

Quantum Mechanics C (130C) Winter 2015 Midterm exam

Thursday, February 19, 2015, 11am-12:20pm

Please remember to put your name on your exam booklet. This is a closed-book exam. There are **4** problems, each with several parts, of varying levels of difficulty; make sure you try all of the parts, as there are many opportunities for partial credit. None of the problems should require very extensive calculation; if you find yourself involved in a morass of calculation, step back and think. Good luck.

Possibly useful information:

$$\boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\uparrow_{\tilde{n}}\rangle = e^{-i\varphi/2} \cos \frac{\theta}{2} |\uparrow_z\rangle + e^{+i\varphi/2} \sin \frac{\theta}{2} |\downarrow_z\rangle \quad \text{satisfies} \quad \vec{\sigma} \cdot \tilde{n} |\uparrow_{\tilde{n}}\rangle = |\uparrow_{\tilde{n}}\rangle$$

$$e^{-i\alpha\tilde{n}\cdot\vec{\sigma}} = \mathbb{1} \cos \alpha - i\tilde{n} \cdot \vec{\sigma} \sin \alpha.$$

1. **Short answers and conceptual questions.** [3 or 4 points each]

For the first **six** sub-problems (a-f), the Hilbert space of our system has an orthonormal basis $\{|\uparrow\rangle, |\downarrow\rangle, |\otimes\rangle\}$.

- (a) In the state $|\phi\rangle = \frac{1}{\sqrt{3}}|\uparrow\rangle + \sqrt{\frac{2}{3}}|\otimes\rangle$ what is the expectation value of the operator $|\otimes\rangle\langle\otimes|$?
- (b) Show that $|\psi\rangle \equiv |\uparrow\rangle + |\downarrow\rangle$ is an eigenvector of the operator

$$\mathbf{W} \equiv 2(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) + |\otimes\rangle\langle\otimes|$$

and find the associated eigenvalue.

- (c) Which of the following operators can implement symmetry operation? Which can be a representation of an observable?

$$\begin{aligned} \mathbf{A} &\equiv |\otimes\rangle\langle\downarrow| + |\downarrow\rangle\langle\otimes|, & \mathbf{B} &\equiv |\downarrow\rangle\langle\downarrow| + \mathbf{i} \cos\theta |\uparrow\rangle\langle\otimes|, \\ \mathbf{C} &\equiv \cos\theta |\otimes\rangle\langle\otimes| + \sin\theta (|\otimes\rangle\langle\uparrow| - |\uparrow\rangle\langle\otimes|) + \cos\theta |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|, \end{aligned}$$

(θ is real).

Suppose we (i) prepare the 3-state system in the state $|\downarrow\rangle$, then (ii) measure an observable \mathbf{M} . We do steps (i) and (ii) repeatedly and determine that there are two possible outcomes m_1, m_2 , which occur with probabilities $p_1 = p_2 = \frac{1}{2}$.

- (d) Does \mathbf{M} commute with $|\downarrow\rangle\langle\downarrow|$?
- (e) What can we infer about the form of the operator \mathbf{M} ?
- (f) Given these experiments, what is the most general form of $|\downarrow\rangle$, written in terms the eigenvectors of \mathbf{M} ?

- (g) Is this matrix

$$\rho \stackrel{?}{=} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a possible density matrix? If not, why not?

- (h) A two-state system A is part of a larger system, $\mathcal{H}_A \otimes \mathcal{H}_B$. The subsystem A is described by the density matrix $\rho = |\uparrow_z\rangle\langle\uparrow_z|$. Which of the following could be the state of the whole system? Please explain.

$$|\text{choice 1}\rangle = |\uparrow_z\rangle_A \otimes |\uparrow_z\rangle_B \quad |\text{choice 2}\rangle = \frac{1}{\sqrt{2}} (|\uparrow_z\rangle_A \otimes |\uparrow_z\rangle_B + |\downarrow_z\rangle_A \otimes |\downarrow_z\rangle_B).$$

Mixture versus superposition. In the next four sub-problems (ijkl), consider the following two density matrices for a qbit:

$$\rho_1 \equiv \frac{1}{2} |\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2} |\uparrow_x\rangle\langle\uparrow_x|$$

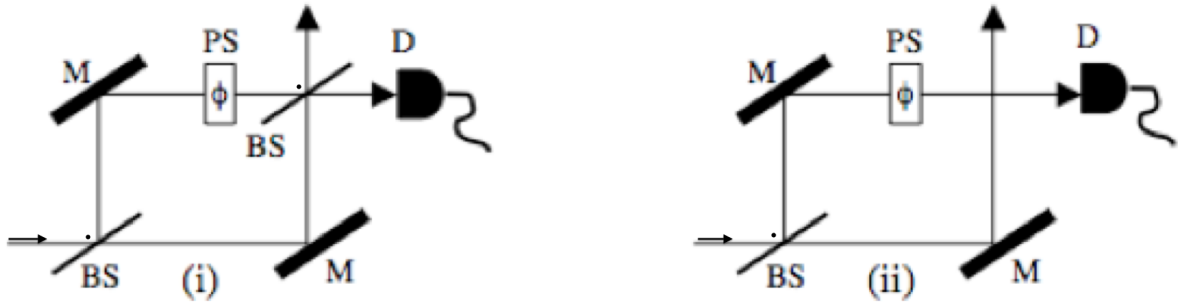
$$\rho_2 \equiv |\phi\rangle\langle\phi|$$

where $|\phi\rangle$ is a ket that you will determine.

- (i) Which represents a mixed state?
- (j) Compute the probability to measure spin up along z in the state ρ_1 .
- (k) Find a state $|\phi\rangle$ for which ρ_2 produces the same probability to measure spin up along z as you found in the previous sub-problem.
- (l) With the choice of $|\phi\rangle$ from the previous problem, tell me a measurement which can distinguish the states ρ_1 and ρ_2 and describe its outcome in each case.

2. **Interferometry** [from Boccio] [20 points]

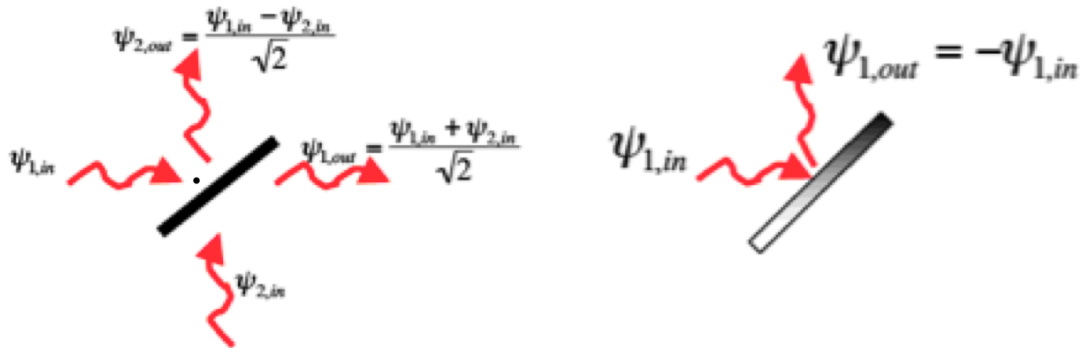
Consider the following two interferometers:



The first one (i) is a ‘balanced’ Mach-Zehnder interferometer, with equal path lengths, perfectly reflecting mirrors (M) and 50-50 beam splitters (BS). The input beam is at the lower left. A transparent piece of glass which imparts a phase shift (PS) ϕ is placed in one arm. Photons are detected (D) at one port. The second interferometer (ii) is the same as the first except that the final beam splitter is omitted.

For each device, determine and sketch the probability for detecting a photon at D, as a function of ϕ . Explain your answer, and explain clearly how you label the intermediate states.

The rules and conventions for the beamsplitters and mirrors (respectively) are depicted in this figure (recall that the dot indicates the orientation of the beamsplitter):



3. **Oscillations.** [20 points]

A *giggleblatz* is a fictional quantum system which can be detected either as a giggle (g) or a blatz (b). That is, its Hilbert space is

$$\mathcal{H} = \text{span}\{|g\rangle, |b\rangle\}$$

(this is an orthonormal basis).

Its Hamiltonian is

$$\mathbf{H} = a\mathbb{1} + b\mathbf{M}$$

where a, b are constants and \mathbf{M} is an observable with eigenvalues m_1, m_2 and respective eigenvectors

$$\begin{aligned} |m_1\rangle &= \cos\theta|g\rangle + \sin\theta|b\rangle \\ |m_2\rangle &= -\sin\theta|g\rangle + \cos\theta|b\rangle . \end{aligned}$$

- (a) Write the spectral decomposition of \mathbf{M} .
- (b) At $t = 0$, a giggleblatz is in the state $|g\rangle$. If we were to measure the observable \mathbf{M} , what are the possible outcomes and with what probabilities will they occur?
- (c) At $t = 0$, a giggleblatz is in the state $|g\rangle$. At time t it reacts with matter inside of a detector. What is the probability that it is detected as a $|b\rangle$?
[Hint: I found it useful to introduce the notation $\alpha_i \equiv t(a + bm_i)$.]

4. **An electric dipole in an electric field.** [20 points]

Consider a system which has an electric dipole moment \vec{d} ; this means that in the presence of an electric field $\vec{\mathcal{E}}$ the system is governed by the Hamiltonian

$$\mathbf{H} = -\vec{d} \cdot \vec{\mathcal{E}} + \mathbf{H}_0$$

where \mathbf{H}_0 is independent of the electric field.

Quantum mechanically, the dipole moment is an observable. Consider a two-state system with Hilbert space $\mathcal{H} = \text{span}\{|+\rangle, |-\rangle\}$ (these states are orthonormal). Suppose the dipole moment operator is

$$\vec{\mathbf{d}} = d(|+\rangle\langle -| + |-\rangle\langle +|)\tilde{x}$$

where d is a constant and \tilde{x} is a unit vector in 3-space. (This is a simple model for *e.g.* an ammonia molecule.)

Suppose further that the field-independent part of the Hamiltonian is

$$\mathbf{H}_0 \equiv E_0|-\rangle\langle -| \tag{1}$$

for some $E_0 > 0$.

- (a) Show that with this choice of field-independent energetics (1) the ground state (state of lowest energy) in the absence of an electric field is $|+\rangle$.
- (b) Compute the expectation value of the dipole moment in the groundstate, in zero electric field.
- (c) Suppose we put the system in a time-independent electric field $\vec{\mathcal{E}} = \mathcal{E}\tilde{x}$. What is the ground state energy as a function of \mathcal{E} ?
- (d) Assume that the electric field is weak ($d\mathcal{E} \ll E_0$). To leading non-vanishing order in $\frac{d\mathcal{E}}{E_0}$, calculate the expectation value of the dipole moment in the groundstate.
- (e) Suppose we put the system in an electric field which varies in space as $\vec{\mathcal{E}} = \mathcal{E}(y)\tilde{x}$. Continuing to assume a weak field ($d\mathcal{E} \ll E_0$), calculate the force on the system. (Recall that a force results from an energy which depends on position.)