

University of California at San Diego – Department of Physics – Prof. John McGreevy
Physics 215C QFT Spring 2026
Assignment 8

Due 11:59pm Monday, May 25, 2026

1. Galilean transformation of non-relativistic fields.

Show that the action

$$S = \int dt d^d x \left(\Phi^* \mathbf{i} \partial_t \Phi - \frac{1}{2m} \vec{\nabla} \Phi^* \cdot \vec{\nabla} \Phi - V(|\Phi|) \right) \quad (1)$$

is invariant under Galilean boosts, in the form

$$\Phi(\vec{x}, t) \rightarrow \Phi'(\vec{x}', t') \quad \text{with} \quad \Phi(\vec{x}, t) = e^{-\frac{1}{2}mv^2 t + im\vec{v} \cdot \vec{x}} \Phi'(\vec{x}', t') \quad (2)$$


with $t' = t$, $x'_i = x_i - v_i t$.

Note that this is also how the nonrelativistic single-particle wavefunction must transform in order to preserve the Schrödinger equation.

Don't forget that $\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu}$.

How does the boost act on the Goldstone mode in the symmetry-broken phase?

2. Diagrammatic understanding of BCS instability of Fermi liquid theory.

- (a) Recall that only the four-fermion interactions with special kinematics are marginal. Keeping only these interactions, show that cactus diagrams (like this: ) dominate.
- (b) To sum the cacti, we can make bubbles with a corrected propagator. Argue that this correction to the propagator is innocuous and can be ignored.
- (c) Armed with these results, compute diagrammatically the Cooper-channel susceptibility (two-particle Green's function),

$$\chi(\omega_0) \equiv \left\langle \mathcal{T} \psi_{\vec{k}, \omega_3, \downarrow}^\dagger \psi_{-\vec{k}, \omega_4, \uparrow}^\dagger \psi_{\vec{p}, \omega_1, \downarrow} \psi_{-\vec{p}, \omega_2, \uparrow} \right\rangle$$

as a function of $\omega_0 \equiv \omega_1 + \omega_2$, the frequencies of the incoming particles. Think of χ as a two point function of the Cooper pair field $\Phi = \epsilon_{\alpha\beta} \psi_\alpha \psi_\beta$ at zero momentum.

Sum the geometric series in terms of a (one-loop) integral kernel.

- (d) Do the integrals. In the loops, restrict the range of momenta to $|\epsilon(k)| < E_D$, the Debye energy, since it is electrons with these energies that experience attractive interactions.

Consider for simplicity a round Fermi surface. For doing integrals of functions singular near a round Fermi surface, approximate the dispersion relation as $\epsilon(k) \simeq v_F(|k| - k_F)$, so that $d^d k \simeq k_F^{d-1} \frac{d\xi}{v_F} d\Omega_{d-1}$. I recommend doing to the frequency integral first (by residues).

- (e) Show that when $V < 0$ is attractive, $\chi(\omega_0)$ has a pole. Does it represent a bound-state? Interpret this pole in the two-particle Green's function as indicating an instability of the Fermi liquid to superconductivity. Compare the location of the pole to the energy E_{BCS} where the Cooper-channel interaction becomes strong.
- (f) **Cooper problem.** [optional] We can compare this result to Cooper's influential analysis of the problem of two electrons interacting with each other in the presence of an inert Fermi sea. Consider a state with two electrons with antipodal momenta and opposite spin

$$|\psi\rangle = \sum_k a_k \psi_{k,\uparrow}^\dagger \psi_{-k,\downarrow}^\dagger |F\rangle$$

where $|F\rangle = \prod_{k < k_F} \psi_{k,\uparrow}^\dagger \psi_{k,\downarrow}^\dagger |0\rangle$ is a filled Fermi sea. Consider the Hamiltonian

$$H = \sum_k \epsilon_k \psi_{k,\sigma}^\dagger \psi_{k,\sigma} + \sum_{k,k'} V_{k,k'} \psi_{k,\sigma}^\dagger \psi_{k,\sigma} \psi_{k',\sigma'}^\dagger \psi_{k',\sigma'}.$$

Write the Schrödinger equation as

$$(\omega - 2\epsilon_k) a_k = \sum_{k'} V_{k,k'} a_{k'}.$$

Now assume (following Cooper) that the potential has the following form:

$$V_{k,k'} = V w_{k'}^* w_k, \quad w_k = \begin{cases} 1, & 0 < \epsilon_k < E_D \\ 0, & \text{else} \end{cases}.$$

Defining $C \equiv \sum_k \omega_k^* a_k$, show that the Schrödinger equation requires

$$1 = V \sum_k \frac{|w_k|^2}{\omega - 2\epsilon_k}. \quad (3)$$

Assuming V is attractive, find a bound state. Compare (3) to the condition for a pole found from the bubble chains above.

3. Fermion propagator in a metal. [bonus problem]

Starting from

$$G(p, t) = -\frac{1}{2\pi\mathbf{i}} \langle \text{gs} | \mathcal{T} c_p(t) c_p^\dagger(0) | \text{gs} \rangle \quad (4)$$

and using the free fermion time evolution operator, and the fact that the ground-state has all levels filled up to the Fermi level:

$$\langle \text{gs} | c_p^\dagger c_p | \text{gs} \rangle = \begin{cases} 1, & \epsilon_p < 0 \\ 0, & \epsilon_p > 0 \end{cases} \quad (5)$$

show that the free fermion propagator can be written as

$$G(p, \omega) = \frac{a}{\omega - \epsilon_p - \mathbf{i}\eta b \text{sgn}(\epsilon_p)} \quad (6)$$

or

$$G(p, \omega) = \frac{a'}{\omega(1 + \mathbf{i}b'\eta) - \epsilon_p} \quad (7)$$

where $\eta = 0^+$ is an infinitesimal for some constants a, b, a', b' to be determined.