University of California at San Diego – Department of Physics – Prof. John McGreevy

# Physics 215C QFT Spring 2026
# Assignment 5

**Due 11:59pm Monday, May 4, 2026**

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1. **Correlators of composite operators made of free bosons in 1+1 dimensions.**

   This problem is a continuation of the last problem on the previous homework.

   Consider a collection of $n$ two-dimensional free bosons $X^\mu$ governed by the action

   $$S = -\frac{1}{4\pi g}\int d^2\sigma \partial_a X_\mu \partial^a X^\mu .$$

   [The coupling $g$ can be absorbed into the definition of $X$ if we prefer, but it is useful to leave this coupling constant arbitrary since different physicists use different conventions for the normalization and as you will see this affects the appearance of the final answer.]

   Until further notice, we will assume that $X$ takes values on the real line.

   (a) Rotate $e^{\mathbf{i}S}$ to Euclidean space ($d^2\sigma = -\mathbf{i}(d^2\sigma)_E$) and compute the Euclidean generating functional

   $$Z[J] = \left\langle e^{\int (d^2\sigma)_E J^\mu X_\mu} \right\rangle \equiv Z_0^{-1}\int [dX] e^{\mathbf{i}S} e^{\int (d^2\sigma)_E J^\mu X_\mu}$$

   (where $Z_0^{-1} \equiv Z[J=0]$ but please don't worry too much about the normalization of the path integral).

   [Hint: use the Green function from the previous problem, and Wick's theorem. Or use our general formula for Gaussian integrals with sources.]

   [Warning: In the problem at hand, even the euclidean kinetic operator has a kernel, namely the zero-momentum mode. You will need to do this integral separately.]

   [Cultural remark 1: this field theory describes the propagation of featureless strings in $n$-dimensional flat space $\mathbb{R}^n$ – think of $X^\mu(\sigma)$ as the parametrizing the position in $\mathbb{R}^n$ to which the point $\sigma$ is mapped.

   Cultural remark 2: this is an example of a *conformal field theory*. In particular recall that massless scalars in $D = 2$ have engineering dimension zero.]

(b) Show that

$$\left\langle \prod_{i=1}^{N} : e^{-i\sqrt{2\alpha'}k_i \cdot X(\sigma^{(i)})} : \right\rangle = \delta^n \left( \sum_i k_i^\mu \right) \prod_{i,j=1}^{N} |z_i - z_j|^{+\alpha' g k_i \cdot k_j} \tag{1}$$

where $\sigma^{(i)}$ label points in 2d Euclidean space, $z_i \equiv \sigma_1^{(i)} + i\sigma_2^{(i)}$, $\alpha'$ is a parameter with dimensions of $[X^2/g]$ (called the 'Regge slope'), and $k_i^\mu$ are a set of arbitrary $n$-vectors in the target space. The : ... : indicate the following prescription for *defining* composite operators. The prescription is simply to leave out Wick contractions of objects within a pair of : ... :. Give a symmetry explanation of the delta function in $k$.

[Cultural remark: this calculation is the central ingredient in the *Veneziano amplitude* for scattering of bosonic strings at tree level.]

(c) Conclude that the composite operator $\mathcal{O}_a \equiv: e^{\mathbf{i}aX}$ : has *scaling dimension* $\Delta_a = \frac{ga^2}{2}$, in the sense that

$$\left\langle \mathcal{O}_a(z) \mathcal{O}_b^\dagger(0) \right\rangle = \delta(a-b) \frac{1}{|z|^{2\Delta_a}}.$$

Notice that the correlation functions of these operators do not describe the propagation of *particles* in any sense. The operator $\mathcal{O}$ produces some power-law excitation of the CFT soup.

(d) Suppose we have one field ($n = 1$) $X$ which takes values on the circle, that is, we identify

$$X \simeq X + 2\pi R\ .$$

What values of $a$ label single-valued operators : $e^{\mathbf{i}aX}$ : ? How should we modify (1)?

2. **Order parameter exponent at the Wilson-Fisher fixed point.** [bonus problem] In lecture we outlined the computation of $\eta$ using position space diagrams. Find the coefficient $c$ in $\eta = c\epsilon^2 + \mathcal{O}(\epsilon^3)$ as a function of $n$ at the $\mathsf{O}(n)$ Wilson-Fisher fixed point. Check that the factors of $r_2$ drop out.

   [Hint: The answer for the Ising model ($n = 1$) is $\eta = \frac{\epsilon^2}{54}$.]

3. **OPE.** [bonus problem] Consider the Gaussian fixed point with $\mathsf{O}(n)$ symmetry. Compute the OPE coefficients for the operators $\mathcal{O}_2 \equiv: \phi_a\phi_a$ :, $\mathcal{O}_4 \equiv: (\phi_a\phi_a)^2$ :, and the identity operator (here $a = 1..n$ and the repeated index is summed). Use this information to compute the beta function, find the Wilson-Fisher fixed point and the correlation length critical exponent $\nu$ there.

4. **RG analysis of less-symmetric spin systems.**

   Suppose that we break the rotation symmetry of the $\mathsf{O}(n)$ model to the subgroup of $\pi/2$ rotations, *i.e.* the cubic symmetry, (for example, for $n = 2$, $(s_1, s_2) \to (s_2, -s_1)$.) If the spins live on the cubic lattice, a spin-orbit coupling could do this.

   (a) Don't look at the next part of the problem yet! What functions of an $n$-vector $\phi_a$ are invariant under $\pi/2$ rotations, but not general rotations?

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$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

(b) Show that in addition to the usual $\mathsf{O}(n)$-symmetric interaction

$$\int d^d x \sum_{a,b=1}^{n} u\phi_a^2\phi_b^2,$$

the LG free energy should include a term of the form

$$\int d^d x \sum_{a=1}^{n} v\phi_a^4.$$

Argue that this is the only new term (preserving cubic symmetry but not the full $\mathsf{O}(n)$ symmetry) which can be a relevant perturbation of the Gaussian fixed point near $d = 4$.

(c) Treating $\mathcal{O}(u) = \mathcal{O}(v) = \mathcal{O}(\epsilon)$, redo the analysis of the running couplings in $d = 4 - \epsilon$ dimensions to derive beta functions for $u$ and $v$ up to corrections of order $\mathcal{O}(u^3) = \mathcal{O}(v^2 u) = ... = \mathcal{O}(\epsilon^3)$.

(d) Your answer to the previous part will be of the form

$$\begin{aligned} -s\partial_s u &= -\epsilon u + A_1 u^2 + A_2 uv + A_3 v^2 + \mathcal{O}(u^3) \\ -s\partial_s v &= -\epsilon v + B_1 u^2 + B_2 uv + B_3 v^2 + \mathcal{O}(u^3). \end{aligned} \tag{2}$$

You should find that $A_3 = B_1 = 0$. Find four fixed points:

- The gaussian fixed point.
- A fixed point where only $u \neq 0$.
- A fixed point where only $v \neq 0$. Describe the physics of this fixed point. (Hint: the action is a sum of $n$ terms.)
- A fixed point where both $(u, v)$ are nonzero.

(In every case, the assumption of $u \sim v \sim \epsilon$ is self-consistent.)

(e) Analyze the stability of these fixed points (by computing the matrix of derivatives of the beta functions at each fixed point). Draw the phase diagram. Which fixed point dominates the critical behavior? You will want to consider different cases depending on whether $n > 4$ or $n < 4$.

(f) When $n > 4$ you may find that $v$ wants to become negative. This means that the effective potential for $m$ becomes unbounded, within our approximation. What have we left out that will restore sanity? What does this mean for the order of the phase transition? (Notice that mean field theory predicts a continuous transition, so any change in this conclusion is a dramatic effect of the fluctuations, more dramatic than just changing the values of critical exponents by a little.)