating out a field d_i with charges q_i^α under gauge field a^α filling chern bands with total chern number c_i leads to an effective

Lagrangian

$$L_i = \frac{c_i}{4\pi} \sum_{\alpha\beta} (q_i^\alpha a_\alpha) d(q_i^\beta a_\beta). \quad (2.73)$$

A useful mnemonic is the following diagram:



For convenience of writing I'll call $a_0 \equiv A$, so the couplings to the external gauge field are included in (2.73). We ignore any other interactions between the partons, so the contributions of multiple fields just add. So integrating out $d_{1,2}$ leaves us with

$$L_{\text{eff}} = \frac{c_1}{4\pi} (a_2 + q_1 A) d(a_2 + q_1 A) + \frac{c_2}{4\pi} (a_1 - a_2 + q_2 A) d(a_1 - a_2 + q_2 A) + L(f, a) \quad (2.74)$$

We can think of these CS terms as attaching flux to the remaining parton f , which is the composite fermion.

Now we can decide what to do with f ; depending on what we do, we'll find different states, with different Hall response. If we let f fill Chern bands with total Chern number ν^* , we can integrate out f , too, in the same way, and get $L_{\text{eff}} = \sum_i L_i$ with L_i as in (2.73). If we set $\nu^* = 1$ we reproduce exactly our earlier construction of the Laughlin state. To find the general Hall response, we can just solve the equations of motion for $a_{1,2}$; these are linear equations that determine $a_{1,2}$ in terms of A . Plugging back into (2.74) then gives an action of the form $L = \frac{\nu}{4\pi} A dA$ with, setting $c_1 = c_2 = c^{38}$,

$$\nu = \frac{c\nu^*}{c + 2\nu^*}. \quad (2.75)$$

If I set $c = 1$, this is exactly (2.68). And the parton trial wavefunction is just (2.69).

Hierarchy and K -matrices. There is another way to construct the hierarchy, by combining the logic with which we began §2.2 with the picture with which we began our discussion of the hierarchy states. So we've argued that the EFT in terms of CS gauge fields is an inevitable consequence of $U(1)$ symmetry, a gap, and broken time-reversal symmetry. That's the same situation we're in when we add a density of quasiparticles to move away from the center of the plateau (assuming we make a gapped state). So the same logic suggests that we write also the quasiparticle current

$$j_{qp}^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu \tilde{a}_\rho \quad (2.76)$$

³⁸In this calculation, it is an extremely useful check to leave the electric charges of the partons arbitrary, and make sure that the answers only depend on the sum of their charges, which is fixed by the charge of the electron.

in terms of a (new) CS gauge field! You can't stop me. The minimal-coupling term $a_\mu j_{qp}^\mu$ becomes a mixed CS term. The full action (epsilon tensors are implicit) is

$$4\pi L = kada + 2Ada + 2ad\tilde{a} + \tilde{k}\tilde{a}d\tilde{a} + \dots,$$

where I've added a CS term to describe the dynamics of the quasiparticles, by the same logic as before. This is of the form (2.14) with a 2×2 K -matrix, $K = \begin{pmatrix} k & 1 \\ 1 & \tilde{k} \end{pmatrix}$ with charge vector $t = (1, 0)$. If we integrate out \tilde{a} and a in this action, we'll find Hall conductivity

$$\nu = \frac{1}{k - \frac{1}{\tilde{k}}}.$$

If $\tilde{k} = 2$, we reproduce our previous hierarchy states. These FQH states of composite fermions are observed [here](#). This CS description determines the groundstate degeneracy (to be $|\det K|^g$ on a genus- g surface) and the charges and statistics of the quasiparticle excitations above this state: minimally couple particles to a and \tilde{a} . The quasiparticle charges that one computes this way are [observed experimentally](#), *e.g.* at $\nu = \frac{2}{5}$.³⁹

Exercise: explain in this language how to get the states with filling larger than a half that we described earlier.

Now, why should we stop here? Moving away from the middle of the plateau of one of these states, there is a density of those quasiparticles whose charges I just mentioned. Why can't they, too, form a QH state? Then we would write their current in terms of yet a third CS gauge field, and we would arrive at a description with a 3×3 K -matrix, and a Hall conductivity that was a continued fraction with a third level. You can also see these states in clean samples.

[\[End of Lecture 9\]](#)

This construction gives K -matrices with integers on the diagonal and ones on the next-to-diagonal. The K -matrix $K = \begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix}$ and charge vector $t = (1, 1)$ also describes states that exist (named after Halperin), and whose filling fraction, quasiparticle data and GSD you can now compute.

We can relate theories with different K -matrices by a relabelling of fields, $a^I \rightarrow W_J^I a^J$. But in order to preserve the flux quantization, the matrix W must have $|\det W| = 1$, that is $W \in \text{GL}(n, \mathbb{Z})$. This resulting equivalence relation acts by

$$K^{IJ} \rightarrow W_K^I K^{KL} W_L^J, \quad t^I \rightarrow W_J^I t^J. \quad (2.77)$$

³⁹A nice resource for the literature on experimental studies of FQHE up to 2004 is the slides of Willett [here](#).

Quantum Hall Metal. Now comes some real magic: look again at the limit as $\nu^* \rightarrow \infty$ of the plot above. What happens when $\nu = \frac{1}{2}$ (or more generally $\nu = \frac{1}{m-1}$) in the composite fermion construction? Look at (2.65): when $\nu = \frac{1}{m-1}$, the composite fermion sees no magnetic field at all! This is despite the fact that half-filling happens (for realistic electron densities in a 2DEG) at enormous magnetic fields like 25 Tesla. This suggests that in our parton ansatz (2.72) the composite fermion f can just fill a Fermi sea, and would move in straight lines in response to an external electric field. This would be a gapless, metallic state. And indeed that is what is seen near $\nu = \frac{1}{2}$! A trial wavefunction for this amazing state is then just

$$\tilde{\Psi}(z) = \mathcal{P}_{LLL} \prod_{i < j} z_{ij}^2 \det_{ij} e^{ik_i \cdot r_j} \quad (2.78)$$

where the $\{k_i\}$ label the N lowest energy modes. I think this wavefunction was first written down by Read and Rezayi, but Halperin, Lee and Read explained a lot of the physics, so I call it the HLR state. A similar story occurs (in experiments as well) at other fillings of the form $\nu = \frac{1}{m-1}$ for m an odd number.

But there's more: let's start at the HLR state at $\nu = \frac{1}{2}$ and vary the magnetic field (at fixed electron density) away from the special value where the composite fermions see no field $B = B_{\nu=\frac{1}{2}} + \delta B$. What happens when we subject a Fermi surface to a magnetic field? We get quantum oscillations: various quantities, including the conductivity, are periodic with period $1/\delta B$. But this periodic structure is exactly where the hierarchy plateaux appear! (That is: if $\frac{1}{\delta B} = \pm \frac{\nu^*}{\rho\Phi_0}$ for some integer ν^* , the actual filling fraction is $\nu = \frac{\nu^*}{2\nu^* \pm 1}$.) This means we can regard all of the hierarchy states, including the original Laughlin states, as extreme manifestations of quantum oscillations.

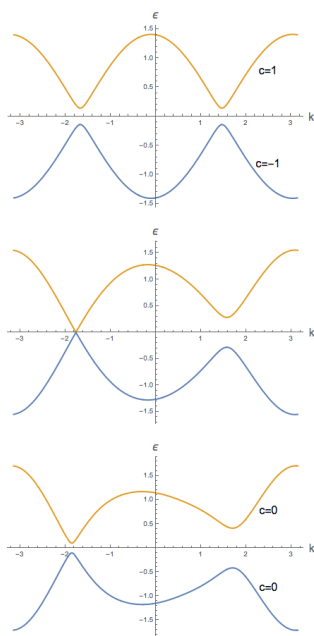
Notice that the composite fermions at the Fermi surface still interact with a Chern-Simons gauge field. This is a non-Fermi liquid: a metallic system that is not described by ordinary Fermi liquid theory.

Incompressible states at even denominators. Finally, one thing that fermions at a Fermi surface like to do is pair up and superconduct. What happens if the composite fermions in the HLR state form a superconductor? Well, by Fermi statistics, it has to be a p -wave superconductor because the composite fermions are spinless (we're assuming the electrons are completely spin-polarized). If they form a $p + \mathbf{i}p$ (nodeless) superconductor, the projection of the BCS wavefunction is the Moore-Read state, which is a non-abelian topological order. (Perhaps more on this wavefunction later. This is not how it was first discovered.) Whether or not the composite fermions pair up depends on the consequences of the CS interactions, and on microscopic details. (For an analysis of this drama, see [here](#).) In the half-filled lowest Landau level, we see a metallic state, but in the half-filled *third* Landau level (*i.e.* at $\nu = \frac{5}{2}$), there is indeed

an incompressible state, which could be the Moore-Read state. (At $\nu = \frac{3}{2}, \frac{7}{2}$, one finds instead states that break translation symmetry – stripes. For why this happens, see [Misha Fogler’s article here.](#))

Recently there has been some further development in our understanding of what happens near $\nu = \frac{1}{2}$. The state (2.78) makes a distinction between a half-full and a half-empty Landau level – it is not symmetric under the interchange of particles and holes. But experiments show that the physics is particle-hole invariant. [Son](#) suggested a way to write down a particle-hole symmetric state by starting with Dirac composite fermions, rather than the non-relativistic ones that I’ve been talking about. This line of inquiry has also had some important consequences for our understanding of the gapped and likely non-abelian state at $\nu = \frac{5}{2}$. Recent measurements (of [thermal Hall conductivity](#) (which counts the number of edge modes, including neutral ones) and of [interfaces with other states](#)) favor a particle-hole symmetric version of the Moore-Read state (called the PH-Pfaffian).

Transitions to neighboring phases. What happens if the Chern numbers of the bands occupied by the partons $d_{1,2}$ are not both 1 (let’s call it $(c_1, c_2) = (1, 1)$)?



It is not hard to describe a transition where Chern numbers change: just vary the bandstructure until the bands touch at a Dirac point. Moving past that point, the two touching bands exchange Chern numbers. (I like to think of it like a crossover event during meiosis. The pictures are very similar.) If one is filled and one is empty, this will change the resulting contribution to the Hall response. For example, imagine that in the figures at right the blue band is filled. And we see that a theory of the transition is a Dirac fermion coupled to Chern-Simons gauge fields.

Consider just the theory of a boson b , with the ansatz $b = d_1 d_2$, where d_1 and d_2 fill Chern bands with Chern numbers (c_1, c_2) . The effective Lagrangian for the gauge field a_1 gluing them together is

$$L = \frac{c_1}{4\pi} (a_1 + q_1 A) d(a_1 + q_1 A) + \frac{c_2}{4\pi} (-a_1 + q_2 A) d(-a_1 + q_2 A) . \quad (2.79)$$

Integrating out a_1 gives Hall response

$$\nu_b = \frac{c_1 c_2}{c_1 + c_2} . \quad (2.80)$$

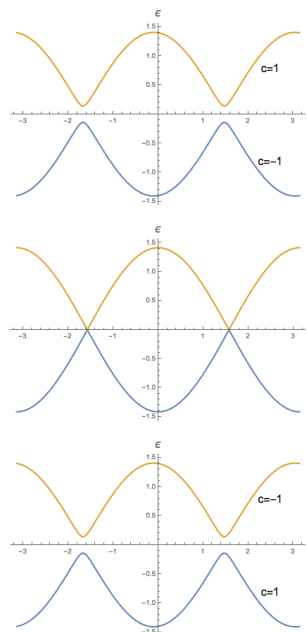
Several cases are interesting. If $(c_1, c_2) = (1, 0)$, the calculation above gives $\nu = 0$, no Hall response. But the state of the boson $b = d_1 d_2$ is still an insulator, there is still an energy gap. This is a (weird!) description of a featureless Mott insulator.

Another very interesting case is when $(c_1, c_2) = (1, -1)$, in which case (2.80) blows up. This is a signal that the theory is actually gapless. In fact it is a dual description of the superfluid phase of the boson. To see this, go back to (2.79). When $c_1 = -c_2 = 1$, it reduces to

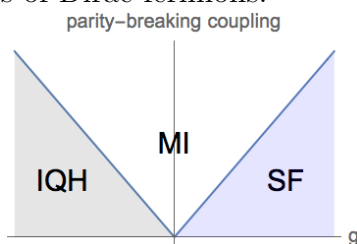
$$L = \frac{1}{2\pi} adA + f^2 \quad (2.81)$$

which says that there is no CS term for a , and I've added back the Maxwell term, since it's now the leading term governing the dynamics of a . Earlier I said that pure compact $U(1)$ gauge theory in $D = 2 + 1$ confines because of monopole instantons. But the first term in (2.81) says that magnetic flux of a carries charge under the $U(1)$ global symmetry. This means the operator $e^{i\sigma}$ is not $U(1)$ symmetric (σ shifts) and can't be added to the action. This describes a gapless theory, where the photon is the goldstone boson, $\partial a = \epsilon \partial \sigma$, for spontaneously breaking the $U(1)$ symmetry.⁴⁰

In the presence of some lattice symmetry (like parity, $k \rightarrow -k$) that forces two band-touching points at the same point in the phase diagram, it can force the Chern number of the filled band to change from $c = -1$ to $c = 1$. If this happens to d_1 (fixing $c_2 = 1$) we have a direct transition from the Laughlin $\nu = \frac{1}{2}$ state to a superfluid. Notice that the critical theory involves two species of Dirac fermions.



We can thereby produce a boson phase diagram that includes these three phases.



(The state with $(c_1, c_2) = (-2, 1)$ is also interesting – it is a boson IQH state with $\nu = 2$.)

Now we can use this to make a theory of electronic transitions out of the HLR phase, for example to a Mott insulator. In our description of the HLR phase, the Hall response comes entirely from that of b . When b goes from the $\nu = \frac{1}{2}$ state to the Mott insulator, the full electron state goes from HLR to a (fermionic) Mott insulator. (And

⁴⁰Another way to think about (2.81) is to integrate out a and see what effective action we get for A . The equation of motion says roughly $\partial^2 a \sim \partial A$, and plugging back in we get $L_{\text{eff}}[A] \sim A \partial \left(\frac{1}{\partial^2} \right) \partial A \sim A^2$, a Meissner mass for A , as we should have in a superfluid.

indeed, the general formula, if f also fills Chern bands with Chern number ν^* , is

$$\nu = \frac{c_1 c_2 \nu^*}{(c_1 + c_2) \nu^* + c_1 c_2}. \quad (2.82)$$

Again if $(c_1, c_2) = (1, 0)$, we find no Hall response.)

With the fractionalization $c = bf$, if b instead forms a superfluid state, it completely higgses the $U(1)$ gauge field gluing together b and f . The result is that we can forget about both b and the gauge field, and we get an ordinary Fermi liquid. So in this way we can describe a phase diagram containing the HLR state, Mott insulator and ordinary metal. You could imagine moving around in such a phase diagram by applying a periodic potential to a 2DEG, at fixed electron density and magnetic field. Recently such a transition seems to actually [have been realized](#) in twisted bilayer dichalcogenides. See [here](#) for an update of the theory.

