Physics 230: Quantum phases of matter
Lightning summary of algebraic topology
Spring 2024

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1 Homology

1.1 Cell complexes and homology

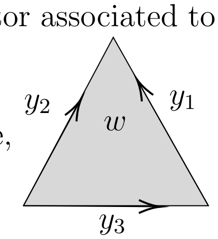
Take a d-dimensional manifold X whose topology is of interest and chop it up into simply-connected *cells*. By "simply-connected" here I just mean that each cell can be deformed into a ball. For $d = 2$ *e.g.* this means a triangulation (or squarulation or $\cdots$) into a set of 2-cells which are triangles (or squares...), 1-cells which are intervals, and 0-cells which are points. It is what physicists might call a lattice, though no translation symmetry is actually required or assumed here. But it has more structure – it knows how it is glued together. This gluing data is encoded in a *boundary map* ∂, which we define next. Let Δ_k be the set of k-cells in the triangulation of X, and choose an an abelian group A (*e.g.* $\mathbb{Z}_2$). Define a vector space

$$\Omega_k \equiv \Omega_k(\Delta, A) \equiv \text{span}_A\{\sigma \in \Delta_k\}$$

to be spanned by vectors associated with k-cells σ, with coefficients in A. (We are writing the group law of A additively, so $e.g.$ for $\mathbb{Z}_2$ it is $1+1=0$.) It does no harm to introduce an inner product where these vectors σ are orthonormal. An element $C \in \Omega_k$ is then a formal linear combination of k-cells, and is called a k-*chain* – it's important that we can *add* (and subtract) k-chains, $C + C' \in \Omega_k$. A k-chain with a negative coefficient can be regarded as having the opposite orientation.

The boundary map takes the vector space Ω_k to the corresponding vector space for the (k-1)-cells, Ω_{k-1}:

$$\partial_k : \Omega_k \to \Omega_{k-1}$$

This map ∂ is linear and takes a basis vector associated with a k-cell to the linear combination (with signs for orientation and multiplicity) of cells the union of which lie in its boundary. (And it takes a basis vector associated to a *collection* of k-cells to the sum of vectors.) For example, in this figure, 

we have $\partial w = y_1 - y_2 + y_3$ (where I denote the vectors associated with the simplices by the names of the simplices, why not?). This construction is called a cell complex[1]. A chain C satisfying $\partial C = 0$ is called a *cycle*, and is said to be *closed*.

[1] There are many very closely related constructions (such as simplicial complex or Δ-complex or semi-simplicial complex or CW-complex) but I will not distinguish between them. One distinction is that we don't care that all the cells are triangles or their higher-dimensional generalizations.

The fact that the boundary of a boundary is empty makes this series of vector spaces connected by linear maps into a chain complex, meaning that $\partial^2 = 0$. So the image of $\partial_{p+1} : \Omega_{p+1} \to \Omega_p$ is a subspace of $\ker(\partial_p : \Omega_p \to \Omega_{p-1})$. This allows us to define the *homology* of this chain complex – equivalence classes of p-cycles, modulo boundaries of $p+1$ chains:

$$H_p(\Delta, A) \equiv \frac{\ker(\partial : \Omega_p \to \Delta_{p-1}) \subset \Omega_p}{\mathrm{im}\,(\partial : \Omega_{p+1} \to \Omega_p)}.$$

These objects depend only on the topology of X and not on how we chopped it up. Below we'll discuss several points of view on this independence of homology on the triangulation.

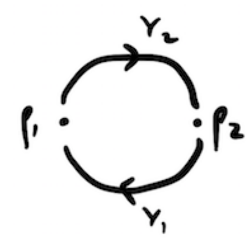
$H_p(\Delta, A)$ is a vector space over A. In the case when A is a field (such as $\mathbb{Z}_p$ for p prime) the dimensions of these vector spaces over A are called the Betti numbers of X. When A is not a field there can be more information called *torsion*, which we'll discuss.

Note that $H_p(X, A)$ is also itself a group. The group law is just addition of representatives: if C and C' are cycles, then the sum of their equivalence classes modulo boundaries is $[C]+[C'] = [C+C']$. This is independent of the choice of representatives.

1.2 More examples than you need

The simplest possible example is complex with only a single 0-cell, a point. This has $H_0(\mathrm{pt}, A) = A$, and all other $H_{n>0}$ vanish. If our cell complex were k 0-cells, we would find $H_0(k \text{ pts}, A) = A^k$ in agreement with the discussion above about ferromagnets.

Circle. Consider the cell complex at top right. This is a cellulation of a circle with one 1-cell and one 0-cell. The boundary map is $\partial e_1 = e_0 - e_0 = 0$. The kernel is everyone and the image is no one. So the homology (with integer coefficients) is $H_0(S^1, A) = A = H_1(S^1, A)$. Another cell decomposition of the circle is the bottom figure at right. Now there are two 1-cells and two 0-cells with boundary map $\partial y_1 = p_1 - p_2 = -\partial y_2$. Now the complex looks like



$$0 \to A^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} A^2 \to 0.$$

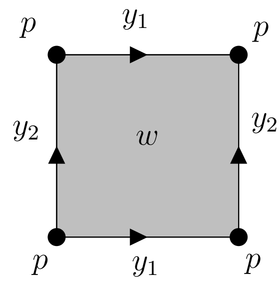
The kernel of ∂ is generated by $y_1 + y_2$. The complement of the image is generated by $p_1 \simeq p_2 \bmod \partial$. So we find the same answer for the homology as above, $b_0(S^1) = b_1(S^1) = 1$.

Ball. Consider what happens if we add to the first example a 2-cell e^2 whose boundary is e^1 – *i.e.* fill in the interior of the circle in the picture. This makes a cellulation of a 2-ball. The complex is

$$0 \to A \xrightarrow{1} A \xrightarrow{0} A \to 0.$$

In that case, $e^1 \in \mathrm{Im}\,\partial_2$, so it kills the first homology – Ω_1 and Ω_2 eat each other. This complex has the same homology as a point. We'll see later that this is because they are related by homotopy – a family of continuous maps which starts at one and ends at the other.

An important point: a demand we make of our cellulations is that each k-cell is topologically a k-ball.

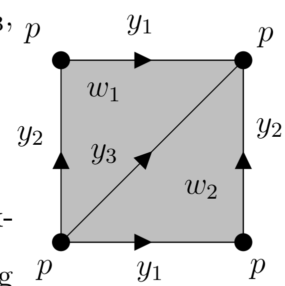


Torus. Consider the cell complex at right: It has one 2-cell w, two 1-cells y_1, y_2 and two 0-cells p_1, p_2. Opposite sides are identified. This is a minimal cell complex for the 2-torus, $T^2 = S^1 \times S^1$.
The boundary map on 2-cells is $\partial w = y_2 + y_1 - y_2 - y_1 = 0$. One 1-cells it is $\partial y_1 = p - p = 0, \partial y_2 = p - p = 0$.

All the boundary maps are zero! The chain complex is

$$0 \to A \xrightarrow{0} A^2 \xrightarrow{0} A \to 0$$

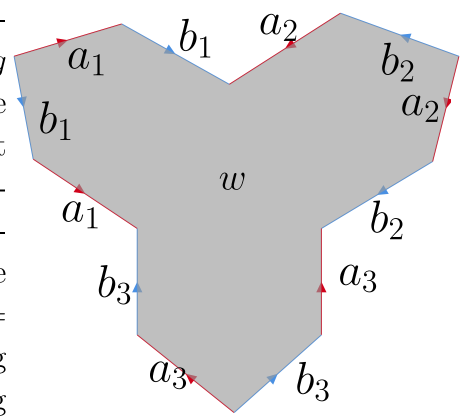
This means that every generator of the cell complex is a generator of homology, and we have $H_0(T^2, A) = A, H_1(T^2, A) = A^2, H_2(T^2, A) = A$ (the betti numbers are $b_0(T^2) = 1, b_1(T^2) = 2, b_2(T^2) = 1$.
We can also choose a less-minimal cellulation, as at right. The boundary maps are $\partial w_1 = y_3 - y_1 - y_2, \partial w_2 = y_1 + y_2 - y_3$, $\partial y_i = 0$. Now the complex is

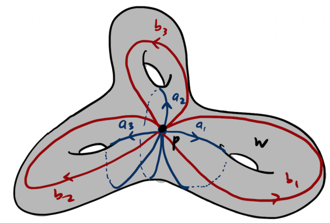


$$0 \to A^2 \xrightarrow{\partial_2} A^3 \xrightarrow{0} A \to 0$$

with $\partial_2 = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$. Clearly ∂_2 has rank 1, so the extra 2-chain and the extra 1-chain just eat each other leaving behind the same homology as before.

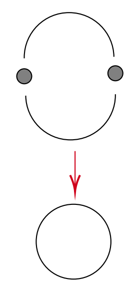
More generally, we can make a cellulation of a genus g Riemann surface Σ_g using a single plaquette, $2g$ 1-cells, and a single 0-cell. (The torus is the case $g = 1$.) At right is a cellulation of a genus 3 Riemann surface. Again the boundary maps are all trivial, and we see that $b_0(\Sigma_g) = 1 = b_2(\Sigma_g), b_1(\Sigma_g) = 2g$. You can see that we're losing some information here by choosing an abelian group.

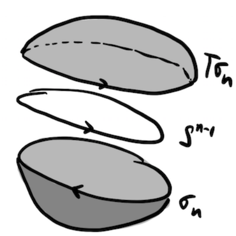




Spheres. Generalizing in another direction, we can make a sphere $S^n, n \geq 1$ starting with an n-dimensional ball B_n– a single n-cell – and identifying all the points on its boundary[2] to make a single 0-cell: $S^n = B_n/\partial B_n$. The boundary map for this complex is again trivial. So $b_0(S^n) = b_n(S^n) = 1$ and all others are zero.

Alternatively, we can make a sphere iteratively. Start with an S^0 (two points ($S^0 = \{x|x^2 = 1\}$) which I'll call σ_0 and $T\sigma_0$, where T stands for anTipodal map), and glue in two 1-cells (intervals, B_1, which I'll call σ_1 and $T\sigma_1$) as in the figure at right, so that $\partial\sigma_1 = \sigma_0 - T\sigma_0$ and $\partial(T\sigma_1) = T\sigma_0 - \sigma_0$. This makes an S^1 as before. Now glue on two 2-cells (disks, B_2, which I'll call σ_2 and $T\sigma_2$) so that $\partial\sigma_2 = \sigma_1 + T\sigma_1 = -\partial(T\sigma_2)$. You see that this can go on forever with an alternation in the sign $\partial\sigma_k = \sigma_{k-1} + (-1)^k T\sigma_{k-1}$ so that $\partial^2 = 0$.





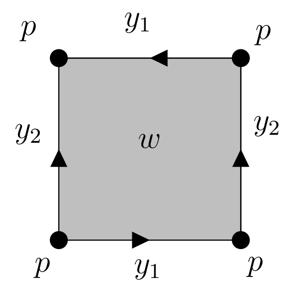
For example, for the 4-sphere we find the complex

$$0 \to A^2 \overset{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}}{\to} A^2 \overset{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{\to} A^2 \overset{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}}{\to} A^2 \overset{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{\to} A^2 \to 0.$$

Now the boundary map in each dimension but the first and last has a 1d kernel and a 1d image, so no homology. So we get the same homology as above.

[2]Note that a single n-cell is not by itself an acceptable complex, since that n-cell has a boundary and the boundary map needs somewhere to go.

An example with torsion. Consider the cell complex at right: It has one 2-cell w, two 1-cells y_1, y_2 and two 0-cells p_1, p_2. Opposite sides are identified, but top and bottom are identified with a twist. This is a minimal cell decomposition for the Klein bottle, an example of an unoriented closed surface.



The boundary map on 2-cells is $\partial_2 w = y_2 + y_1 - y_2 + y_1 = 2y_1$. On 1-cells it is $\partial_1 y_1 = p - p = 0 = \partial_1 y_2$.

Here is the first place where we have to say something about the choice of A. If our coefficient group were $\mathbb{Z}_2$, the map ∂_2 would just be zero, and we would find the same answers for $H_{0,1,2}(\Delta, \mathbb{Z}_2)$ as for the torus. With *e.g.* $\mathbb{Z}_3$ coefficients, however, $2y_1 = y_1$ mod 3, so we find no generator of $H_2(\Delta, \mathbb{Z}_3)$, and only one generator of $H_1(\Delta, \mathbb{Z}_3)$. With integer coefficients, we find

$$H_2(\Delta, \mathbb{Z}) = 0, H_1(\Delta, \mathbb{Z}) = \langle y_1, y_2 | 2y_1 = 0 \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}, H_0(\Delta, \mathbb{Z}) = \mathbb{Z} = \langle p \rangle \, .$$

(Here I am using an additive notation for these abelian groups, since we add the coefficients.) The finite-group summands are called torsion homology.

With $A = \mathbb{Z}_6$ we find

$$H_2(\Delta, \mathbb{Z}_6) = \langle 3w | 6w = 0 \rangle = \mathbb{Z}_2, H_1(\Delta, \mathbb{Z}_6) = \langle y_1, y_2 | 2y_1 = 0 \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_6, H_0(\Delta, \mathbb{Z}_6) = \mathbb{Z}_6 = \langle p | 6p = 0 \rangle \, .$$

The reason $A = \mathbb{Z}_6$ can detect the torsion is because $\mathbb{Z}_6$ contains zero-divisors, a nontrivial *torsion subgroup* $TG = \{g \in G | ng = 0, n \geq 1\}$. In contrast, if we choose the abelian group to be a field (such as $\mathbb{Z}_p$ with p prime or the rationals $\mathbb{Q}$), which by definition has no zero-divisors, the information about torsion is lost, as you can see in the examples above.

You can see that the homology with coefficients in $\mathbb{Z}_n$ is not just the integer homology mod n. Below I'll say a little more about how they are related.

It is sometimes useful to think about the data specifying the boundary map as an *attaching map* describing how the cell complex is assembled starting from the 0-cells and working up in dimension, as the following examples illustrate. These examples also show that torsion homology can occur for oriented manifolds.

$\mathbb{RP}^n$**.** Real projective space $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Such a line is specified by a vector up to rescaling by a nonzero real number: $\mathbb{RP}^n = \{\vec{v} \in \mathbb{R}^{n+1}\} / (\vec{v} \sim \lambda \vec{v}), \lambda \in \mathbb{R} \backslash \{0\}$. By rescaling, we can pick a gauge where $|\vec{v}| = 1$; this leaves just the sign of λ unfixed, so $\mathbb{RP}^n = S^n / (\hat{v} \sim -\hat{v})$ – the sphere with antipodal points identified. The upper hemisphere (a B_n) is a fundamental domain for this $\mathbb{Z}_2$ action, but the $\mathbb{Z}_2$ still acts on the equator, which is a S^{n-1}:

$$\mathbb{RP}^n = B_n / \left(\hat{v} \sim -\hat{v} \text{ on } \partial B_n = S^{n-1} \right) .$$

So we see that the boundary of the ball is itself $\mathbb{RP}^{n-1}$.

So we obtain a cell complex for $\mathbb{RP}^n$ from one for $\mathbb{RP}^{n-1}$ by attaching a single n-cell. What is the attaching map? Well, we're going to again divide up S^{n-1} into two hemispheres, each of which will be associated with a single $(n-1)$-cell, σ_{n-1}. This one $(n-1)$-cell is obtained from the cell complex we made above for S^n by identifying its two $(n-1)$-cells, σ_{n-1} and $T\sigma_{n-1}$. There is one tricky point about the orientation here. Let's do the first couple: $\partial\sigma_1 = \sigma_0 - T\sigma_0 = \sigma_0 - \sigma_0 = 0$. But as you can see from the figure at right $\partial\sigma_2 = \sigma_1 + T\sigma_1 = 2\sigma_1$.

In fact $\partial\sigma_3 = \sigma_2 - T\sigma_2 = \sigma_2 - \sigma_2 = 0$ – it couldn't be a plus sign because then we'd get $\partial^2\sigma_3 = 4\sigma_1 \neq 0$, not a chain complex. The point is that the antipodal map in dimension n reverses the orientation if n is even. So $\Delta(\mathbb{RP}^n) = \sigma_0 \cup \sigma_1 \cdots \cup \sigma_n$ where $\partial\sigma_i = (1+(-1)^i)\sigma_{i-1}$. Torsion up the wazoo. So the complex is

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

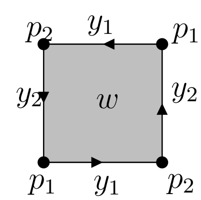
This gives

$$H_i(\mathbb{RP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}_2, & i \text{ odd}, < n, \\ \mathbb{Z}, & i = n, n \text{ odd}, \\ 0, & \text{else} \end{cases}.$$

You can check this answer for $n = 2$ with the cell complex at right, which gives the complex

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z}^2 \to 0$$

with $\partial_2 = (2, 2), \partial_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.



This is overkill on this example, but one way to compute the homology of a complex is using the software Macaulay2. Here are the necessary commands for this example, with integer coefficients:

```
d1=matrix{{1,-1}, {-1,1}}
d2=matrix{{2},{2}}
C = new ChainComplex; C.ring = ZZ;
C #0 = target d1; C #1 = source d1; C #2 = source d2;
C.dd #1 = d1; C.dd #2 = d2;
answer = HH C;
prune answer
```

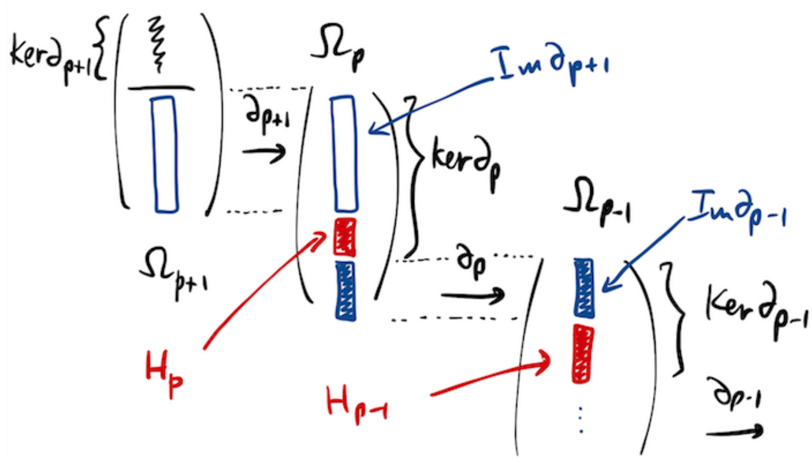
Incidentally, the group manifold of the rotation group $\mathsf{SO}(3)$ is $\mathbb{RP}^3$.

$\mathbb{CP}^n$**.** Complex projective space $\mathbb{CP}^n$ is the space of complex lines (copies of $\mathbb{C}$) through the origin in $\mathbb{C}^{n+1}$, $\mathbb{CP}^n = \{\vec{z}\}/\left(\vec{z} \sim \lambda\vec{z}\right), \lambda \in \mathbb{C}\setminus\{0\}$. We can choose a gauge where $|\vec{z}| = 1$, leaving just a phase ambiguity: $\mathbb{CP}^n = S^{2n+1}/\left(\vec{z} \sim \lambda\vec{z}\right), |\lambda| = 1$. To fix the phase, consider the region where $z^{N+1} \neq 0$. Then we can use λ to set $z^{N+1} > 0$, so that a general point is of the form $\vec{z} = (\vec{w}, \sqrt{1 - |\vec{w}|^2}), |\vec{w}|^2 \leq 1$. But the set of points $\{\vec{w} \in \mathbb{C}^n, |\vec{w}|^2 \leq 1\}$ is a B_{2n}. Its boundary occurs when $z^{N+1} = 0$, which means $|\vec{w}| = 1$, which is an S^{2n-1}. On this locus, the phase redundancy still acts. So:

$$\mathbb{CP}^n = B_{2n}/\left(\hat{w} \sim \lambda\hat{w} \text{ on } \partial B_{2n} = S^{2n-1}\right).$$

Therefore the boundary is a copy of $\mathbb{CP}^{n-1}$. So a cell complex for $\mathbb{CP}^n$ is $\Delta(\mathbb{CP}^n) = \sigma_0 \cup \sigma_2 \cup \cdots \cup \sigma_{2n}$, and the boundary map is just zero. $b_i(\mathbb{CP}^n) = 1$ for i even and $b_i(\mathbb{CP}^n) = 0$ for i odd.

At right is a visualization of homology which I find useful.



1.3 Euler-Poincaré theorem

The euler character is

$$\chi(X) \equiv \sum_{p=0}^{d}(-1)^p I_p = \sum_{p=0}^{d}(-1)^p b_p.$$

Here I_p is simply the *number* of p-simplices in the triangulation. We've seen that this is sometimes saturated by the minimal cellulation, *i.e.* no cancellation is required at all.

Proof: $I_p = \dim\Omega_p = \dim\ker\partial_p + \dim\mathrm{Im}\,\partial_p$. This is made clear by the visualization above. Now when we add these up with alternating coefficients, we get the alternating sum of the betti numbers $b_p = \dim\ker\partial_p - \dim\mathrm{Im}\,\partial_{p+1}$, using the fact that $0 =$

$\dim \mathrm{Im}\, \partial_{d+1}$. This gives a proof that the euler character is a topological invariant, independent of the triangulation. ■

1.4 Exact sequences

Suppose we have a short exact sequence of chain maps:

$$0 \to A_\bullet \stackrel{i}{\to} B_\bullet \stackrel{\pi}{\to} C_\bullet \to 0.$$

'Chain maps' means that they commute with the boundary operator ∂. 'Exact sequence' means that the kernel of one map is equal to the image the previous: $\ker(i) = 0, \ker(\pi) = \mathrm{im}(i), \mathrm{im}(\pi) = C$. The $\bullet$ just means that we have such a map for each value of the label.

Such a short-exact sequence on the chains produces a long exact sequence on the homology:

$$\ldots \stackrel{\partial_\star}{\to} H_p(A) \stackrel{i_\star}{\to} H_p(B) \stackrel{\pi_\star}{\to} H_p(C) \stackrel{\partial_\star}{\to} H_{p-1}(A) \stackrel{i_\star}{\to} \cdots$$

is exact. This involves three statements: exactness at each of the three kinds of nodes. The only tricky part is the definition of the 'connecting homomorphism' $\partial_\star$. For more, see §1.4 of these notes.

2 Cohomology

2.1 de Rham cohomology

[Bott and Tu, early sections, Polchinski §B.4, Nash and Sen §2.3] A p-form on a smooth manifold $\mathcal{M}$ is made from a completely-antisymmetric p-index tensor $A_{i_1 \cdots i_p}$

$$A = \frac{1}{p!} A_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{2.1}$$

Here dx^i are coordinate differentials, *i.e.* a basis of cotangent vectors associated with a set of coordinates on $\mathcal{M}$ ($dx^i(\partial_{x^j}) = \delta^i_j$)[3]. The set of p-forms on $\mathcal{M}$ (perhaps with some smoothness and integrability properties) is a vector space $\Omega^p(\mathcal{M})$. The coefficients can

[3]In case you are not familiar with these notions, here is brief recap to explain the notation. A tangent vector on a manifold $\mathcal{M}$ at a point p, $v \in T_p\mathcal{M}$, is of the form $v = v^i \frac{\partial}{\partial x^i}$ in terms of some local coordinates x^i. It is a differential operator in the following sense. For any function f and for any curve $x^i(t)$ through p, the rate of change of f along the curve is

$$\frac{df}{dt}|_p = \frac{dx^i}{dt} \frac{\partial}{\partial x^i} f|_p.$$

be from any field, but we will think mostly about $\mathbb{R}$. These vector spaces enjoy a product, the wedge product, which we've already written in (2.1). A good way to define the wedge product is in terms of the coordinate differentials: the wedge product $dx^{i_1} \wedge dx^{i_p}$ is completely antisymmetric and separately linear in each factor.

$$(A_p \wedge B_q)_{i_1 \cdots i_{p+q}} = \frac{(p+q)!}{p!q!} A_{[i_1 \cdots i_p} B_{i_{p+1} \cdots i_{p+q}]}$$

where the square brackets indicate antisymmetrization of indices, *i.e.* average over permutations weighted by $(-1)^\sigma$, the sign of the permutation. The wedge product is graded antisymmetric, meaning

$$A_p \wedge B_q = (-1)^{pq} B_q \wedge A_p.$$

The exterior derivative is a linear differential operator $d : \Omega^p \to \Omega^{p+1}$ defined by $d = dx^\nu \wedge \partial_\nu$, or more explicitly

$$(dA_p)_{i_1 \cdots i_{p+1}} = (p+1)\partial_{[i_1} A_{i_2 \cdots i_{p+1}]}.$$

It satisfies $d^2 = 0$ by equality of mixed partials: $[\partial_i, \partial_j] = 0$ on smooth functions.

The main job in life of a p-form on $\mathcal{M}$ is to be integrated over p-dimensional submanifolds $X_p \subset \mathcal{M}$: $\int_{X_p} A_p$ is a coordinate-invariant number. Stokes' theorem says

$$\int_{X_p} dA_{p-1} = \int_{\partial X_p} A_{p-1}.$$

So far, the metric has not been involved. The Hodge star operation requires the metric:

$$(\star\omega)_{i_{p+1} \cdots i_n} = \frac{\sqrt{\gamma}}{p!} \epsilon_{i_1 \cdots i_n} \omega^{i_1 \cdots i_p}.$$

(Indices are raised using the metric, $\omega^i \equiv g^{ij}\omega_j$.)

To get some familiarity with the above language let's think about the case $\mathcal{M} = \mathbb{R}^3$ for a moment. Then $\Omega^0(\mathbb{R}^3)$ and $\Omega^3(\mathbb{R}^3)$ are both spanned by ordinary functions, while $\Omega^1(\mathbb{R}^3)$ and $\Omega^2(\mathbb{R}^3)$ are both spanned by vector fields – functions with a single index. On functions, $df = \partial_i f dx^i$. On 1-forms,

$$d(f_i dx^i) = (\partial_y f_z - \partial_z f_y)\, dy \wedge dz + (\partial_x f_y - \partial_y f_x)\, dx \wedge dy + (\partial_z f_x - \partial_x f_z)\, dz \wedge dx = \frac{1}{3!} \epsilon_{ijk} \partial_i f_j \epsilon_{ilm} dx^l \wedge dx^m.$$

Since this is true for any $\frac{dx^i}{dt} = v^i$ and any point p and any f, the important part is the $\frac{\partial}{\partial x^i}$.

The second ingredient is that a cotangent vector is an element of the dual vector space $T_p^\star \mathcal{M}$ – an object which eats a tangent vector and gives a number. A basis for such things is given by the coordinate differentials dx^i, which satisfy $dx^i(\partial_{x^j}) = \delta^i_j$.

On 2-forms

$$d(f_x dy \wedge dz + f_y dz \wedge dx + f_z dx \wedge dy) = \partial_i f_i dx \wedge dy \wedge dz.$$

So this accounts for all the classic operations of vector calculus:

$$d(\text{0-form}) = \text{gradient}, \quad d(\text{1-form}) = \text{curl}, \quad d(\text{2-form}) = \text{divergence}.$$

A classical physics context where one encounters a cohomological question is in fluid dynamics: given a vector field, say describing the flow of a fluid on some space X, when is it the gradient of a well-defined function on X? Or in electrostatics on some space X, an allowed electric field configuration must be the gradient of a scalar potential on $X\setminus$ the locations of the charges.

The de Rham cohomology $H^\bullet(X)$ of a smooth manifold X of dimension n is the cohomology of the complex

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(X) \to 0.$$

That is

$$H^p(X) \equiv \frac{\ker\, d : \Omega^p \to \Omega^{p+1}}{\text{im}\, d : \Omega^{p-1} \to \Omega^p}.$$

It is called *co*homology rather than homology because d *increases* the index p. More significantly there is some reversal of arrows relative to homology: 'cohomology is a contravariant functor'.

There are close relations between homology and cohomology, which come under the name 'Poincaré duality'. See section 2.5 of these notes for more.

2.2 Cech cohomology

Another, more powerful, cohomology theory is defined as follows.

The simplest version of the idea is to think about *locally constant* functions on patches. Locally constant means that on each connected component of its domain, the function takes a constant value. Cover the manifold X with open sets U_α. These open sets intersect in e.g. $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta$, and triple-intersections $U_{\alpha\beta\gamma} \equiv U_\alpha \cap U_\beta \cap U_\gamma$ and so on. Define C^k to be the vector space of A-valued locally-constant functions on the disjoint union of the $(k+1)$-overlaps for some abelian group A:

$$C^k \equiv \{\text{locally constant functions}, \coprod_{\alpha_0\cdots\alpha_k} U_{\alpha_0\cdots\alpha_k} \to A\}.$$

So an element of C^0 just assigns an element of A to each U_α. Then there is an analog of the boundary map (actually coboundary map, since it goes in the other direction) which makes this into a chain complex, $\delta : C^k \to C^{k+1}$. It is defined as a difference of restrictions, as follows. For example, given $f : U_{\alpha\beta} \to A$ on all double-overlaps, this defines a function $f : U_{\alpha\beta\gamma} \to A$ on all triple-overlaps just by restriction. The coboundary map $\delta : C^0 \to C^1$ is

$$(\delta f)_{\alpha\beta} = f_\alpha - f_\beta.$$

The idea is that δf checks agreement, *i.e.* whether or not f can be regarded as a function on the union. The map for $C^1 \to C^2$ is:

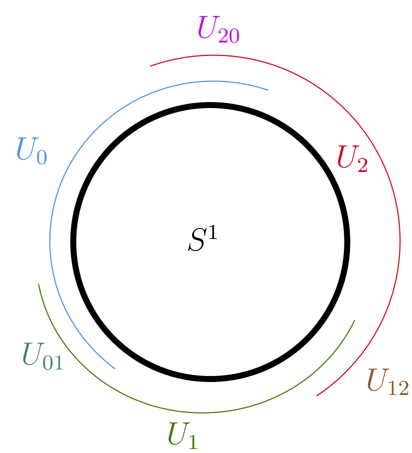
$$(\delta f)_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma}.$$

Note that we define $f_{\alpha\gamma} \equiv -f_{\gamma\alpha}$. There is a similar definition for general k, so that $\delta^2 = 0$, and the cohomology of the complex is well-defined, and again is a topological invariant (actually the same data as above). The definition for general k is

$$(\delta f)_{\alpha_1 \cdots \alpha_{k+1}} = \sum_i (-1)^i f_{\alpha_1 \cdots \widehat{\alpha}_i \cdots \alpha_{k+1}}$$

where $\widehat{\alpha}$ indicates that α is missing. In this expression, we've chosen an (arbitrary) order for the subsets.

Cech cohomology is very simple to actually calculate in reasonable examples. Consider a cover of a circle by three patches U_0, U_1, U_2, with overlaps U_{01}, U_{12}, U_{20}. The space of 0-cochains is $C^0 = \{\omega_\alpha, \alpha = 0,1,2 | \omega_\alpha \text{ is constant on } U_\alpha\} = A^3$, while the space of 1-cochains is $C^1 = \{\eta_{\alpha\beta}, \alpha, \beta = 0,1,2 | \eta_{\alpha\beta} \text{ is constant on } U_{\alpha\beta}\} = A^3$. There are no triple-overlaps (and because the space is one-dimensional, there can be no triple-overlaps in any open cover) so $C^2 = 0$.



The coboundary map $\delta : C^0 \to C^1$ acts by $(\delta\omega)_{\alpha\beta} = \omega_\alpha - \omega_\beta$. The Cech complex is

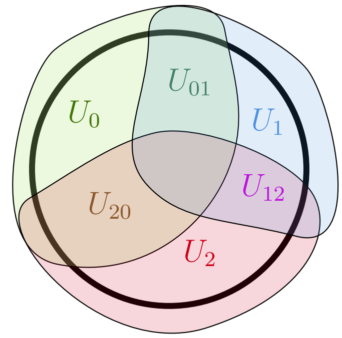
$$0 \to A^3 \xrightarrow{\delta} A^3 \to 0$$

with

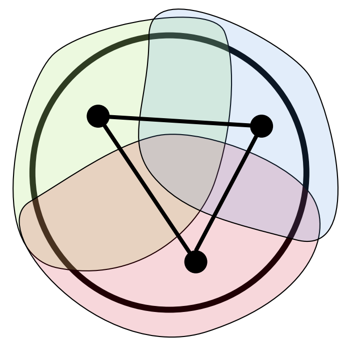
$$\delta = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

More, explicitly $H^0(S^1) = \ker(\delta) = \{\omega_0 = \omega_1 = \omega_2\} = A$. And $H^1(S^1) = A^3/\mathrm{im}(\delta) = A$. A 1-cocycle $\eta = (\eta_{01}, \eta_{12}, \eta_{20})$ is a coboundary if $\eta_{01} + \eta_{12} + \eta_{20} = 0$. So a generator of $H^1(S^1)$ is of the form $(g, 0, 0)$ where $A = \langle g \rangle$.

Not all covers of a manifold will give the same answer. A *good cover* is one for which every intersection $U_{\alpha_1 \cdots \alpha_k}$ is topologically a ball. So the example above is a good cover of the circle. An example of good cover of the 2-sphere has four open sets. One is the northern hemisphere. The other three each cover an enlarged one-third pie-slicing of the southern hemisphere, as at right.



A good cover of $\mathcal{M}$ is associated with a cell decomposition of $\mathcal{M}$: associated a 0-cell to each open set, if $U_{\alpha\beta}$ is non-empty, a 1-cell connects the 0-cells α and η. If $U_{\alpha\beta\gamma}$ is non-empty, we fill in the face of the triangle $\alpha\beta\gamma$ with a 2-cell. Keep going. There is a close relation between δ and the boundary map for this cell complex. On the homework you can work out the Cech homology for the 2-sphere and you will see close parallels with the homology of the tetrahedron.



3 Homotopy

[Hatcher, beginning] First some definitions. A *homotopy* is a family of maps $f_t : X \to Y, t \in I$ (I is the interval) such that

$$f : \begin{array}{ccc} X \times I & \to & Y \\ (x,t) & \mapsto & f_t(x) \end{array}$$

is continuous. (In the following everything in sight is assumed to be continuous.) Two maps $f_{0,1} : X \to Y$ are said to be *homotopic* if there exists a homotopy f_t with the obvious boundary conditions. In this case we will write $f_0 \simeq f_1$.

An important class of examples is the following. A *deformation retraction* of X into $A \subset X$ is a homotopy from $f_0 = \text{id} : X \to X$ to $f_1 =$ a *retraction* $r : X \to X$ with $r(X) = A$ and $r^2 = r$ like a projector. For example, we can find a deformation retraction of a disk X to a point A, or an annulus X to a circle A, or a disk with two holes to ∞ or θ or two circles attached by a line segment, like eyeglasses:

There are retractions which are not deformation retractions, such as from $X =$ two points to $X =$ one point, or from $X =$ annulus to $X =$ one point.

An important definition: X is *homotopy equivalent* to Y (or X and Y have the same homotopy type or $X \simeq Y$) if there exist $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity map.

For example, if X deformation retracts to $A \subset X$ via $f : X \times I \to X$ with $r : X \to A$ the retraction and $i : A \to X$ the inclusion, then $ri = 𝟙$, and $ir \simeq 𝟙$ by the homotopy f. Therefore $X \simeq A$. So the two-hole disk and ∞ and θ and the eyeglasses are all homotopy equivalent. Another example is $\mathbb{R}^n$ and the n-dimensional open ball.

As the name suggests, homotopy equivalence is an equivalence relation, as you can check. Deformation retraction is not. $X \simeq Y$ iff there exists Z which deformation retracts to X or Y. If $X \simeq$ a point, then we say X is *contractible*.

3.1 Notions of 'same'

There are many possible notions of when two spaces are 'the same'. They differ by what structure we care about. For example, if we are interested in spaces with metrics, we regard two spaces to be equivalent if they are related by an invertible isometry – a smooth map which preserves the metric. If we are just interested in doing calculus, two smooth manifolds are equivalent if they are related by an invertible smooth map (diffeomorphism). If we are only doing topology, continuous maps – homeomorphisms are enough.

Homotopy vs homeomorphism. Homotopy equivalence is yet another notion of equivalence, which depends only on continuity, but is weaker than homeomorphism. There are manifolds which are homotopy equivalent but not homeomorphic: a ball and a point are homotopy equivalent by the deformation retraction, but they have different dimension. The dimension of a manifold is a homeomorphism invariant, but not a homotopy invariant.

3.2 Homotopy equivalence and homology/cohomology

If two manifolds are homotopy equivalent, they have the same homology and cohomology groups. See section 3.2 and 3.3 of these notes for proofs of these statements.

3.3 Homotopy groups

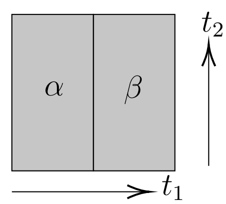
Let X be a topological space with a *base point*, $p \in X$, just a point in X that we like for some reason. The homotopy groups of X are:

$$\pi_q(X) \equiv \text{homotopy classes of maps} : (I^q, \partial I^q) \to (X, p).$$

Here $I^q \equiv I \times I \times \cdots I$ is the inside of the q-dimensional cube, homotopic to a q-ball, and its boundary is the unit cube, homotopic to a $(q-1)$-sphere. Since all points in ∂I^q map to the same point in X, an equivalent definition would consider maps from $I^q/\partial I^q \simeq S^q$ taking the north pole to the base point.

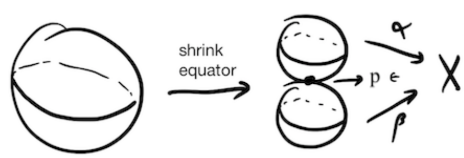
For $q > 0$, π_q is a group under the following product operation: Given α, β : $(I^q, \partial I^q) \to (X, p)$ (so that $[\alpha], [\beta] \in \pi_q(X)$), define $[\alpha][\beta] = [\alpha \star \beta]$ where $\alpha \star \beta$ is the map

$$(\alpha \star \beta)(t_1, \cdots, t_q) = \begin{cases} \alpha(2t_1, t_2, \cdots, t_q) & , 0 \leq t_1 \leq \frac{1}{2} \\ \beta(2t_1 - 1, t_2, \cdots, t_q) & , \frac{1}{2} \leq t_1 \leq 1 \end{cases} \tag{3.1}$$



(I draw the picture for $q = 2$. All the black lines map to the base point.)

If instead we were using maps from S^q, we first map S^q to two S^qs attached at their north poles by shrinking the equator to a point, then map the top sphere by α and the bottom sphere by β.

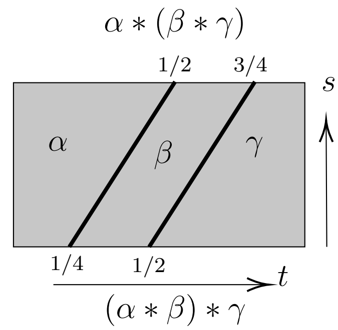


For $q = 0$, $\pi_0(X) = \text{maps} : \text{point} \to X/ \simeq \equiv [\text{point}, X]$ is not in general a group. Rather $\pi_0(X)$ is the set of path components of X. (For the special case where $X = G$ is a Lie group, $\pi_0(G) = G/G_0$, where G_0 is the component of G containing the identity element; this is a group.)

Basic facts about homotopy groups:

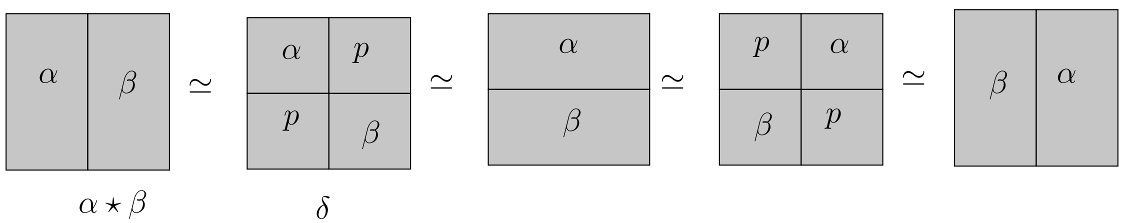
1. $\pi_q(X)$ is a group. The identity operation is the constant map to the base point. The inverse is $[f^{-1}(t_1 \cdots t_q)] = [f(1 - t_1, t_2, \cdots t_q)]$.

The product is associative in the sense that $\alpha * (\beta * \gamma) \simeq (\alpha * \beta) * \gamma$ are homotopy equivalent. Here is the homotopy:



2. $\pi_q(X)$ is abelian for $q > 1$. $\pi_1(X)$, called the *fundamental group of X*, is special in that it can be non-abelian.

   Proof of statement 2:



   $\alpha \star \beta$ in the first step is the map in (3.1). Here the definition of the map δ:

$$\delta(t_1, \cdots, t_q) = \begin{cases} \alpha(2t_1, 2t_2 - 1, \cdots, t_q) & , 0 \leq t_1 \leq \frac{1}{2}, \frac{1}{2} \leq t_2 \leq 1 \\ \beta(2t_1 - 1, 2t_2, \cdots, t_q) & , \frac{1}{2} \leq t_1 \leq 1, 0 \leq t_2 \leq \frac{1}{2} \\ p, \text{otherwise} \end{cases}$$

   – the points in the lower left and upper right all map to the base point. ■

3. If $X \simeq Y$ then $\pi_q(X) \cong \pi_q(Y)$.

4. $\pi_q(X \times Y) = \pi_q(X) \times \pi_q(Y)$. Even simpler than the Kunneth formula.

   Basic fact 4 follows from the fact that any map $I^q \to X \times Y$ is of the form (f_x, f_y) with $f_x : I^q \to X, f_y : I^q \to Y$. And it is a group homomorphism since $(f_x, f_y) \star (g_x, g_y) = (f_x \star g_x, f_y \star g_y)$.

5. Let $\Omega_p X \equiv \{\text{continuous maps} : (I^1, \partial I^1) \to (X, p)\} \equiv$ the *loop space* of X. So the definition of $\pi_1(X, p)$ is just $\pi_0(\Omega_p(X))$. For $q > 2$ also[4], $\pi_{q-1}(\Omega_p X) = \pi_q(X, p)$.

[4] This is a teleological footnote using ideas we'll develop below to prove this statement more. Please ignore it on a first pass. I learned it from Bott and Tu and here. The following sequence (for X path connected) defines a fiber bundle

$$\Omega X \to PX \to X \tag{3.2}$$

called the *path fibration* where $PX \equiv \{\text{maps } \mu : I \to X, \mu(0) = p\}$ (p is the base point). The first

The idea is that a representative of $\pi_q(X)$, $f : I^q \to X$ can be viewed instead as a map $I^{q-1} \to \Omega X$ just by picking a slice of I^q.

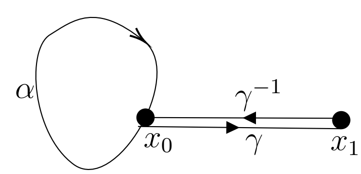
A corollary of this statement is that $\pi_1(\Omega X)$ is always abelian.

6. Like homology, π_q is a covariant functor from the category of topological spaces (and continuous maps) to the category of groups (and group homomorphisms). To see this, consider a map $\phi : (X, x_0) \to (Y, y_0)$. Given a representative of $\pi_q(X)$, $\alpha : (I^q, \partial I^q) \to (X, x_0)$, we can use ϕ to make a representative of $\pi_q(Y)$, namely $\phi \circ f : (I^q, \partial I^q) \to (Y, y_0)$. So we can define an induced map on the homotopy groups
$$\phi_\star[\alpha] \equiv [\phi \circ f].$$
This is a group homomorphism in the sense that $\mathbb{1}_\star = \mathbb{1}$, $\phi \circ (\alpha \star \beta) = (\phi \circ \alpha) \star (\phi \circ \beta)$ and given also $\psi : (Y, y_0) \to (Z, z_0)$, we have $\psi_\star \circ \phi_\star = (\psi \circ \phi)_\star$.

   One consequence of this is the obvious-sounding statement (basic fact 3) that homotopy-equivalent spaces have the same homotopy groups (Hatcher Proposition 1.18). If $f : X \to Y$ and $g : Y \to X$ are the relevant maps then the induced map $f_\star : \pi_q(X, x_0) \to \pi_q(X, f(x_0))$ is an isomorphism with inverse $g_\star$.

7. Making the choice of base point explicit, $\pi_q(X, p) \cong \pi_q(X, p')$ (*i.e.* they are isomorphic groups) if X is path connected. So we don't need to make the choice of base point explicit.

About the dependence on the base point (item 7): Suppose that X is path connected. A path γ from x_0 to x_1 induces a map on the loop spaces $\Omega_{x_0} X \to \Omega_{x_1} X$ by $\alpha \to \gamma \star \alpha \star \gamma^{-1}$ (where γ^{-1} means the path γ traversed backwards).



This induces a map
$$\gamma_\star : \pi_{q-1}(\Omega_{x_0} X, \overline{x_0}) \to \pi_{q-1}(\Omega_{x_1} X, \overline{x_1})$$

map is just inclusion ($\Omega X = \{\mu \in PX | \mu(1) = p\}$), and the second map is the projection $\pi(\mu) = \mu(1)$.

The statement that (3.2) is a fiber bundle means the second map in (3.2) satisfies the 'homotopy lifting property' or 'covering homotopy property' (this elaborate-seeming statement is explained on page 198-199 of Bott and Tu; the fact that it holds for $\pi : PX \to X$ is simple when X is path-connected). This means the short-exact sequence (3.2) induces a long-exact sequence on the homotopy groups
$$\cdots \to \pi_q(\Omega X) \to \pi_q(PX) \to \pi_q(X) \to \pi_{q-1}(\Omega X) \to \cdots$$

And finally PX is contractible because each path can be deformation retracted to the base point, or as it says in the link above: "the picture is that of sucking spaghetti into one's mouth?.

So the exactness of the long-exact sequence means $\pi_q(X) \cong \pi_{q-1}(\Omega X)$.

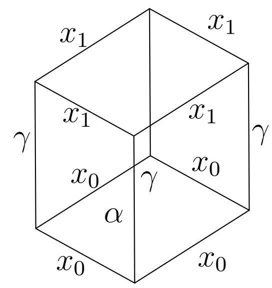
where $\overline{x_t}$ is the constant map to x_t. But this is the same as a map

$$\gamma_\star : \pi_q(X, x_0) \to \pi_q(X, x_1).$$

This map is an isomorphism, with $(\gamma^{-1})_\star = (\gamma_\star)^{-1}$.
More explicitly, for $[\alpha] \in \pi_q(X, x_0)$ define a homotopy

$$F : I^{q+1} = I^q \times I \to X$$

as follows. For $u \in I^q$, we want $F(u, 0) = \alpha(u)$ and $F(u, t) = \gamma(t), \forall u \in \partial I^q$. This defines the map on all but one face of ∂I^{q+1}. Now I quote a theorem ('the box principle of obstruction theory') that such an F can be extended to all of I^{q+1}, and we define $[F(u, 1)] = \gamma_\star[\alpha]$.



If we take $x_0 = x_1$, this defines an action of $\pi_1(X, x_0)$ on $\pi_q(X, x_0)$ describing the result of moving the base point around in a non-contractible loop. Its nontriviality measures something about how much the choice of base point matters. Bott and Tu Prop. 17.6.1 shows that

$$\pi_q(X, x_0)/\pi_1(X, x_0) \cong [S^q, X]$$

where the quotient on the LHS is by the action defined above, and the RHS is homotopy classes of maps from S^q to X without any notion of base point ('free homotopy'). This is not a group. The map from the LHS to the RHS is just inclusion of base-point-preserving maps into the set of all maps.

What's special about I_q or spheres? It is actually possible to define homotopy groups of maps from other spaces. But in general the set of homotopy classes of maps from one space to another is not a group.

Higher homotopy groups are hard to compute. Even for spheres $\pi_q(S^n)$ are not all known for large-enough q and n. For small q, n here is the table:

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2{}^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$

(from here; that page also has a larger table).

Fundamental group. Let's focus on $q = 1$, the fundamental group, for a bit. In this case the product is simpler to understand: a representative of $\pi_1(X)$ is just a closed path in X starting and ending at the base point. The product of two paths is just one followed by the other, with the parameter rescaled so that total duration is still 1.

There is information in $\pi_q(X)$ that is not present in the homology. For example, $\pi_1(X)$ has more information than $H_1(X)$. Here is an example of a space X with trivial $H_1(X, \mathbb{Z})$ (such a space is called 'acyclic') but nontrivial $\pi_1(X)$. Take a figure eight and glue in two 2-cells whose boundaries are $a^5 b^{-3}$ and $b^3(ab)^{-2}$ where a and b are the two loops. The cell complex is then

$$0 \to \mathbb{Z}^2 \xrightarrow{M} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \to 0$$

with

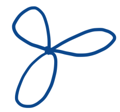
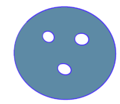
$$\partial_2 = M = \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} . \quad \text{Since } \det M = -1$$

there is not even any torsion first homology. But

$$\pi_1(X) = \left\langle a, b | a^5 b^{-3} = 1, b^3 (ab)^{-2} = 1 \right\rangle = \left\langle a, b | a^5 = b^3 = (ab)^2 \right\rangle = I^\star \tag{3.3}$$

the binary icosahedral group, a double cover of the icosahedral group (the symmetry group of the icosahedron and dodecahedron) $I \cong A_5$, also known as the alternating group on (*i.e.* the even permutations of) five elements, $S_5/\mathbb{Z}_2$. Under the surjection $I\star \to I$ a maps to a $2\pi/5$ rotation through center of a pentagon, and b maps to a $2\pi/3$

rotation through a vertex. The double cover of I arises by the inclusion $I \subset \mathsf{SO}(3)$, and is induced by the double cover $\pi : \mathsf{SU}(2) \to \mathsf{SO}(3)$.

Why is (3.3) the answer for $\pi_1(X)$? Well, π_1 of a bouquet of circles – q circles attached at a point (which we take to be the base point) – is the *free group* on q elements. Each circle provides a generator, and there are no relations. A bouquet of q circles is a deformation retract of $\mathbb{R}^2 \setminus \{q \text{ points}\}$ (or $\mathbb{R}^3 \setminus \{q \text{ lines}\}$),  $\simeq$ , so this is also π_1 of the latter spaces.

This free group on ≥ 2 elements is a deeply horrible object. The elements are all words made of the letters $a_1, a_2 \cdots a_q$ and $a_1^{-1}, a_2^{-1} \cdots a_q^{-1}$, and the only relation is that you can cancel an a_i and an a_i^{-1} if they are right next to each other. It contains copies of itself as a subgroup. Ick.

When we glue in a 2-cell we introduce a relation in π_1 according to the gluing map; this gives (3.3).

This example is related to the existence of the Milnor homology sphere M – a 3-manifold with the same homology as a sphere, but different homotopy groups. M can be defined as $S^3/I^\star$ (with $I^\star$ the binary icosahedral group as above). Therefore $\pi_1(M) = I'$. There is a lot to say about this space.

General Fact: $H_1(X, \mathbb{Z})$ is the abelianization of $\pi_1(X)$, *i.e.*

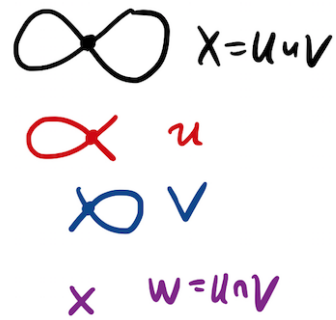
$$H_1(X, \mathbb{Z}) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$$

where $[G, G] \equiv \langle ghg^{-1}h^{-1}, g, h \in G\rangle$ is the commutator subgroup of G, the subgroup generated by (multiplicative) commutators of elements of G. This is not hard to see: the whole difference between π_1 and H_1 is that in the latter we keep track of the order in which the closed loops are traversed. Modding out by commutators is erasing exactly this information. [For a more explicit proof, see Hatcher Theorem 2A.1.]

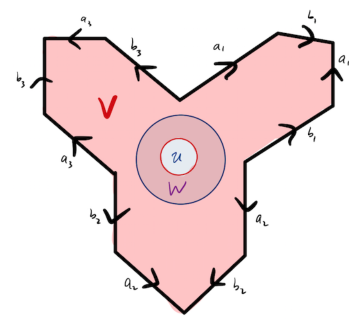
van Kampen Theorem. [Justin Roberts' knot knotes has a very nice discussion with a proof sketch] Here is a way to compute the fundamental group of a space by chopping up the space. Let $X = U \cup V$, two open sets, and let $W \equiv U \cap V$. Denote $\pi_1(Y) \equiv \langle s_Y | r_Y \rangle$ for $Y = U, V, W$, so s_Y is a set of generators of $\pi_1(Y)$ and r_Y is a set of relations. Let $i^{U,V}$ be the inclusion maps of W into U and V. Then

$$\pi_1(X) = \left\langle s_U \cup s_V | r_U \cup r_V \cup \{i^U_\star(g) = i^V_\star(g)\}_{g \in s_W} \right\rangle .$$

That is: a set of generators of $\pi_1(X)$ is just those of U and those of V. This double-counts the generators on the overlap. The theorem says that it's enough to add one relation for each generator of the fundamental group of the overlap.



Example 1: Cut a bouquet of two circles ($S^1 \wedge S^1$, two circles attached at a point) into two open sets U and V where each of U and V removes a single point from one of the circles. The overlap can be deformed to an X which is contractible. $\pi_1(U)$ and $\pi_1(V)$ each have one generator, and there are no relations, so $\pi_1(S^1 \wedge S^1)$ is the free group on two elements.



Example 2: Consider a genus-g Riemann surface Σ_g, described as a polygon with $2g$ sides, with the identifications given in §1.2. Let U be a disk inside the polygon and V a little more than its complement. Then $U \cap V$ is an annulus, homotopy equivalent to a circle, with $\pi_1(S^1) = \langle g \rangle$. The inclusion of g into U maps it to $i_\star^U(g) = 0$, since every loop in U is trivial. The inclusion of g into V is homotopic (in V) to $\prod_{i=1}^g [a_i, b_i]$ with $[a, b] \equiv aba^{-1}b^{-1}$. We conclude that $\pi_1(\Sigma_g) = \langle \{a_i, b_i\}_{i=1}^g | \prod_{i=1}^g [a_i, b_i] \rangle$.

Example 3: The answer we found above for π_1 of the acyclic space X can also be obtained by these methods. It is best to do it in two steps, first removing both disks and then removing just one.

Higher homotopy groups and homology. For general q, there is a natural homomorphism

$$i : \begin{array}{ccc} \pi_q(X) & \to & H_q(X) \\ [f] & \mapsto & f_\star(u) \end{array}$$

where u is a generator of $H_q(S^q)$. For better or worse, this map is neither injective nor surjective.

One more fact about the relation between homotopy groups and homology, however, is the Hurewicz isomorphism theorem: The first nontrivial homotopy and homology groups of a path-connected manifold occur in the same dimension q. ($q = 0$ doesn't count.) If $q > 1$ then they are isomorphic. (In symbols: if $q > 1$, and $\pi_k(X) = 0$ for $1 \leq k < q$, then $H_q(X) = 0$ for $1 \leq k < q$ and $H_q(X) = \pi_q(X)$.)

Consider the case $q = 2$. Then the claim is that $H_2(X) \doteq H_1(\Omega X) = \pi_1(\Omega X)$ (since $\pi_1(\Omega X) = \pi_2(X)$ is abelian), and therefore $H_2(X) = \pi_2(X)$. The first equality, with the $\doteq$, follows from $H_1(X) = 0$ but I will not explain it here. There is a general proof on page 225 of Bott and Tu which uses a spectral sequence and induction from the above case.

This theorem implies that $\pi_q(S^n) = \delta^{n,q}\mathbb{Z}$ for $q \leq n$.

3.4 Fiber bundles and covering maps

When does an 'exact sequence of spaces' like

$$0 \to F \xrightarrow{i} E \xrightarrow{\pi} B \to 0 \tag{3.4}$$

induce a long exact sequence on their homotopy groups

$$\cdots \to \pi_q(F) \xrightarrow{i_\star} \pi_q(E) \xrightarrow{\pi_\star} \pi_q(B) \xrightarrow{\partial} \pi_{q-1}(F) \to \cdots \to \pi_1(B) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_\star} \pi_0(E) \xrightarrow{\pi_\star} \pi_0(B) \to 0 \tag{3.5}$$

? In general it does not. But with some extra assumptions on the sequence of continuous maps (3.4) it does. The extra assumption says that E is a *fiber bundle*; B is the base, and $F = \pi^{-1}(b_0)$ is the fiber. I mention this here also because this notion will play an important role in the interpretation of the quantum double model.

Part of the assumption is that a neighborhood every fiber $\pi^{-1}(U)$ is homeomorphic to $U \times F$. Such a map π is called a *covering map*.

The further condition for E to be a fiber bundle is that for each open set U_α in a cover of B, the diagram at right commutes. The vertical map is just forgetting about the fiber.

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\phi"] \arrow[dr, "\pi"'] & U_\alpha \times F \arrow[d] \\ & U_\alpha \end{tikzcd}$$

These maps $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ are then called *local trivializations*, analogous to local coordinates on a manifold. For the pathologists among you: try to come up with an example of a covering map which does not produce a fiber bundle; I don't want to do it.

A *section* of a fiber bundle is a map $s : B \to F$ with $\pi \circ s = \mathbb{1}$.

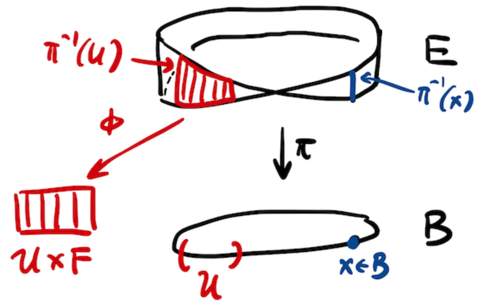
Transition functions. Now on the double-overlaps $U_{\alpha\beta}$ of an open cover of B, we have maps

$$\phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F.$$

These are called transition functions. They lie in a subgroup of the group of homeomorphisms of the fiber F called the *structure group* of the bundle.

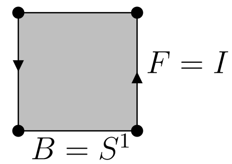
Of course a product manifold, like $T^2 = S^1 \times S^1$, is a fiber bundle, but a trivial one, where the transition functions can all be chosen to be the identity. (If we were keeping track of more information, such as the *complex structure* on the torus, the boundary conditions which identify S^1 with S^1 by a shift gives a nontrivial operation called a Dehn twist.)

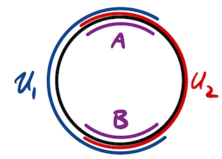
Example: Mobius band. Take $B = S^1$ and $F = I$. If we impose boundary conditions that the orientation of F reverses when we go around the circle, we get the Mobius band.



Cover $B = S^1$ with two open sets $U_{1,2}$. They overlap in $U_{12} = A \cup B$, with two components. The nontrivial transition functions are

$$\phi_{12}(x) = \begin{cases} 1 & \text{if } x \in A \\ g & \text{if } x \in B \end{cases}$$





where g is the orientation-reversal of the fiber.

A very similar example is obtained by replacing the fiber by S^1; this is the Klein bottle.

Example: Hopf bundle. Take $E =$ the unit quaternions, or $\mathsf{SU}(2) \simeq S^3$, and $F = S^1$, the unit complex numbers inside the unit quaternions. Taking $S^3 \subset \mathbb{C}^2 = \{(z_0, z_1)\}$, the base is

$$\left(S^3 = \{|z_0|^2 + |z_1|^2 = 1\} \subset \mathbb{C}^2\right) / (z_0, z_1) \sim e^{\mathbf{i}\alpha}(z_0, z_1)$$

which is $\mathbb{CP}^1 \simeq S^2$. This is just the Bloch sphere of normalized pure states of a qubit, and the projection map is just forgetting the overall phase of the wavefunction. In stereographic projection $S^2 \simeq \mathbb{C} \cup \{\infty\}$, the projection map is $\pi : (z_0, z_1) \to z_0/z_1$, where the range of the map is $B \simeq S^2$ is the Riemann sphere (complex plane union the point at infinity). In polar coordinates ($r_0^2 + r_1^2 = 1$ defines the S^3) the projection is $\pi(r_0 e^{\mathbf{i}\theta_0}, r_1 e^{\mathbf{i}\theta_1}) = \frac{r_0}{r_1} e^{\mathbf{i}(\theta_0 - \theta_1)}$. Fixed $\rho = r_0/r_1$ is a $T^2 \subset S^3$ which degenerates at $\rho = \pm\infty$ to two linked circles. A visualization from Wikipedia is at right.

Another way to present the Hopf bundle projection, which arises all the time in physics is:

$$\pi : \begin{cases} \mathbb{C}^2 & \to & \mathbb{R}^3 \\ S^3 & \to & S^2 \\ z & \mapsto & z^\dagger \vec{\sigma} z \end{cases}$$

The top row applies to general $z = (z_0, z_1)$, and the middle row applies to the subspace where $|z_0|^2 + |z_1|^2 = 1$. We can make local sections of this bundle by finding the ± 1 eigenvectors of $\hat{n} \cdot \vec{\sigma}$.

One reason to care about the Hopf bundle, besides its ubiquity in theoretical physics, is that it gives relations between homotopy groups of spheres. The exact homotopy sequence is

$$\cdots \pi_q(S^1) \to \pi_q(S^3) \to \pi_{q-1}(S^2) \to \pi_{q-1}(S^1) \to \cdots \tag{3.6}$$

Fact: $\pi_q(S^1) = \mathbb{Z}\delta^{q,1}$ for $q \geq 1$. We conclude from (3.6) that $\pi_q(S^3) \cong \pi_q(S^2)$ for $q \geq 3$. In particular $\pi_3(S^2) = \mathbb{Z}$ is generated by the Hopf projection itself.

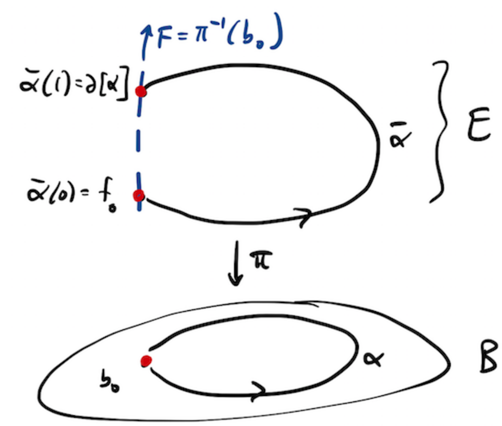
Universal cover. How do we know the homotopy groups of the circle? One way is to write $S^1 = \mathbb{R}/\mathbb{Z}$. Now we appeal to the following
General Fact: if $X = C/G$ and C is simply connected ($\equiv \pi(C) = 0$) then $\pi_1(X) = G$. (If you are not happy with this level of detail, Hatcher has a long section on $\pi_1(S^1)$.)
Of intermediate generality between a quotient and a fiber bundle is the existence of a covering map $\pi : C \to X$. If C is simply connected, then the space C is called the *universal cover* of X. For example, we saw in §1.2 that a Riemann surface Σ_g of genus $g \geq 1$ can be made by taking a disk and making identifications along its boundary. Since the disk is simply connected, it is the universal cover of Σ_g, and

$$\pi_1(\Sigma_g) = \left\langle a_i, b_i | a_1^{-1}b_1^{-1}a_1b_1a_2^{-1}b_2^{-1}a_2b_2 \cdots a_g^{-1}b_g^{-1}a_gb_g = 1 \right\rangle$$

where the one relation comes from the disk filling in the boundary. (By this notation I mean the group generated by the list of things before the |, modulo the list of relations after the |.) For $g = 1$, this says that a and b commute, which they'd better since $\pi_1(T^2) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$ is abelian. For $g > 1$, $\pi_1(\Sigma_g)$ is non-abelian. You can see that its abelianization is $\mathbb{Z}^{2g}$ in agreement with our previous result for $H^1(\Sigma_g)$.

Brief and insufficient words about the connecting homomorphism. [Bott and Tu p. 209] I said that for fiber bundles there *is* a long exact sequence on the homotopy groups, but what is the mysterious map $\partial : \pi_q(B) \to \pi_{q-1}(F)$ in (3.5)?

The idea is that a map $\alpha : I^q \to B$ can be lifted to a map $\tilde{\alpha} : I^q \to E$, but it does not necessarily end at the base point of E, since the base point of B can have many pre-images in E, only one of which is the base point in E. Regard $F = \pi^{-1}(b_0)$ as the fiber over the base point. First the case $q = 1$: the lift of α can be chosen so that $\bar{\alpha}(0)$ is the base point in F. Then $\partial[\alpha] = [\bar{\alpha}(1)]$.



For general q, the properties of a fiber bundle guarantee that α can be lifted so that $\bar{\alpha}(t_1, \cdots t_{q-1}, 0)$ lifts to the constant map to the base point of F. Then its image under the connecting homomorphism is

$$\partial[\alpha] = [(t_1 \cdots t_{q-1}) \mapsto \bar{\alpha}(t_1 \cdots t_{q-1}, 1)].$$

α in the same homotopy class have the same image. The keyword is 'covering homotopy property'.

One final comment about covering maps: If $\pi : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering map, then $\pi_\star : \pi_q(\tilde{X}, \tilde{x}_0) \to \pi_q(X, x_0)$ is an isomorphism for $q \geq 2$. The idea is that every map $S^n \to X$ lifts to a map $S^n \to \tilde{X}$ for $q \geq 2$. This is Hatcher Prop. 4.1 on page 342, and also follows from the 'covering homotopy property'. In the case of $q = 1$, the induced map is merely injective, and embeds π_1 of the covering space as a subgroup of $\pi_1(X)$ (Hatcher Prop. 1.31) – it's just the subgroup of loops which lift to closed loops in $\tilde{X}$ (unlike the one α in the figure above). There is therefore a correspondence between covers of X and subgroups of $\pi_1(X)$.

3.5 Relative homotopy.

It is possible define a notion of relative homotopy. Given $Y \subset X$ a closed subspace containing the base point p, the relative homotopy group is

$$\pi_q(X, Y, p) \equiv \pi_{q-1}\,(\text{paths from } p \text{ to } Y)$$

or slightly more explicitly

$$\pi_q(X, Y, p) = \{\alpha : (I^q, \partial I^q) \to (X, p \text{ or } Y)\}/\simeq . \tag{3.7}$$

What I mean by this is: all of the faces of ∂I^q get mapped to the base point as usual, except for the bottom face of ∂I^q, which gets mapped anywhere into Y. By the 'bottom face' of ∂I^q I mean $\{t_1 = 0\}$. So $\alpha|_{t_1=0} : (I^{q-1}, \partial I^{q-1}) \to (Y, p)$. The product is defined as usual for $q > 1$, but $\pi_1(X, Y)$ is not a group.

The inclusion map $i : Y \to X$ induces a map $i_\star : \pi_k(Y, p) \to \pi_k(X, p)$ as usual. Noting that $\pi_k(X, p) = \pi_k(X, p, p)$, we also have a map

$$j_\star : \pi_k(X, p) = \pi_k(X, p, p) \to \pi_k(X, Y, p).$$

Finally, we can define

$$\partial_\star : \begin{array}{ccc} \pi_k(X, Y, p) & \to & \pi_{k-1}(Y, p) \\ \alpha & \mapsto & \alpha|_{\text{bottom face}} \end{array}.$$

This produces a long exact sequence on homotopy:

$$\cdots \to \pi_k(Y, p) \xrightarrow{i_\star} \pi_k(X, p) \xrightarrow{j_\star} \pi_k(X, Y, p) \xrightarrow{\partial_\star} \pi_{k-1}(Y, p) \xrightarrow{i_\star} \cdots \,.$$

Nash and Sen (page 117) use this sequence to show $\pi_q(S^n) = 0$ for $q < n$. I must admit, though, that I do not understand what they mean by $B \supset S^n$.