

University of California at San Diego – Department of Physics – Prof. John McGreevy
Physics 215C QFT Spring 2026
Assignment 4

Due 11:59pm Monday, April 27, 2026

1. **All possible terms.**

Perturb the gaussian fixed point of the $O(N)$ model in $D \geq 3$ by a term

$$\delta S_6 = \int d^D x g_6 (\phi^a \phi^a)^3 .$$

This term is a marginal perturbation of the gaussian fixed point in $D = 3$. Should we worry that we left it out of our study of the Wilson-Fisher fixed point? At leading order in $\epsilon = 4 - D$, what does it do to the Wilson-Fisher fixed point?

2. **An example of the power of the RG logic.**

Consider quantum mechanics of a single particle in d dimensions, with Hamiltonian

$$H = \frac{p^2}{2m} + V(q), \quad [q, p] = i.$$

Consider the (say, euclidean) path integral for this problem,

$$Z = \int [dq] e^{-S[q]}, \quad S[q] = \int dt \left(\frac{m}{2} \dot{q}^2 - V(q) \right).$$

To be more precise, with periodic boundary conditions, $Z(\beta) = \int_{q(t+\beta)=q(t)} [dq] e^{-S[q]} = \text{tr} e^{-\beta H}$ is the thermal partition function. Alternatively, instead of Z , we could consider the Green's function $G(q_1, t_1; q_2, t_2) = \int_{q(t_1)=q_1}^{q(t_2)=q_2} [dq] e^{-S[q]}$.

Working by analogy with our treatment of field theory, show that any **smooth**¹ potential V is a *relevant* perturbation of the free particle, *i.e.* the Gaussian fixed point with $H = \frac{p^2}{2m}$.

Hint: change variables to $\phi(t) \equiv \sqrt{m} q(t)$.

Use this to explain in words why the high energy asymptotics of the density of states

$$N(E) \equiv \{\# \text{ of eigenvalues of } H \text{ less than } E\}$$

¹Some singular potentials are also relevant perturbations. If $V(q) \sim q^{-\alpha}$, how big can α be for my statement to remain true? Thanks to Brian Vermilyea for reminding me that a singular enough potential will cause trouble.

is given by the *Weyl formula* (even for $V(q) \neq 0$):

$$N(E) = E^{d/2} K_d L^d + \dots$$

where $K_d = \frac{\Omega_{d-1}}{(2\pi)^d}$ as usual, and L is the linear size of the box in which we put the particle (an IR cutoff).

Hint: we can represent the density of states by a path integral using an inverse Laplace transform:

$$\text{tr} \frac{1}{\omega - H} = \int d\beta e^{\beta\omega} Z(\beta)$$

and the relation

$$\text{Im} \frac{1}{\omega + i\epsilon - H} = \pi\delta(\omega - H).$$

In a series of problems starting with the next one, we will study free massless bosons in two dimensions. This system is solvable and has many physical applications – *e.g.* in string theory, and at the edge of quantum Hall systems. It is an example of a *conformal field theory*. It is a field theory where the excitations are *not particles*. And it's an example where we can understand all the composite operators exactly.

3. There is no continuous symmetry-breaking in two dimensions.

(a) Consider a massless scalar X in 2d, with action

$$S[X] = -\frac{1}{4\pi} \int d^2\sigma \partial_a X \partial^a X. \quad (1)$$

Show that the euclidean Green function G_2 satisfies

$$\nabla^2 G_2(z, z') = -2\pi\delta^2(z - z') \quad (2)$$

($z = \sigma_1^E + i\sigma_2^E$)² and is given by

$$G_2(z, z') = -\ln |z - z'|,$$

for example by Fourier transform.

(b) [Perhaps this part is more of a diatribe than a problem.] The long-distance behavior of G_2 has important implications for the physics of massless scalars in two dimensions. Thinking of G_2 as the two point function of the massless scalar

$$G_2(z, z') = \langle X(z)X(z') \rangle$$

²Hint: this is a Schwinger-Dyson equation.

let's ask the following question:

There is no potential energy for the field X in (1). Someone used to doing physics in 3+1 dimensions might think that this means that there is a vacuum for every value of X . Let's try to fix the expectation value of the scalar $\langle X \rangle = x$ and see what happens. Perturb the putative vacuum $|x\rangle$ a little bit at the position z by inserting the operator X there. To measure what happens, insert the operator X at z' . The correlator G_2 can thus be interpreted as a measurement of how the effects of our perturbation fall off with distance at large $|z - z'|$. What happens? Contrast this with the behavior you would see for a scalar field with a flat potential in more than two dimensions. Note that the case of (0+1) dimensional QFT (*i.e.* quantum mechanics) is even more problematic in the infrared.

One way to arrive at an action like (1) is if the field X arises as a Goldstone boson associated with a symmetry $X \rightarrow X + a$, which would be spontaneously broken by fixing the vacuum $|x\rangle$. Then it is guaranteed by Goldstone's theorem that the action can only depend on derivatives of X . Note that the Goldstone-ness of the massless bosons (*i.e.* whether they are massless because of a broken symmetry) is not crucial for this discussion. One might expect massless bosons whose masslessness is not protected by a symmetry to be lifted (to acquire a mass) quantumly, but there are special cases (such as in supersymmetric theories) where different points on the space of minima of the potential need not be related by a symmetry.

This result is called the Hohenberg-Coleman-Mermin-Wagner Theorem. Coleman's paper on the subject is S. Coleman, "There are no Goldstone bosons in two-dimensions," *Commun. Math. Phys.* **31**:259-264 (1973). It should be called "There is no continuous symmetry-breaking in two dimensions."

Free massless scalars in $D = 1 + 1$ are important for many reasons and we'll study them some more.