

Physics 215C QFT Spring 2026

Assignment 1 – Solutions

Due 11:59pm Monday, April 6, 2026

1. An application of effective field theory in quantum mechanics.

[I learned this example from Z. Komargodski.]

Consider a model of two canonical quantum variables ($[\mathbf{x}, \mathbf{p}_x] = \mathbf{i} = [\mathbf{y}, \mathbf{p}_y], 0 = [\mathbf{x}, \mathbf{p}_y] = [\mathbf{x}, \mathbf{y}]$, etc) with Hamiltonian

$$\mathbf{H} = \mathbf{p}_x^2 + \mathbf{p}_y^2 + \lambda \mathbf{x}^2 \mathbf{y}^2.$$

(This is similar to the degenerate limit of the model we studied in lecture with two QM variables where both natural frequencies are taken to zero.)

- (a) Based on a semiclassical analysis, would you think that the spectrum is discrete or continuous?

The potential has flat directions along the coordinate axes, $\{x = 0\} \cup \{y = 0\}$. This means there are unbounded classical orbits, which suggests that the spectrum should be continuous. This conclusion is in fact wrong. (An excuse for discounting it is that the set of initial conditions which follow unbounded orbits have measure zero.) The idea of why it's wrong is that the valleys of flat directions get narrower and narrower as one goes farther from the origin; this means that the kinetic energy cost in the transverse direction gets too expensive.

- (b) Study large, fixed x near $y = 0$. We will treat x as the slow (= low-energy) variable, while y gets a large restoring force from the background x value. Solve the y dynamics, and find the groundstate energy as a function of x :

$$V_{\text{eff}}(x) = E_{\text{g.s. of } y}(x).$$

If we treat x as a constant, the hamiltonian for y is a harmonic oscillator problem. The groundstate energy is

$$V_{\text{eff}}(x) = E_{\text{g.s. of } y}(x) = \sqrt{\lambda}|x|$$

- (c) Presumably you did the previous part using your knowledge of the spectrum of the harmonic oscillator Hamiltonian. Redo the previous part using path integral methods.

We can also do it using path integrals. Let's do it in euclidean time. The Lagrangian that gives $H = p^2 + \omega^2 x^2$ is $L = \frac{1}{4}\dot{x}^2 - \frac{\omega^2}{4}x^2$, with $\omega^2 = 4\lambda x^2$. The integral we need is

$$\int [Dy] \exp\left(-\int dt y M y\right) = \det M^{-1/2} = e^{-\frac{1}{2}\text{tr} \log M}$$

with $M \equiv \left(-\frac{\partial^2}{4} + \lambda x^2\right)$. This gives a correction to the effective action for x

$$e^{-\delta S_{\text{eff}}[x]} = e^{-\frac{1}{2}\text{tr} \log M}.$$

If we treat x as constant, and consider a time interval T , this corrects the effective potential by

$$V_{\text{eff}}(x) = +\frac{1}{2}\text{tr} \log M/T = +\frac{1}{2} \int_{-\Lambda}^{\Lambda} \ddagger\omega \log\left(\frac{\omega^2}{4} + \lambda x^2\right) \quad (1)$$

$$= \frac{1}{2} \left(-2\Lambda(2 + \log 4 - 2 \log \Lambda) + 2\sqrt{\lambda}|x| + \mathcal{O}(1/\Lambda)\right). \quad (2)$$

We need to regulate the frequency integral and ignore the meaningless additive constant, but we get the same answer as with the canonical method. (We can also differentiate with respect to x under the integral sign to make the integral finite, and then integrate with respect to x ; this makes clear that the UV divergence is just additive garbage.) Note that doing this calculation with non-constant x , we can do a derivative expansion and some additional terms involving derivatives of x will also be produced. These don't change the conclusion below about the spectrum.

- (d) The result for $V_{\text{eff}}(x)$ is not analytic in x at $x = 0$. Why did that happen? At $x = 0$, y becomes massless (i.e. it is a spring whose natural frequency goes to zero there). Integrating out massless degrees of freedom produces singularities in the effective action.
- (e) Is the spectrum of the resulting 1d model with

$$\mathbf{H}_{\text{eff}} = \mathbf{p}_x^2 + V_{\text{eff}}(\mathbf{x})$$

discrete? Is this description valid in the regime that matters for the semi-classical analysis?

[Bonus: determine the spectrum of \mathbf{H}_{eff} .]

The potential $V \sim |x|$ bounds the trajectories and has a discrete spectrum. Integrating out y is a better approximation at larger $|x|$, which is where the dangerous flat directions occur. That is: this approximation is valid outside of a compact region of field space near $x = y = 0$ in which the potential is bounded below. Such a region cannot produce a continuum in the spectrum.

The actual spectrum of the absolute value potential is fun. The solutions of the Schrödinger problem (we can rescale x to get rid of the constant prefactor in the potential) $\psi(x) = \psi_{>}(x)\theta(x) + \psi_{<}(x)\theta(-x)$ satisfy

$$\begin{cases} (-\partial_x^2 + (x - E)) \psi_{>} = 0, & x > 0 \\ (-\partial_x^2 + (-x - E)) \psi_{<} = 0, & x > 0 \end{cases}.$$

The solutions for $x > 0$ are the two Airy functions

$$\psi_{>}(x) = a_{>}\text{Ai}(x - E) + b_{>}\text{Bi}(x - E)$$

of which the second blows up at large argument and hence cannot be normalized so we must set $b_{>} = 0$. Similarly, for $x < 0$, we have

$$\psi_{<}(x) = a_{<}\text{Ai}(-x - E) + b_{<}\text{Bi}(-x - E)$$

and again we must set $b_{<} = 0$. Since the potential has finite measure near $x = 0$ (i.e. no delta function) the wavefunction and its first derivative must be continuous at $x = 0$ and we have

$$\begin{aligned} \psi_{>}(0) = \psi_{<}(0) &\implies a_{>}\text{Ai}(-E) = a_{<}\text{Ai}(-E) \\ \psi'_{>}(0) = \psi'_{<}(0) &\implies a_{>}\text{Ai}'(-E) = -a_{<}\text{Ai}'(-E) \end{aligned} \quad (3)$$

which means either $a_{>} = a_{<} = 0$ OR $\text{Ai}(-E) = 0$ OR $\text{Ai}'(-E) = 0$. This means that the boundstates occur at zeros of the airy function or its derivative:

$$\{\text{boundstate energies}\} \propto \{E | \text{Ai}(-E) = 0 \text{ or } \text{Ai}'(-E) = 0\}.$$

2. Entanglement and EFT. [Bonus problem]

In a given state ρ_{AB} , one measure of the entanglement between two parts of a bipartite Hilbert space $\mathcal{H} = A \otimes B$ is the entanglement entropy, the von Neumann entropy $S_A \equiv -\text{tr}_A \rho_A \log \rho_A$ of the reduced density matrix of one of the parts,

$\rho_A = \text{tr}_B \rho_{AB}$. When the whole system is in a pure state $\rho_{AB} = |\psi\rangle\langle\psi|$, this is a useful measure of their entanglement.

Consider the problem of two oscillators we discussed in lecture as a parable about integrating out high-energy modes. Compute the entanglement entropy S_q of the light mode q in the groundstate of the combined system, by whatever means necessary.

If $S_q \neq 0$, it means that q does not have its own wavefunction, only a probability distribution on wavefunctions. Show that S_q is very small in the limit $\omega_0 \ll \Omega$, so that it is not a terrible idea to approximate the low-energy physics of q as being described by a pure state.

3. **Emergence of the Dirac equation.** (Some words of motivation: we're going to study some interesting phenomena involving the field theory of a Dirac fermion; some of them involve questions about regularizing this field theory. So it will be instructive to have in mind a very concrete and physical regularization of this field theory.)

Consider a chain of free fermions with

$$H = -t \sum_n c_n^\dagger c_{n+1} + h.c.$$

Show that the low-energy excitations at a generic value of the filling are described by the massless Dirac lagrangian in 1+1 dimensions. [Hint: the problem has (discrete) translation-invariance and is linear.]

Find an explicit choice of 1 + 1-d gamma matrices that matches the answer from the lattice model. Show that the right-movers are right-handed $\gamma^5 \equiv \gamma^0 \gamma^1 = 1$ and the left-movers are left-handed.

This system has a conserved charge $N \equiv \sum_n c_n^\dagger c_n$ counting the number of fermions, which we get to pick. The easiest way to do this is to add a chemical potential $H \rightarrow H - \mu N$ and choose μ to get the desired number of particles on average. (This is the same as fixing the number of particles in the thermodynamic limit.) In that case we have

$$H = -t \sum_n c_n^\dagger c_{n+1} + h.c. - \mu \sum_n c_n^\dagger c_n = \int_{\text{BZ}} dk c_k^\dagger c_k \epsilon_k$$

with $\epsilon_k = -2t \cos ka - \mu$, and the integral is over the Brillouin zone. $a = 1$ is the lattice spacing. By 'generic filling' I mean choose the number of particles per site to be between 0 and 1. The former and latter correspond to choosing $\mu = \pm 2t$

at the bottom or top of the band, where the dispersion is quadratic, rather than linear.

We can focus on the physics at the two Fermi points $k = \pm k_F$ (where k_F solves $\epsilon_{k_F} = 0$) by plugging in

$$\psi(x) \simeq \int_R \bar{d}k e^{(k_F+k)x} \psi_R + \int_R \bar{d}k e^{(-k_F+k)x} \psi_L$$

where R is a small-enough region in momentum space that the two domains don't overlap. This gives

$$H = \int_R \bar{d}k \left(v_F k \psi_R^\dagger \psi_R - v_F k \psi_L^\dagger \psi_L \right)$$

where $v_F \equiv \partial_k \epsilon_k |_{k=k_F}$. Translating into an action, setting $v_F = 1$, and pretending R goes on forever (this is how we can fool ourselves that the chiral current is conserved), this is

$$S = \int dx dt \left(\psi_R^\dagger (\partial_t - \partial_x) \psi_R + \psi_L^\dagger (\partial_t + \partial_x) \psi_L \right) = \int d^2x \left(\bar{\Psi} \gamma^\mu \partial_\mu \Psi \right)$$

with

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

and

$$\gamma^0 = \sigma^1, \gamma^1 = \mathbf{i}\sigma^2, \gamma^5 \equiv \gamma^0 \gamma^1 = -\sigma^3.$$

This gives

$$\gamma^5 \Psi = -\sigma^3 \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} -\psi_L \\ \psi_R \end{pmatrix}$$

so indeed the left-moving particle has left-handed chirality.