

University of California at San Diego – Department of Physics – Prof. John McGreevy  
**Physics 215C QFT Spring 2026**  
**Assignment 6**

Due 11:59pm Monday, May 11, 2026

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1. **Topological terms in QM.**

The purpose of this problem is to demonstrate that total derivative terms in the action (like the  $\theta$  term in QCD) do affect the physics.

The Euclidean path integral for a particle on a ring with magnetic flux  $\theta = \int \vec{B} \cdot d\vec{a}$  through the ring is given by

$$Z = \int [D\phi] e^{-\int_0^\beta d\tau \left( \frac{m}{2} \dot{\phi}^2 - i \frac{\theta}{2\pi} \dot{\phi} \right)} .$$

Here

$$\phi \equiv \phi + 2\pi \tag{1}$$

is a coordinate on the ring. Because of the identification (1),  $\phi$  need not be a single-valued function of  $\tau$  – it can wind around the ring. On the other hand,  $\dot{\phi}$  is single-valued and periodic and hence has an ordinary Fourier decomposition. This means that we can expand the field as

$$\phi(\tau) = \frac{2\pi}{\beta} Q\tau + \sum_{\ell \in \mathbb{Z} \setminus 0} \phi_\ell e^{i \frac{2\pi}{\beta} \ell \tau} . \tag{2}$$

- (a) Show that the  $\dot{\phi}$  term in the action does not affect the classical equations of motion. In this sense, it is a topological term.
- (b) Using the decomposition (2), write the partition function as a sum over topological sectors labelled by the *winding number*  $Q \in \mathbb{Z}$  and calculate it explicitly.

[Hint: use the Poisson resummation formula

$$\sum_n f(n) = \sum_l \hat{f}(2\pi l)$$

where  $\hat{f}(p) = \int dx e^{-ipx} f(x)$  is the fourier transform of  $f$ .]

- (c) Use the result from the previous part to determine the energy spectrum as a function of  $\theta$ .
- (d) Above we wrote the Euclidean path integral. Perform a Wick rotation to find the real-time action.

(e) Derive the canonical momentum and Hamiltonian from the action above and verify the spectrum.

(f) [bonus] Consider what happens in the limit  $m \rightarrow 0, \theta \rightarrow \pi$  with  $X \equiv \frac{\theta - \pi}{m} \sim \beta^{-1}$  fixed. Interpret the result as the partition function for a spin 1/2 particle. What is the meaning of the ratio  $X$  in this interpretation?

2. **O(N) model at large N.** In lecture we studied the O(N) model in an expansion in  $\epsilon = 4 - D$ . When  $N$  is large, there is another small parameter in which to expand. We'll see that the results are consistent with the  $\epsilon$  expansion. It is also an example that illustrates the manipulations we'll do in describing the BCS phenomenon.

Consider the (Euclidean) partition function for an  $N$ -vector of scalar fields in  $D$  dimensions:

$$Z = \int [d\phi] e^{iS[\phi]}, \quad S[\vec{\phi}] = \int d^D x \left( \partial_\mu \phi^a \partial^\mu \phi^a - r \phi^a \phi^a - \frac{g}{N} (\phi^a \phi^a)^2 \right).$$

(a) At the free fixed point, what is the dimension of the coupling  $g$  as a function of the number of spacetime dimensions  $D$ ? Show that it is classically marginal in  $D = 4$ , so that this action is (classically) scale invariant for  $r = 0$ .

(b) [optional] Show that the definition above of  $u = g/N$  is a good idea if we want to take  $N \rightarrow \infty$ , at fixed  $g$ . Do this by considering the  $N$ -dependence of diagrams that contribute to, say, the free energy, and demanding that in the large- $N$  limit the interaction terms contribute with the same power of  $N$  as the leading term.

(c) Analyze, at large  $N$ , the critical behavior of the model as  $r$  is varied. You'll need to consider separately the regimes  $D > 4, D = 4, 2 < D < 4, D \leq 2$ . Here are the steps: first use the Hubbard-Stratonovich trick to replace  $\phi^4$  by  $\sigma \phi^2 + \sigma^2$  (up to factors) in the action, where  $\sigma$  is a new scalar field<sup>1</sup>. Then integrate out the  $\phi$  fields. Find the saddle point equation for  $\sigma$ ; argue that the saddle point dominates the integral for large  $N$ . Regulate the integrals in a convenient way. Find a translation invariant saddle point (*i.e.* where  $\sigma$  is constant). Plug the saddle point configuration of  $\sigma$  back into the action for  $\phi$  and describe the resulting dynamics.

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<sup>1</sup>To be more explicit, use (a path-integral version of) the identity

$$e^{u\phi^4} = \frac{1}{\sqrt{\pi u}} \int_{-\infty}^{\infty} d\sigma e^{-\sigma^2/u - 2\phi^2\sigma}.$$

Compare with the results of the  $\epsilon$  expansion.

- (d) [bonus] Compute the correlation-length critical exponent  $\nu$  at leading order in large  $N$ . Compare with the epsilon expansion results.