University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2025
Assignment 3

Due 11:59pm Wednesday, April 23, 2025

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1. **Polyacetylene returns.**

   On the previous homework, you may have wondered what is the connection between the field theory we were studying (a scalar coupled to fermions in $D = 2$) and polyacetylene. I'd like to explain that connection a bit.

   On the first homework, you showed that the Dirac equation emerges from a chain of spinless fermions. Consider an extension of that model to include also *phonon* modes, *i.e.* degrees of freedom encoding the positions of the ions in the solid. (Again we ignore the spins of the electrons for simplicity.)

$$H = -t\sum_n (1+u_n)c_n^\dagger c_{n+1} + h.c. + \sum_n K(u_n - u_{n+1})^2 \equiv H_F + H_E.$$

   Here u_n is the deviation of the nth ion from its equilibrium position (in the $+x$ direction), so the second term represents an elastic energy. Assume periodic boundary conditions and an even number of sites. We will treat the phonons as slow modes.

   (a) Consider a configuration

$$u_n = \phi(-1)^n \tag{1}$$

   where the ions move closer in pairs. Compute the electronic spectrum. (Hint: this enlarges the unit cell. Define $c_{2n} \equiv a_n, c_{2n+1} \equiv b_n$, and solve in Fourier space, $a_n \equiv \oint đk e^{2\mathbf{i}kn} a_k$ *etc* with an ensmallened Brillouin zone.) You should find that when $\phi \neq 0$ there is a gap in the electron spectrum (unlike when $\phi = 0$). Expand the spectrum near the minimum gap and include the effects of the field ϕ in the continuum Dirac theory (for small ϕ).

   (b) **Peierls' instability.** Compute the groundstate energy of the electrons H_F in the configuration (1), at half-filling (*i.e.* the number of electrons is half the number of available states). Check that you recover the previous answer when $\phi = 0$. Interpret the answer when $\phi = 1$.

   Compute H_E in this configuration, and plot the sum of the two as a function of ϕ. Choosing the parameters so the minimum is in the small-ϕ region, find the minimum.

(c) You should find that the energy is independent of the *sign* of ϕ. This means that there are two groundstates. We can consider a domain wall between a region of $+$ and a region of $-$. Show that this domain wall carries a fermion mode whose energy lies in the bandgap and whose filled and empty states have charge $\pm\frac{1}{2}$.

(d) [bonus] Verify the result of the previous part by diagonalizing the relevant tight-binding matrix.

(e) [bonus] Time-reversal played an important role here. If we allow complex hopping amplitudes, we can make a domain wall without midgap modes. Explain this from field theory. Bonus: explain this from the lattice hamiltonian.

2. **Anomaly cancellation in the Standard Model.** If we try to gauge a chiral symmetry (such as hypercharge in the Standard Model (SM)), it is important that it is actually a symmetry, *i.e.* is not anomalous. In $D = 3+1$, a possible anomaly is associated with a choice of three currents, out of which to make a triangle diagram. We'll call a "$\mathsf{G}_1\mathsf{G}_2\mathsf{G}_3$ anomaly" the diagram with insertions of currents for $\mathsf{G}_1, \mathsf{G}_2$ and G_3. Generalizing a little, we showed that the divergence of the current for G_1 is

$$\partial_\mu j_1^{A\mu} = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F^{2B}_{\mu\nu}F^{3C}_{\rho\sigma}\sum_f(-1)^f\mathrm{tr}_{R(f)}\{T_1^A, T_2^B\}T_3^C.$$

The sum is over each Weyl fermion, $R(f)$ is its representation under the combined group $\mathsf{G}_1 \times \mathsf{G}_2 \times \mathsf{G}_3$, and T_1^A are a basis of generators of the Lie algebra of G_1 etc. in the representation of the field f. By $(-1)^f$ I mean $\pm$ for left- and right-handed fermions respectively.

We consider the possibilities in turn.

(a) Convince yourself that the divergence of the $\mathsf{U}(1)_Y$ hypercharge current gets a contribution of the form

$$\partial_\mu J_Y^\mu = \left(\sum_{\text{left}} Y_l^3 - \sum_{\text{right}} Y_r^3\right)\frac{g'^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}B_{\rho\sigma}$$

from the triangle with three insertions of the current itself (here B is the hypercharge gauge field strength). The sum on the RHS is over all left- and right-handed Weyl spinors weighted by the cube of their hypercharge. Check that this sum evaluates to zero in the SM.

(b) Show that any anomaly of the form $\mathsf{SU}(N)\mathsf{U}(1)^2$ or $\mathsf{SU}(N)G_1G_2$ is zero.

(c) (Easy) Convince yourself that there is no $\mathsf{SU}(3)^3$ anomaly for QCD.

(d) Check that there is never an $\mathsf{SU}(2)^3$ anomaly. (Hint: the generators satisfy $\{\tau^a, \tau^b\} = 2\delta^{ab}$.)

(e) Show that the $\mathsf{SU}(3)^2\mathsf{U}(1)_Y$ anomaly demands that $2Y_Q - Y_u - Y_d = 0$. Check that this is true in the SM.

(f) Show that a necessary condition for hypercharge to not have an anomaly with the Electroweak gauge bosons on the RHS is $Y_L + 3Y_Q = 0$, where Y_L and Y_Q are the hypercharges of the left-handed leptons and quarks. Check that this works out in the SM.

(g) There is another kind of anomaly called a *gravitational anomaly*. This is a violation of current conservation in response to coupling to curved space. An example is of the form

$$\partial_\mu j^\mu_Y = a \mathrm{tr} \mathcal{R} \wedge \mathcal{R}$$

where $\mathcal{R}$ is a two-form related to the curvature of spacetime (analogous to the field strength F). The coefficient a is proportional to $\sum_{\text{left}} \mathrm{tr} Y_l - \sum_{\text{right}} \mathrm{tr} Y_r$. Check that this too vanishes for hypercharge in the Standard Model.

These conditions, plus the assumption that the right-handed neutrino is neutral, actually determine all the hypercharge assignments.

(h) [bonus] Show that the previous statement is true.

There are various points of view from which the anomalies determine the charge assignments.

One is: Given the $\mathsf{SU}(3) \times \mathsf{SU}(2)_L$ representations, the actual hypercharges are the only way to satisfy all the anomaly constraints that is chiral. From this point of view, the fact that the hypercharges are all integer multiples of $1/6$ (so that $\mathsf{U}(1)_Y$ is compact) is an outcome of anomaly cancellation.

Another is: Assuming that the hypercharges are quantized (in some units), the choice in the SM is the only chiral choice, even without using the gravitational chiral anomaly constraint. This is a consequence of Fermat's Last Theorem.

(i) [bonus] Show that $\mathsf{U}(1)_B$ and $\mathsf{U}(1)_L$ are anomalous, but have all opposite anomalies, so that $\mathsf{U}(1)_{B-L}$ is non-anomalous. Here all quarks (antiquarks) have charge $1/3$ $(-1/3)$ under $\mathsf{U}(1)_B$, and all leptons (antileptons) have charge 1 (-1) under $\mathsf{U}(1)_L$.