

Physics 217: The Renormalization Group

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0.1 Introductory comments

The ‘renormalization group’ (RG) is a poor name for the central concept in many-body physics. It is a framework for addressing the question: what is the relationship between microscopic laws and macroscopic observations?

Or, closer to home, it allows us to answer questions such as: Why don’t you need to understand nuclear physics to make your tea in the morning?¹

There is a lot more to say to contextualize the RG, which, as you can see from the previous question, is really a piece of metaphysics, that is, it is a framework for how to do physics. But since it is such a broad and far-reaching concept, in order to avoid being vague and useless, it will be better to start with some concrete and simple ideas, before discussing of some of its many consequences.

A word about prerequisites: The official prerequisite for this course is graduate statistical mechanics. I think you would be able to get by with a good undergrad class.

The historical origins of the RG (at least its name) are tied up with high-energy particle physics and quantum field theory. That stuff involves quantum mechanics in a serious way. This course will not. I think I can say honestly that all of our fluctuations will be thermal.

(Advertisement: Next quarter I am teaching a special topics course for which at the moment the plan is a continuation of this course to the realm of quantum many body physics.)

Also, I think our discussion will all be non-relativistic, $v \ll c$.

Initial Tentative Plan:

1. Scaling and self-similarity
2. RG treatment of random walks
3. Ising model
4. RG treatment of iterative maps and the period-doubling approach to chaos
5. RG treatment of percolation and lattice animals
6. Critical phenomena (perhaps it is a mistake to postpone this great victory of the RG).

¹This framing of the question I heard from Savas Dimopoulos.

7. $4 - \epsilon$ expansions
8. RG understanding of the method of matched asymptotic expansions
9. RG treatment of stochastic PDEs

As the title indicates, this is a very rough guess for what we'll do. An early target will be a renormalization-group understanding of the central limit theorem.

Sources for these notes (anticipated):

Introduction to Renormalization Group Methods in Physics, by R. J. Creswick, H. A. Farach, C. P. Poole.

Lectures on Phase Transitions and the Renormalization Group, by Nigel Goldenfeld.

Statistical Physics of Fields, by Mehran Kardar.

Introduction to Statistical Field Theory, by Eduard Brézin.

Renormalization Group Methods, a guide for beginners, by W. D. McComb.

Scaling and Renormalization in Statistical Physics, by John Cardy.

Modern Theory of Critical Phenomena, by S.-K. Ma (UCSD).

0.2 Conventions

The convention that repeated indices are summed is always in effect unless otherwise indicated.

A useful generalization of the shorthand $\hbar \equiv \frac{\hbar}{2\pi}$ is

$$\mathfrak{d}k \equiv \frac{dk}{2\pi}.$$

I will also write $\delta(q) \equiv (2\pi)^d \delta^d(q)$.

I will try to be consistent about writing fourier transforms as

$$\int \frac{d^d k}{(2\pi)^d} e^{ikx} \tilde{f}(k) \equiv \int \mathfrak{d}^d k e^{ikx} \tilde{f}(k) \equiv f(x).$$

RHS \equiv right-hand side.

LHS \equiv left-hand side.

BHS \equiv both-hand side.

IBP \equiv integration by parts.

$+\mathcal{O}(x^n)$ \equiv plus terms which go like x^n (and higher powers) when x is small.

I reserve the right to add to this page as the notes evolve.

Please tell me if you find typos or errors or violations of the rules above.

1 Scaling and self-similarity

[This discussion largely follows the early chapters of the book by Creswick et al.] First some somewhat-vague definitions to get us started. An object is *self-similar* if its parts, when magnified by a suitable scale factor λ , look like the whole. Something is *scale-invariant* if this is true for every λ . An important generalization is the notion of *statistical self-similarity* – something which is sampled from a distribution which is self-similar.

The point in life of the renormalization group is to provide a way of thinking about (and ideally relating quantitatively) what’s going on at different scales. So something which is self-similar is a simple special case for the RG.

A symptom of scale invariance is a power law.

1.1 Fractal dimension

The word ‘dimension’ is used in many ways in this business. Let’s consider a set of points in d -dimensional Euclidean space, \mathbb{R}^d . In the previous sentence ‘dimension’ is the minimum number of coordinates needed to specify the location of a point (this is usually called ‘Euclidean dimension’). It’s an integer. A subset of \mathbb{R}^d specified by some algebraic equations on the coordinates generically has a Euclidean dimension which is an integer (though it may not be the same integer for every point). That is, locally around almost every solution of the equations, the object will look like a piece of \mathbb{R}^{d_T} for some $d_T \leq d$ (sometimes this notion is called ‘topological dimension’).

Here is a different, RG-inflected definition of the dimension of an object $O \subset \mathbb{R}^d$, called fractal dimension or Hausdorff dimension: cover the object O with d -balls of radius a ,

$$B_{r_0}(a) \equiv \{\vec{r} \in \mathbb{R}^d \text{ such that } |\vec{r} - \vec{r}_0| \leq a\}. \quad (1.1)$$

Let

$N(a) \equiv$ the minimum number of such balls required to cover O ,

minimizing over the locations of their centers. Do this for various values of a . Then, if this function is a power law,

$$N(a) \sim a^{-D} \quad (1.2)$$

then D is the fractal dimension of O .

A few observations:

- Notice that D may itself depend on the range of ball-sizes a we consider, that is the same scaling relation may not hold for all a . Often (always) there is a

short-distance cutoff on the scaling relation (1.2) – if our object is the coastline of France, it is maybe not so useful to consider femtometer-sized balls.

- For objects defined by algebraic equations, $D = d_T$.
- O is a set of points in \mathbb{R}^d , the dimension of the objects composing O (maybe points themselves, maybe line segments...) have some topological dimension d_T , and

$$d_T \leq D \leq d.$$

Where does the right bound come from? By placing d -balls centered at the sites of a cubic lattice (with a radius a proportional to the lattice spacing) we can cover a whole region of \mathbb{R}^d with a number that goes like a^{-d}

It behooves us to give some examples with $D \notin \mathbb{Z}$. Such things are called *fractals* and are generally defined recursively.

1. A Cantor set in $d = 1$ can be defined beginning with a line segment of length a_0 . Divide it in thirds and remove the middle segment. Repeat for each sub-segment. At the end of this process, we get a bunch of points ($d_T = 0$) in $d = 1$. According to our definition (1.1), 1-ball of radius a is an interval of length $2a$. After n steps of the above procedure, we end up with $2^n \equiv N$ line segments of length $a_n = a_0 3^{-n}$, $n = 1, 2, 3, \dots$ (that's what I'm drawing). Since we only remove stuff, we can cover the whole thing with these, so we have a lower bound of $N(a_n) = 2^n$, and you can't do better. Eliminate $n : n = -\frac{\log \frac{a_n}{a_0}}{\log 3}$, so

$$N(a) = 2^n = 2^{-\frac{\log \frac{a_n}{a_0}}{\log 3}} = \left(\frac{a}{a_0} \right)^{-\frac{\log 2}{\log 3}}$$

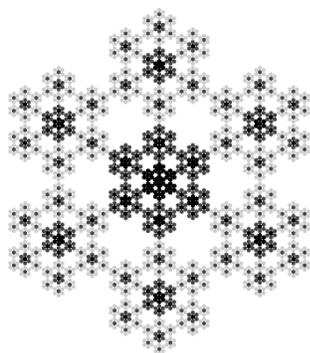
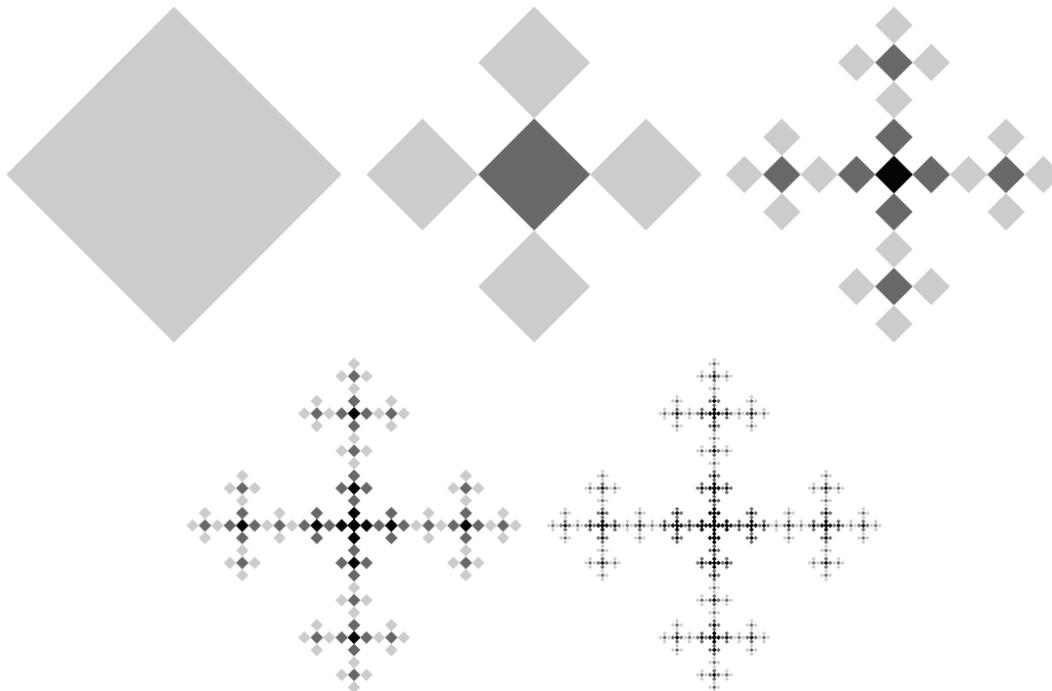
which gives fractal dimension

$$D = \frac{\log 2}{\log 3} \simeq .63 \in (0, 1) . \tag{1.3}$$

Notice that this object is self-similar with scale factor $\lambda = 3$: the two remaining thirds are identical to the original up to a rescaling of the length by a factor of three. This fact can be used to infer the power-law, since it means $N(a) = 2N(3a)$. So if $N(a) \sim a^{-D}$, we must have $a^{-D} = 2(3a)^{-D} \implies 1 = 2 \cdot 3^{-D}$ which is (1.3).

2. Here's an example in $d = 2$. Take a square with side length a_0 . Now divide it into nine squares by dividing each side by three. Remove every other sub-square,

leaving the corners. Repeat. This procedure gives the accompanying figures. The resulting figure is again self-similar with $\lambda = 3$ and has $N(a) = 5N(3a)$ – we need only five times as many balls of radius a to cover the region as balls of radius $3a$. Therefore, if there is a scaling relation $N(a) \sim a^{-D}$, we need $D = \frac{\log 5}{\log 3} \simeq 1.46$. Note that this is sensibly between 1 and 2.



The figure at left is defined by a similar procedure. I don't know how I'm ever going to get any physics done if I don't stop making these pictures. Lots of interesting fractals come from other procedures where the fractal dimension is not so easy to find.

1.2 Fractal dimension of a random walk

So far we've been discussion fractals defined by an artificial procedure. Consider a *random walk* in Euclidean space of dimension $d \geq 2$. Starting from some origin, we take discrete steps, where the displacement \vec{r} of each step is chosen independently from some distribution $p(\vec{r})$. For example, we could take our steps to have fixed length a_0 , and uniformly distributed direction. For now, we assume this distribution and that the walk is otherwise unrestricted.

Of interest is the net displacement after M steps

$$\vec{R}_M \equiv \sum_{i=1}^M \vec{r}_i.$$

This is a random variable with average

$$\langle \vec{R}_M \rangle_M \equiv \int d^d r_1 \cdots d^d r_M p(\vec{r}_1) \cdots p(\vec{r}_M) \sum_{i=1}^M \vec{r}_i$$

which vanishes by our assumption of rotation invariance of the individual distributions. On the other hand, $\langle \vec{r}_i \cdot \vec{r}_j \rangle \propto \delta_{ij}$, so the square-length of the net displacement has

$$\langle |\vec{R}_M|^2 \rangle_M = \sum_i^M \sum_j^M \langle \vec{r}_i \cdot \vec{r}_j \rangle_M = \sum_i^M \langle |r_j|^2 \rangle = M a_0^2.$$

The RMS displacement $R(M) \equiv \sqrt{\langle |\vec{R}_M|^2 \rangle_M} = \sqrt{M} a_0$ goes like the square root of the number of steps, a probably-familiar result which we are going to get some new perspective on now.

A walk of M steps can be regarded as n subwalks of M/n steps (choose M so that these are integers). By the above result, the RMS displacement of the subwalks is $r(n) = \sqrt{n} a_0$. This suggests that we may think of a random walk (RW) of M steps of length a_0 as a RW of M/n steps each of length (approximately) $\sqrt{n} a_0$. Notice that this ‘coarse-grained’ step size is not actually the same for each subwalk. This perspective allows us to infer the fractal dimension of an unrestricted RW. Let $N(a)$ be as above the number of balls of radius a needed to cover a walk (probably) of M steps of length a . When the ball-size is about the same as the step-size, we need one ball for each step (this is over-counting but should give the right scaling), we’ll have

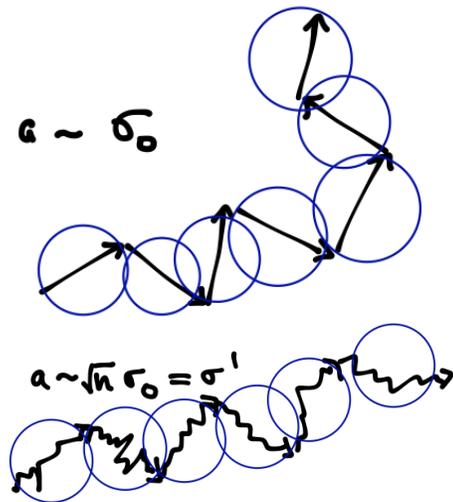
$$N(a) \sim M, \quad \text{for } a \sim \sigma_0.$$

For the sub-steps, the same relation says we should cover each subwalk step (length $\sqrt{n} a_0$, of which there are M/n) with a ball, so

$$N(\sqrt{n} a_0) \sim \frac{M}{n}.$$

Combining the previous two displayed equations (eliminate M) gives

$$M = N(a_0) = n N(\sqrt{n} a_0) \implies N(a) \sim a^{-2}$$



which says that the fractal dimension of the (unrestricted, in $d \geq 2$) random walk is $D = 2$.

As a practical physicist, why should you care about this result? Here's one kind of answer: suppose you have in your hands some object which is locally one-dimensional, but squiggles around in a seemingly random way. It is governed by some microscopic dynamics which are mysterious to you, and you would like to know if you can model it as an unrestricted random walk. One diagnostic you might do is to measure its fractal dimension; if it's not $D = 2$ then for sure something else is going on in there. (If it *is* $D = 2$ something else still might be going on.)

[End of Lecture 1]

I should mention two points regarding the notion of fractal dimension.

1. The Hausdorff dimension we've defined is not the only candidate for such a scale-dependent and possibly-fractional notion of dimension. In fact there are *many* others, and they are not all equivalent. Two that are notable are the *box-counting* dimension, where one covers the whole \mathbb{R}^d with a grid of boxes of side length a and counts the number $N(a)$ of boxes containing an element of the set as a function of a ; if $N(a) \sim a^{-D_{bc}}$ then this defines the box-counting dimension D_{bc} . This one is easier to implement numerically since it doesn't involve a minimization procedure. Another one is the *correlation dimension*, which is related to a problem on the homework.
2. For some statistically self-similar sets, a single fractal dimension does not capture the full glory of their fractaliciousness, and it is useful to introduce a whole spectrum of fractal dimensions. Such a thing is called *multifractal*.

I would like to say more about both of the previous points later on in the course.

1.3 RG treatment of random walk

Now we'll study the random walk with a bit more rigor. To be specific, suppose that each microscopic step is sample from the (Gaussian) distribution

$$p(\vec{r}) = \mathcal{N} e^{-\frac{|\vec{r}|^2}{2\sigma_0^2}}, \quad \mathcal{N} = (2\pi\sigma_0^2)^{-d/2} .$$

As before, the detailed form of the single-step distribution will be unimportant for the questions of interest to us – the technical term is 'irrelevant'; this will be an outcome of the RG analysis. In this case, we have $\langle \vec{r} \rangle = 0$, $\langle \vec{r} \cdot \vec{r} \rangle = \sigma_0^2$.

Let $\vec{r}' \equiv \sum_{i=1}^n \vec{r}_i$. Think of this as a ‘coarse-grained step’ – imagine that the single steps (of RMS size σ_0) are too small to see, but for n big enough, n of them can get somewhere. The distribution for the coarse-grained step is:

$$\begin{aligned}
 P(\vec{r}') &= \int d^d r_1 \cdots d^d r_n p(\vec{r}_1) \cdots p(\vec{r}_n) \delta \left(\vec{r}' - \underbrace{\sum_{i=1}^n \vec{r}_i}_{= \int d^d k e^{i\vec{k} \cdot (\vec{r}' - \sum_i \vec{r}_i)} \right) \\
 &\quad \text{(Do } n \cdot d \text{ Gaussian integrals.)} \\
 &\quad \text{Note that } d^d k \equiv \frac{d^d k}{(2\pi)^d} \text{.)} \\
 &= \int d^d k \exp \left(-n|\vec{k}|^2 \frac{\sigma_0^2}{2} - i\vec{k} \cdot \vec{r}' \right) \\
 &\quad \text{(One more Gaussian integral.)} \\
 &= \mathcal{N}' \exp \left(-\frac{|\vec{r}'|^2}{2n\sigma_0^2} \right), \quad \mathcal{N}' \equiv (2\pi n\sigma_0^2)^{-d/2}. \quad (1.4)
 \end{aligned}$$

This is the same form of the distribution, with the replacement $\sigma_0 \rightarrow \sigma' \equiv \sqrt{n}\sigma_0$. We can make it *actually* the same distribution if we rescale our units (the second half of the RG transformation): rescale $r' \equiv \sqrt{n}r''$, where the zoom factor is chosen to keep the width of the distribution the same after the coarse-graining step. Remembering that distributions transform under change of variables by

$$P(\vec{r}')d^d r' = P(\vec{r}'')d^d r''$$

we have

$$P(\vec{r}'') = \frac{1}{(2\pi\sigma_0)^{d/2}} e^{-\frac{|\vec{r}''|^2}{2\sigma_0^2}}$$

– the same distribution as we had for single step. Therefore, a random walk is (probably) a fractal – it is self-similar on average.

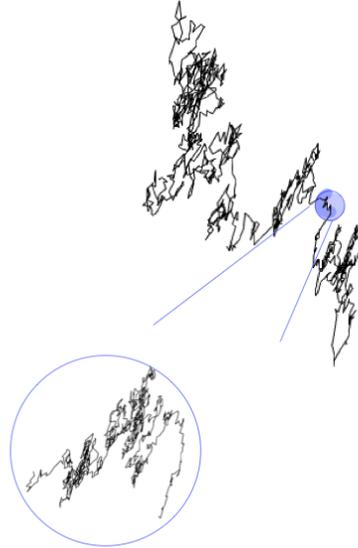
The two steps above – (1) coarse graining and (2) rescaling – constitute a *renormalization group transformation* (more on the general notion next). The ‘coupling constant’ σ_0 transforms under this transformation, in this case as

$$\sigma_0 \mapsto \sigma_{\text{renormalized}} = \sigma_0,$$

i.e. it maps to itself; such a parameter is called *marginal* and is a special case.

Consider the RMS distance covered by a walk in M steps,

$$R(M)^2 \equiv \left\langle \left| \sum_{i=1}^M \vec{r}_i \right|^2 \right\rangle_M.$$



It depends on M and the typical step size, which is σ (since $\sigma^2 = \int d^d r |\vec{r}|^2 p(\vec{r})$). Dimensional analysis tells us that we must have $R(M) \propto \sigma$ and the statistical self-similarity we've just found suggests a power law dependence on M :

$$R(M) \sim \sigma M^\nu$$

which scaling relation defines the exponent ν . The coarse-grained walk (no rescaling) takes $M' = N/n$ steps. Demanding the same outcome for the RMS displacement in both the microscopic description and in the coarse-grained description says

$$\sigma M^\nu = \underbrace{\sigma'}_{\sigma'=n\sigma} (M')^\nu = \sqrt{n}\sigma \left(\frac{M}{n}\right)^\nu = n^{\frac{1}{2}-\nu}\sigma M^\nu. \quad (1.5)$$

(In the context of quantum field theory, a relation with the same logical content is called a Callan-Symanzik equation.) In order for this to be true for all n , we must have

$$\nu = \frac{1}{2}.$$

So we've shown that an unrestricted random walk in $d \geq 2$ has a relationship between the fractal dimension and the RMS displacement exponent $\nu = 1/D$.

1.4 Anatomy of an RG scheme

As we saw in (1.5), we are taking a passive point of view on the RG transformations: the observable physics (whatever it may be) is the same, and we are only changing our description of that physics.

An RG transformation has two steps:

1. **Coarse-graining or decimation:** The idea of this step is familiar from the central idea of how thermodynamics emerges from statistical mechanics: we should average over the stuff we can't keep track of (microscopic configurations of the system), holding fixed the stuff we do keep track of (the thermodynamic variables like energy and volume). In the connection mentioned in the previous sentence, we do it all at once. The key new idea of the RG is to do it a little bit at a time. Integrate out or average over some set of short-distance/fast degrees of freedom, holding fixed the long-wavelength/slow degrees of freedom.

Notice that this step is not necessarily reversible: the value of a definite integral (or sum) does not uniquely determine the integrand (or summand). We lose information in this step. This means that a set of transformations defined this

way is not in fact a group in the mathematical sense, since there is no inverse element (it is a *semigroup*). So much for that.

The idea is that we are squinting, so that the smallest distance Δx we can resolve gets a little bigger, say before the coarse-graining, we had a resolution $\Delta x = \epsilon$, and afterwards we only keep track of stuff that varies on distances larger than $\Delta x = \lambda \epsilon$ for some scale factor $\lambda > 1$.

2. **Rescaling:** Now we change units to map the coarse-grained system back onto the original one, so that $\lambda \epsilon \mapsto \epsilon$. We do this so that we can compare them.

Now we're going to think about the space on which this transformation is acting. Its coordinates are the parameters of the system, such as the parameters defining the probability distribution such as σ_0 for the random walk, or the couplings in the Hamiltonian if $p = e^{-\beta H}/Z$. Let's call the set of such parameters $\{h_j\}$, where j is an index which runs over as many parameters as we need to consider². These parameters get transformed according to

$$\{h_j\} \xrightarrow{\text{steps 1, 2}} \{h'_j \equiv \mathcal{R}_j(\{h\})\}.$$

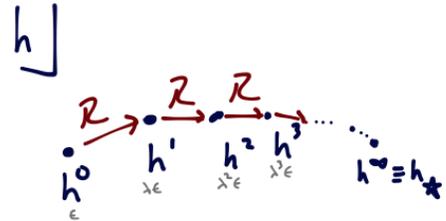
This map is something we can do over and over, coarse-graining (zooming out) by a factor of λ each time, until we get to macroscopic sizes. The repeated application of the map $h'_j \equiv \mathcal{R}_j(h)$ describes a dynamical system on the space of parameters. If we are interested in macroscopic physics, we care about what happens when we do it lots of times:

$$h \mapsto \mathcal{R}(h) \mapsto \mathcal{R}(\mathcal{R}(h)) \equiv \mathcal{R}^2(h) \mapsto \mathcal{R}^3(h) \rightarrow \dots$$

(When studying such a possibly-nonlinear dynamical system more generally, it is a good idea to ask first about the possible late-time behavior.)

What can happen? There are three possibilities:

1. We can reach a *fixed point*, $h_* = \mathcal{R}(h_*)$. (We'll include $h \rightarrow \infty$ in some direction in this case. That just means we chose a bad parametrization.)
2. We can go around in circles forever. This is called a *limit cycle*.



²One of the many crucial contributions of Ken Wilson to this subject was (I think) allowing for the possibility of including arbitrarily many parameters. The terror you feel at this possibility of an infinite-dimensional space of coupling parameters will be allayed when we discover the correct way to organize them two pages from now.

3. We can jump around chaotically forever.

The first case is the one about which we have a lot to say, and fortunately is what seems to happen usually. The distribution described by such a fixed point of the RG is self-similar, by the definition of the RG transformation. (If this is true when our zooming size $\lambda \rightarrow 1$, then it is actually scale-invariant.)

1.5 Scaling behavior near a fixed point.

Now, having found a fixed point h_* of our RG transformation, which is a candidate for describing the macroscopic behavior of our system, it is a good idea to look at the behavior in the neighborhood of the fixed point: linearize about the fixed point. We will see that this analysis immediately spits out the phenomenology of scaling behavior near a critical point. If that is not a familiar notion, don't worry.

First, define the set of points which flows to the fixed point to be the

$$\text{critical surface of } h_* \equiv \{h \mid \lim_{n \rightarrow \infty} \mathcal{R}^n(h) = h_*\}$$

– this is the basin of attraction of the fixed point in question.

Linearizing about the fixed point, let $h_j \equiv h_j^* + \delta_j$, where $|\delta| \ll 1$ will be our small parameter. This maps under the RG step according to

$$h_j \equiv h_j^* + \delta_j \mapsto h'_j = \mathcal{R}_j(h^* + \delta) \stackrel{\text{Taylor}}{=} \underbrace{\mathcal{R}_j(h^*)}_{=h_j^*} + \delta_k \underbrace{\frac{\partial h'_j}{\partial h_k} \Big|_{h_*}}_{\equiv R_{jk}} + \mathcal{O}(\delta^2)$$

where in the last step we assumed that the RG map \mathcal{R} is analytic in the neighborhood of the fixed point, *i.e.* that it has a Taylor expansion. How could it not be? We got it by doing some finite sums of analytic functions. By $+\mathcal{O}(\delta^2)$ I mean plus terms that go like δ^2 and higher powers of delta which are small and we will ignore them. If we ignore them, then the map on the deviation from the fixed point δ is a linear map:

$$\delta_j \mapsto \delta'_k = R_{jk} \delta_k.$$

We know what to do with a linear map: find its eigenvalues and eigenvectors.

$$R_{jk} \phi_k^{(n)} = \rho_n \phi_j^{(n)}.$$

Notice that nothing we've said guarantees that R_{jk} is a symmetric matrix, so its right and left eigenvectors need not be the same (the eigenvalues are), so we'll also need

$$\tilde{\phi}_j^{(n)} R_{jk} = \rho_n \tilde{\phi}_k^{(n)}.$$

Together, these are orthonormal

$$\sum_j \tilde{\phi}_j^{(n)} \phi_j^{(n')} = \delta_{n,n'} \quad (1.6)$$

and complete

$$\sum_n \tilde{\phi}_j^{(n)} \phi_k^{(n)} = \delta_{jk}.$$

About the eigenvalues, notice the following. We've defined the RG transformation $\mathcal{R} \equiv \mathcal{R}_\lambda$ to accomplish a coarse-graining by a scale factor λ . We can imagine defining such a transformation for any λ , and these operations form a semigroup under composition

$$\mathcal{R}_\lambda \mathcal{R}_{\lambda'} = \mathcal{R}_{\lambda\lambda'}.$$

This is useful because it says that the eigenvalues of these operators

$$\mathcal{R}_\lambda \phi^{(n)} = \rho_n(\lambda) \phi^{(n)}$$

must satisfy the same multiplication law

$$\rho_n(\lambda) \rho_n(\lambda') = \rho_n(\lambda\lambda').$$

But a function which satisfies this rule must have the form

$$\rho_n(\lambda) = \lambda^{y_n}$$

for some y_n independent of λ .

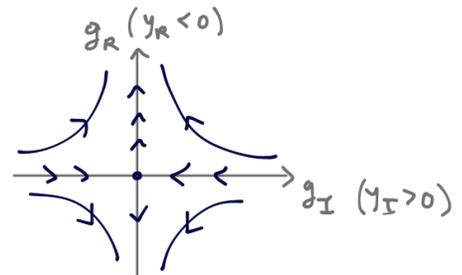
So now we use the eigenvectors of R as a preferred coordinate basis near the fixed point:

$$\delta_j = \sum_n g_n \phi_j^{(n)}, \quad g_n \stackrel{(1.6)}{=} \sum_k \tilde{\phi}_k^{(n)} \delta_k.$$

y_n is called the *dimension* of the coupling g_n .

Now we see a crucial RG dichotomy: If $|\rho_n| < 1$ ($y_n < 0$) then as we act with \mathcal{R} many times to get to long wavelengths, then $g_n \rightarrow 0$. Such a coupling is called *irrelevant*: it goes away upon repeated RG transformations and its effects on macroscopic physics can be ignored. Notice that since the perturbation is getting smaller, the approximation $|\delta| \ll 1$ becomes better and better in this case.

In contrast, if $|\rho_n| > 1$ ($y_n > 0$) then as we act with \mathcal{R} many times to get to long wavelengths, then g_n grows. Such a parameter is called



relevant, and represents an instability of the fixed point: our linearization breaks down after repeated applications of \mathcal{R} and we leave the neighborhood of the fixed point.

[End of Lecture 2]

The case of a coupling with $y_n = 0$ which doesn't change is called *marginal*.

In these terms, the critical surface (actually its tangent space near the fixed point) is determined by

$$\{g_n = 0 \text{ if } y_n > 0\}.$$

In particular, the *codimension* of the critical surface in the space of couplings is the number of relevant perturbations of the fixed point.

2 Random walks

Next we generalize our ensemble of random walks to illustrate some features of the RG that were missing from our simple pure Gaussian example above.

2.1 Biased gaussian walk

First, we can see an example of a relevant operator if we study a *biased* walk, with

$$p(\vec{r}) = (2\pi\sigma^2)^{-d/2} \exp\left(-\frac{|\vec{r} - \vec{r}_0|^2}{2\sigma^2}\right). \quad (2.1)$$

Again define the distribution for the coarse-grained step to be

$$\begin{aligned} P(\vec{r}') &= \int \prod_{i=1}^n (d^d r_i p(\vec{r}_i)) \delta\left(\vec{r}' - \sum_i \vec{r}_i\right) \\ &\stackrel{\text{(more Gaussian integrals)}}{=} (2\pi n\sigma^2)^{-d/2} \exp\left(-\frac{|\vec{r}' - n\vec{r}_0|^2}{2n\sigma^2}\right). \end{aligned} \quad (2.2)$$

So, after the coarse-graining step, we have

$$\begin{cases} \sigma' &= \sqrt{n}\sigma \\ \vec{r}'_0 &= n\vec{r}_0 \end{cases}$$

After the rescaling step, to keep the width of the distribution fixed, we have

$$\begin{cases} \sigma^{(R)} &= \sigma \\ \vec{r}_0^{(R)} &= \sqrt{n}\vec{r}_0 \end{cases}$$

which says that the bias of the walk is a relevant operator of dimension $y_0 = \frac{1}{2} > 0$.

We have here an explicit example of an RG map \mathcal{R} . Let's study its fixed points. There's one at $(\sigma, \vec{r}_0 = 0)$ (for any σ , so actually it is a family of fixed points parametrized by the marginal coupling σ) which is the unbiased walk we studied earlier. This fixed point is unstable because if we turn on a little r_0 it will grow indefinitely.

And there's another fixed point at $(\sigma, \vec{r}_0 = \infty)$. This is where we end up if we perturb the unbiased fixed point. The distribution (2.1) says (by direct calculation) that



$$R(M) = \sqrt{\left\langle |\vec{R}_M|^2 \right\rangle_M} = \sqrt{M^2 |\vec{r}_0|^2 + M\sigma^2} \xrightarrow{M \gg 1} M |\vec{r}_0|.$$

This means that for large a , we'll need $N(a) \sim 1/a$ spheres of radius a to cover the walk – it will be one dimensional.

This means that a system defined by some microscopic distribution of the form (2.1) with some value of \vec{r}_0 and σ will look like a Brownian walk of the type described above, with fractal dimension $D = 2$, if you look at it closely, with a resolution $\delta x \ll \sigma$, but from a distance, it will look like a one-dimensional path.

2.2 Universality class of the random walk

Now let the distribution from which a single step is sampled be any rotation-invariant distribution $p(\vec{r}) = p(|\vec{r}|)$ with finite moments. (For example, the fixed-step length distribution $p(\vec{r}) = \frac{1}{4\pi a^2} \delta(|\vec{r}| - a)$ is a good one to keep in mind.) The distribution for the coarse-grained step is

$$\begin{aligned} P(\vec{r}') &= \int \prod_{i=1}^n d^d r_i p(\vec{r}_i) \delta \left(\vec{r}' - \sum_i \vec{r}_i \right) \\ &= \int d^d k e^{-i\vec{k} \cdot \vec{r}'} \left(\left\langle e^{i\vec{k} \cdot \vec{r}} \right\rangle \right)^n \end{aligned} \quad (2.3)$$

The quantity

$$\left\langle e^{i\vec{k} \cdot \vec{r}} \right\rangle = \int d^d r p(\vec{r}) e^{i\vec{k} \cdot \vec{r}} \equiv g(k)$$

is called the characteristic function of the distribution $p(\vec{r})$, and is a generating function for its moments:

$$\langle r^m \rangle = (-i\partial_k)^m g(k)|_{k=0}.$$

The Taylor expansion in k of its logarithm is the cumulant expansion:

$$\log g(k) = \sum_m \frac{(\mathbf{i}k)^m}{m!} C_m, \quad C_m = (-i\partial_k)^m \log g|_{k=0}.$$

The important point for us is

$$\langle e^{i\vec{k}\cdot\vec{r}} \rangle = \exp \left(i\vec{k} \cdot \langle \vec{r} \rangle - \frac{1}{2} \sum_{\mu,\nu} k_\mu k_\nu \underbrace{(\langle r_\mu r_\nu \rangle - \langle r_\mu \rangle \langle r_\nu \rangle)}_{=\sigma_0^2 \delta_{\mu\nu}} + (i\vec{k})^3 C_3 + \mathcal{O}(k^4) \right).$$

Why should we treat the integration variable k in (2.3)

$$P(\vec{r}') = \int d^d k e^{-i\vec{k}\cdot\vec{r}'} e^{-\frac{n}{2}\sigma_0|\vec{k}|^2 + \mathcal{O}(nk^3)}$$

as small? Because the integrand is suppressed by the Gaussian factor. If the Gaussian bit dominates, then the integrand has support at $k \sim \frac{1}{\sqrt{n\sigma_0^2}}$, at which the m th term in the cumulant expansion contributes to the exponent as

$$nk^m C_m \sim n^{1-\frac{m}{2}} \xrightarrow{n \rightarrow \infty} 0 \quad \text{if } m > 2,$$

where the important thing is just that C_m is finite and independent of n and k . Then we can do the remaining Gaussian integral (ignoring the small corrections which are suppressed by $e^{-n^{1-\frac{m}{2}} C_m}$)

$$P(\vec{r}') = (2\pi n\sigma_0^2)^{d/2} e^{-\frac{1}{2} \frac{|\vec{r}'|^2}{n\sigma_0^2}}.$$

What's this? This is the Gaussian we used at the beginning.

This result, that the distribution for a sum of many random variables independently distributed according to some distribution with finite moments, is usually called the Central Limit Theorem or the Law of Large Numbers. (For more on the derivation I recommend the discussion in Kardar volume 1.)

In the framework of the RG it is an example of *universality*: all such probability distributions are in the basin of attraction of the gaussian random walk – they are said to be in the same *universality class* meaning that they have the same long-wavelength physics. In particular, their RMS displacement goes like $R_M \sim M^{1/2}$ for large number of steps M .

Notice that we did not prove that the Gaussian fixed point is the only one: we had to assume that we were in its neighborhood in order to use the $k \sim n^{-1/2}$ scaling – this scaling is a property of the neighborhood of the fixed point, just like the exponents y we got by linearizing about the general fixed point in §1.5.

In fact there are other fixed points in the space of d -dimensional walk distributions, as we will see next.

One lesson which does generalize, however, is that most of the possible perturbations of the fixed point are irrelevant, and there is only a small number of relevant or marginal perturbations.

[End of Lecture 3]

2.3 Self-avoiding walks have their own universality class

[Still from Creswick! I like this book. According to Amazon, Dover is putting out a second edition.] Suppose that the random 1d objects we are studying are actually *polymers* – a long chain molecule made of ‘monomers’ which cannot be in the same place, *i.e.* they have some short-ranged repulsion from each other. We can model this as lattice paths without self-intersection, or *self-avoiding walks* (SAWs). Does this microscopic modification of our ensemble change the long-wavelength physics?

It certainly changes our ability to do all the sums. If our polymer has n monomers, we’d like to know about the numbers

$$M_n(\vec{R}) \equiv \# \text{ of SAWs with } n \text{ steps connecting } \vec{0} \text{ to } \vec{R}. \quad (2.4)$$

Then we could figure out the RMS displacement from head-to-tail of the n step polymer (actually we are not distinguishing between head and tail):

$$R(n)^2 \equiv \langle |\vec{R}|^2 \rangle_n = \frac{\sum_{\vec{R}} M_n(\vec{R}) |\vec{R}|^2}{\sum_{\vec{R}} M_n(\vec{R})}.$$

As with the unrestricted random walk, we might expect to have (we will) a scaling relation

$$R(n) \sim n^\nu \quad (2.5)$$

with some characteristic exponent ν .

Enumerating M_n is not so easy. For the square lattice, $M_1 = 4$, $M_2 = 4 \cdot 3$ since there are three choices for the second step, $M_3 = 4 \cdot 3 \cdot 3$, but after that we can make a loops (for some of the previous choices) and it gets ugly and grows rapidly. A generating function which packages this information is the grand-canonical-type object

$$G(K) \equiv \sum_n K^n M_n(\vec{R}) \quad (2.6)$$

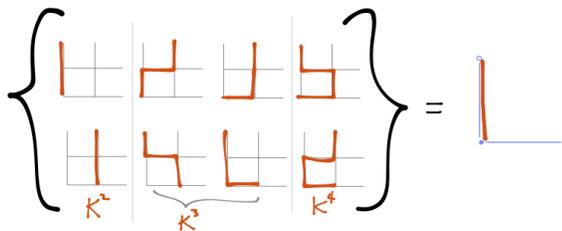
where K is a *fugacity* whose size determines the relative contributions of walks of different lengths to $G(K)$.

Let

$$\xi^2(K) \equiv \frac{1}{G(K)} \sum_R \sum_n |\vec{R}|^2 K^n M_n(\vec{R}) \quad (2.7)$$

The weights are related by

$$W_{\Gamma'}(K') = \sum_{\Gamma \in \Gamma'} W_{\Gamma}(K)$$



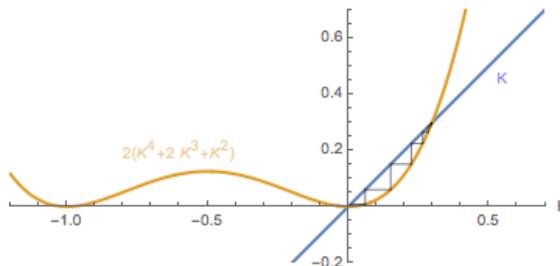
where we regard Γ' as defining an equivalence class of walks on the finer lattice. Here is an example of a possible rule for determining the inclusion $\Gamma \in \Gamma'$, for $b = 2$. It is very non-unique. Depicted is a unit cell of Γ' (blue) and the overlapping unit cells of Γ (black). A SAW which enters the cell at the lower left must leave through the right or the top. The model has a $\pi/2$ rotation symmetry so we can consider just the latter.

The result is

$$K' = 2(K^4 + 2K^3 + K^2). \quad (2.8)$$

Let me emphasize that the details of this real-space RG procedure are not to be taken too seriously, and other similarly-legitimate schemes produce somewhat different polynomials on the RHS here.

At right is a visualization of the map (2.8). Fixed points are (by definition) intersections between the curve (2.8) and the line $K' = K$. The map (2.8) has three fixed points:



1. $K = 0$, which is an ensemble dominated by very short walks, and in particular finite-length ones.
2. $K = \infty$, which is dominated by crazy lattice-filling walks. Maybe interesting.
3. $K = K_c \simeq 0.297$. This third one is where we go from finite walks at K slightly below K_c to infinite walks at $K > K_c$.

The jagged line between $K' = K$ and the curve defined by (2.8) depicts the repeated action of the map with an initial condition near (but slightly below) the fixed point at $K = K_c$. As you can see from the jagged line, the fixed point K_c is unstable – the perturbation parametrized by $K - K_c$ is relevant. Its dimension determines the exponent ν defined in (2.5) as follows.

Near the critical point,

$$\xi(K) \stackrel{K \rightarrow K_c}{\sim} |K - K_c|^{-\nu} \stackrel{!}{=} \underbrace{b}_{=2} \underbrace{\xi'(K')}_{=\xi(K')} = 2 \underbrace{|K'(K) - K_c|^{-\nu}}_{=\frac{\partial K'}{\partial K}|_{K_c}(K - K_c)}$$

Therefore

$$|K - K_c|^{-\nu} = b \left(\left| \frac{\partial K'}{\partial K} \right|_{K_c} \right)^{-\nu} |K - K_c|^{-\nu}$$

from which we conclude

$$\nu = \frac{\ln b}{\ln \left| \frac{\partial K'}{\partial K} \right|_{K_c}} = 0.771.$$

Numerical simulations give $K_c = 0.379$ and $\nu = 0.74$. The real-space RG can be systematically improved by increasing b .

The important conclusion, however, is pretty robust: the $d = 2$ SAW has a different exponent than the unrestricted walk:

$$\nu_{SAW} > \nu_{RW} = 1/2.$$

This makes sense, since it means that $R_{\text{RMS}}(\text{SAW}) > R_{\text{RMS}}$ for many steps – the SAW takes up more space since it can't backtrack. The fractal dimension is therefore smaller $D_{\text{SAW}} = \frac{1}{\nu} \simeq 1.3 < 2$.

Different exponents for the same observable near the critical point means different universality class.

Teaser: This is the $n \rightarrow 0$ limit of $O(n)$ model! More specifically, the critical point in temperature of the latter model maps to the large-walk limit: $T - T_c \sim M^{-1}$. This realization will allow us to apply the same technology we will use for the Ising model and its $O(n)$ generalizations to this class of models.

Measurability of the fractal dimension. One final comment for this section, in the opposite spirit of the previous one: I've spoken above about the fractal dimension of a random walk, for example of a random polymer configuration, as an 'observable'. How could you measure it?

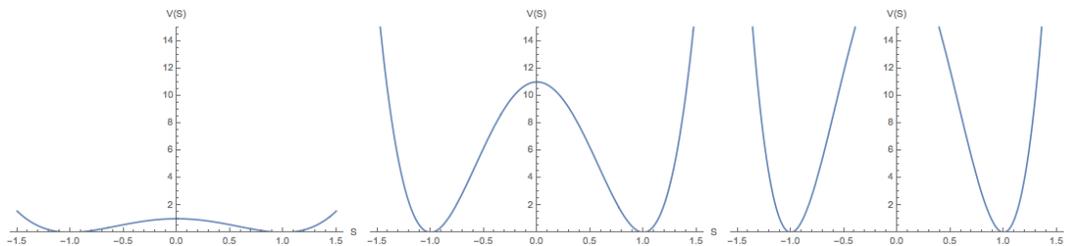
Suppose the monomers making up the polymer scatter light. The fractal dimension can be extracted from the *structure factor* $S(k)$, as measured by the intensity of scattering of light off the object, as a function of the wavenumber k of the light. (This statement is related to the open-ended question on the first homework.)

3 Ising models

Words about the role of models, solvable and otherwise, and universality:

Fixed points of the RG are valuable. Each one describes a possible long-wavelength behavior, and each one has its own basin of attraction. That basin of attraction includes lots of models which are in some sense ‘different’: they differ in microscopic details of values of couplings, and sometimes even more dramatically. Two important examples: (1) a lattice model and a continuum model can both reach the same fixed point. The idea is that if the correlation length is much longer than the lattice spacing, the magnetization looks like a continuous field, and we can interpolate between the points.

(2) a model with two states per site (like an Ising magnet, the subject of this section) and a model with infinitely many states at each site can flow to the same fixed point. Here’s a picture of how that might come about. Suppose we have at each site a variable called S which lives on the real line, and it is governed by the potential energy function $V(S) = g(S^2 - 1)^2$. (So for example the Boltzmann distribution is $e^{-\beta V(S)}$). The parameter g might be relevant, in the sense that $g \rightarrow \infty$ at long wavelengths. This process of making g larger is depicted in the following figures (left to right $g = 1, 11, 21$):



As you can see, it becomes more and more energetically favorable to restrict S to two values $S = \pm 1$ as g grows.

I’ve just made a big deal about universality and the worship of fixed points of the RG. Part of the reason for the big deal is that universality greatly increases the power of simple models: if you can understand the physics of some simple (even ridiculously over-idealized) model and show that it’s in the same universality class as a system of interest then you win.

[End of Lecture 4]

[Goldenfeld §2.5, Creswick §5, lots of other places] The Ising model is an important common ground of many fields of science. At each site $i \in \Lambda$ (Λ may be a chain, or the square lattice, or an arbitrary graph, and $i = 1 \dots |\Lambda| \equiv N(\Lambda) = N$ is the number of sites), we have a binary variable $s_i = \pm 1$ called a spin, whose two states are sometimes called up and down. There are 2^N configurations altogether. (Although I

will sometimes call these ‘states’ I emphasize that we are doing classical physics.)

The name ‘Ising model’ connotes the following family of energy functionals (also known as Hamiltonians):

$$-H(s) = \sum_{i \in \Lambda} h_i s_i + \sum_{ij} J_{ij} s_i s_j + \sum_{ijk} K_{ijk} s_i s_j s_k + \dots \quad (3.1)$$

where this sum could go on forever with terms involving more and more spins at once. (The RG will generically generate all such terms, with coefficients that we can hope do not cause too much trouble.) With this definition, the model may describe magnetic dipoles in a solid, constrained satisfaction problems, neural nets, ... this list also could go on forever³.

Equilibrium statistical mechanics. Why might we care about $H(s)$? We can use it to study the equilibrium thermodynamics of the system, at some temperature $T \equiv 1/\beta$. Let’s spend a few moments reminding ourselves about the machinery of equilibrium statistical mechanics. The key ‘bridge’ equation between the microscopic world (stat mech) and the macroscopic world (thermo) in thermal equilibrium is

$$e^{-\beta F} = \sum_s e^{-\beta H(s)} \equiv Z.$$

Here in our theorists’ paradise, we measure temperature in units of energy, $k_B = 1$. Notice that in classical equilibrium stat mech, the temperature is redundant with the overall scaling of the Hamiltonian, only the combinations $\beta h, \beta J...$ appear in the partition function, so a system with twice the temperature and twice the couplings will have the same physics. The sum here is over the 2^N configurations of the spins:

$$\sum_s \equiv \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \sum_{s_3=\pm 1} \dots \sum_{s_N=\pm 1} \equiv \prod_{i=1}^N \sum_{s_i=\pm 1} \equiv \text{tr}$$

and we will sometimes write tr for ‘trace’. I emphasize that we are doing classical physics here.

Why do we care about the free energy F ? For one thing, it encodes the thermodynamics of the system: the average energy is

$$E \equiv \langle H \rangle \equiv \frac{1}{Z} \text{tr} H e^{-\beta H} = -\partial_\beta \log Z,$$

the entropy is

$$S = -\partial_T F,$$

³Here is an example I learned of recently of how an Ising model is used for data clustering.

the heat capacity is

$$C_V = \partial_T E = \frac{1}{T^2} (\langle H^2 \rangle - \langle H \rangle^2),$$

a dimensionless measure of the number of degrees of freedom. Notice the familiar thermodynamic identity $F = E - TS$ follows by calculus.

More ambitiously, if we knew how F depended on all the coupling parameters $\{h_i, J_{ij}, K_{ijk} \dots\}$ in (3.1), we know all of the correlation functions of the spins, for example

$$\partial_{h_i} F = -T \partial_{h_i} \log Z = -T \frac{1}{Z} \text{tr} \frac{s_i}{T} e^{-\beta H} = -\langle s_i \rangle.$$

And similarly,

$$\partial_{h_i} \partial_{h_j} F = (\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle) T^{-1} \equiv G_{ij} T^{-1}.$$

It is a generating function for these (connected) correlation functions. Also of interest is the spin susceptibility:

$$\chi \equiv \partial_h M = \frac{1}{T} \frac{1}{N^2} \sum_{ij} (\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle) = \frac{1}{N^2} \sum_{ij} G_{ij}.$$

Clean and local Ising models. Two important specializations of (3.1) are quite important in physics (not always in the other applications of the Ising model). We will (usually) restrict to the important special case with the following two assumptions.

1. the couplings (J_{ij} and friends) are *local* in the sense that the coupling between two sites goes away ($J_{ij} \rightarrow 0$) if the sites are far apart ($|r_i - r_j| \rightarrow \infty$).

A reason to care about the two point function in the case where there is a notion of locality, then, is that it allows to define a *correlation length*, ξ :

$$G_{ij} \stackrel{r \gg a}{\sim} e^{-r_{ij}/\xi}$$

– here a is the range of the interactions, or the lattice spacing, and $r_{ij} \equiv |r_i - r_j|$ is the distance between the locations of spins i and j . The correlation length will depend on the parameters in H and on the temperature, and it measures the distance beyond which the spin orientations are uncorrelated. More formally, $\xi^{-1} \equiv -\lim_{r \rightarrow \infty} \partial_r \log G_{i, i+r}$ (but of course the ∞ here has to stay with the box containing the system in question).

2. the couplings are translation invariant $J_{ij} = Jf(|r_i - r_j|)$ for some function of the distance $f(r)$. (If one thinks of variations of J_{ij} with i, j as coming from some kind of microscopic disorder, one refers to this case as *clean*.) We will often consider the case where $f(r)$ only has support when $r = a$ is one lattice spacing. (Notice that $s^2 = 1$ means that we can ignore the case when $r = 0$.)

These two assumptions are independent, but we will usually make both. So: on any graph (of N sites), the nearest-neighbor ‘clean’ Ising model has energy functional

$$-H = h \sum_i s_i + J \sum_{\langle ij \rangle} s_i s_j$$

where $\langle ij \rangle$ means the set of nearest neighbors i, j . An important observable (especially in this case) is the *magnetization*

$$M \equiv \frac{1}{N} \sum_{i=1}^N \langle s_i \rangle = -\frac{1}{N} \partial_h F.$$

When $J > 0$, the energy of a configuration is lower if neighboring spins point the same way; in this ‘ferromagnetic’ case everybody can be happy. In the antiferromagnetic case $J < 0$, neighbors want to disagree. All spins can agree to disagree if the graph has no loops. Any loop with an odd number of sites, like a triangle, leads to a *frustration* of the antiferromagnetic interaction, which requires compromise and leads to drama.

Lack of drama for bipartite lattices. A bipartite lattice is one which can be divided into two distinct sublattices A, B each of which only neighbors sites of the other lattice. That is $\langle ij \rangle$ contains only pairs, one from A and one from B . For example, hypercubic lattices are all bipartite: let the A lattice be those sites (x, y, \dots) whose (integer) coordinates add up to an even number $x + y + \dots \in 2\mathbb{Z}$. The honeycomb lattice is also bipartite. The triangular lattice is not.⁴

A consequence of bipartite is that any loop traverses an odd number of sites, since it must alternate between the two sublattices. Hence there is no frustration for a (nearest-neighbor!) Ising antiferromagnet on a bipartite lattice. In fact, a stronger statement is true. Since

$$H_{h=0, J}(s^A, s^B) = -J \sum_{\langle ij \rangle} s_i^A s_j^B$$

if we flip the spins of one sublattice, we also reverse J :

$$H_{h=0, J}(s^A, -s^B) = +J \sum_{\langle ij \rangle} s_i^A s_j^B = H_{h=0, -J}(s^A, s^B).$$

But for any function ϕ

$$\sum_{\{s\}} \phi(\{s\}) = \sum_{\{s\}} \phi(\{-s\}) \tag{3.2}$$

⁴Notice, by the way, that bipartite does not require that A and B be isomorphic or even that they have the same number of sites. For example, if we simply removed a (periodic) subset of sites (and all the associated links) from the A sublattice of a lattice, we would still have a bipartite lattice. You can find an example by googling ‘Lieb lattice’. Beware confusion in the literature on this point.

by renaming summation variables. Therefore on a bipartite lattice

$$Z(h = 0, J) = \sum_{\{s\}} e^{-\beta H_{h=0, J}(s^A, s^B)} = \sum_{\{s\}} e^{-\beta H_{h=0, -J}(s^A, s^B)} = Z(h = 0, -J).$$

So on a bipartite lattice, a ferromagnet and an antiferromagnet have the same thermodynamics.

The nearest-neighbor restriction is essential here. Even for a one-dimensional chain, we can make frustration by adding antiferromagnetic interactions for $s_i s_{i+1}$ and for $s_i s_{i+2}$.

Symmetry and spontaneous symmetry breaking. Notice that the general Ising Hamiltonian (3.1) enjoys the following property

$$H_{h, J, K \dots}(s) = H_{-h, J, -K \dots}(-s)$$

– flipping all the spins and flipping the coefficients of odd powers of the spins preserves the energy. In particular, if $h = 0, K = 0$, all odd powers do not appear, and flipping the spins is a symmetry of the Hamiltonian. What consequence does this have for thermodynamics?

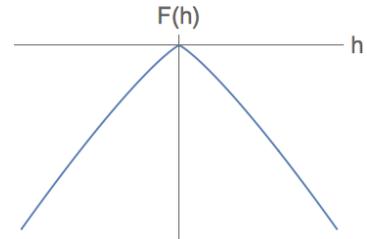
$$Z(-h, J, -K, T) = \sum_{\{s\}} e^{-\beta H_{-h, J, -K}(s)} = \sum_{\{s\}} e^{-\beta H_{h, J, K}(-s)} \stackrel{(3.2)}{=} Z(h, J, K, T).$$

And therefore the free energy in particular satisfies $F(-h, J, -K, T) = F(h, J, K, T)$.

Question: does this mean that when $h = 0$ we must have zero magnetization,

$$M = \frac{1}{N} \sum_i \langle s_i \rangle \propto \partial_h F \stackrel{?}{=} 0 \quad ?$$

Answer: It would if $F(h)$ had to be a smooth, differentiable function. In order for $\langle s \rangle_{h=0}$ to be nonzero, $F(h)$ must have a different derivative coming from positive and negative h , as in the figure at right. This phenomenon is called *spontaneous symmetry breaking* because the symmetry reverses the sign of the magnetization $M \rightarrow -M$.



(Note that only when $h = 0$ does the symmetry map the ensemble to itself.)

But this phenomenon, of $\partial_h F|_{h=0^+} \neq \partial_h F|_{h=0^-}$ requires the function $F(h)$ to be non-analytic in h at $h = 0$. This is to be contrasted with the behavior for a finite system ($N < \infty$) where

$$Z(h) = \sum_{\{s\}} e^{-\beta H(s)} = e^{-\beta h m_1} c_1 + e^{-\beta h m_2} c_2 + \dots + e^{-\beta h m_n} c_n$$

where c_a are positive coefficients (functions of βJ) independent of h , and m_a is proportional to the magnetization in configuration a . The important point is just that there is a finite number of terms in this sum, and Z is therefore a polynomial in $e^{-\beta h}$, so $F(h) = -T \log Z(h)$ is only singular when $Z = 0$ or $Z = \infty$, which doesn't happen for interesting values of β, h , *i.e.* only at zero temperature can we make this vanish.

In conclusion: spontaneous symmetry breaking requires the thermodynamic limit $N \rightarrow \infty$.

3.1 Decimation RG for 1d nearest-neighbor Ising model

Let's step back from these grand vistas and apply the RG for the Ising model in one dimension. Consider a chain of sites $i = 1 \dots N$, arranged in a line with spacing a , and with an even number of sites, $N \in 2\mathbb{Z}$. And for definiteness, if you must, take periodic boundary conditions $s_{N+1} = s_1$. Turn off the magnetic field, so

$$H = -J \sum_{i=1}^N s_i s_{i+1} .$$

We'll speak about the ferromagnetic case, $J > 0$ (though the same results apply to $J < 0$ since the chain is bipartite). The partition function

$$Z = \text{tr} e^{-\beta H} = Z(\beta J)$$

is calculable exactly in many ways, each of which instructive. Since the partition function only depends on the combination βJ , let us set $\beta = 1$.

In the spirit of the RG, let us proceed by a hierarchical route, by decimating the even sites:

$$\sum_{\{s\}_{i,\text{even}}} e^{-H(s)} = e^{-H_{\text{eff}}(s_{\text{odd}})}$$

On the right hand side, we have defined the *effective* hamiltonian for the spins at the odd sites. The odd sites are separated by distance $2a$ and there are half as many of them. We can use this as the first half of an RG implementation (the second half is rescaling). We've zoomed by a factor of $\lambda = a'/a = 2$.

In this 1d case we can actually do these sums:

$$\sum_{s_2=\pm 1} e^{+J s_2 (s_1 + s_3)} = 2 \cosh(J(s_1 + s_3)) \equiv \Delta e^{J' s_1 s_3}$$

where in the last step we defined Δ, J' , constants independent of the spin configuration. Then the result for the whole trace over even spins is

$$e^{-H_{\text{eff}}(s')} = \Delta^{N/2} e^{J' \sum_i s_{2i+1} s_{2(i+1)+1}}, \quad \implies \quad H_{\text{eff}}(s') = \sum_{i, \text{ odd}} J' s_i s_{i+2} + \frac{N}{2} \log \Delta.$$

The Δ business just adds a constant to the (free) energy, which divides out of the partition function and we don't care about it here.

We can figure out what the new parameters are by checking cases, of which only two classes are distinct:

$$\begin{aligned} \text{if } s_1 = s_3: \quad 2 \cosh 2J = \Delta e^{J'} &\xrightarrow{\text{product}} \Delta^2 = 4 \cosh J \\ \text{if } s_1 = -s_3: \quad 2 = \Delta e^{-J'} &\xrightarrow{\text{ratio}} e^{2J'} = \cosh 2J \end{aligned} \quad (3.3)$$

[End of Lecture 5]

The solution can be usefully written as

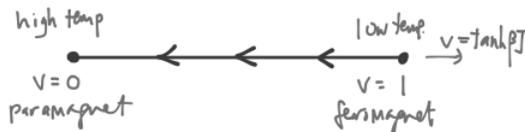
$$v' = v^2, \quad \Delta = 2\sqrt{\cosh 2J} \quad (3.4)$$

where $v \equiv \tanh J \in [0, 1]$ (using hyperbolic trig identities).

Why is the effective hamiltonian of the same form as the original one? The couplings like the magnetic field multiplying odd numbers of spins vanish by the Ising spin-flip symmetry of the original model. (More precisely: because of the locality of H , we can determine H_{eff} by decimating only a finite number of spins. This rules out generation of nonzero h' by some version of spontaneous symmetry breaking. This requires locality of the interactions.) The 4-spin interaction vanishes because in 1d, each site has only two neighbors with whom it interacts, each of which has only one other neighbor.

The map (3.4) is another explicit example of an RG map on the parameters. In this case, unlike the previous SAW example, it happens to be exact.

This map has two fixed points: One is $v^* = 0$, which is $\beta J = 0$ meaning infinite temperature, or no interactions; this one is 'boring' from the point of view of the study of many-body physics and collective phenomena, since the spins don't care about each other at all.



The other fixed point is $v_* = 1$, which is $\beta J = \infty$ meaning zero temperature or infinite interaction strength. This is a ferromagnetic fixed point where it is very urgent for the spins to agree with each other. The fact that there is no fixed point at a finite temperature means that there is no critical behavior in the 1d nearest-neighbor Ising model; only at $T = 0$ do the spins align with each other. More explicitly, how does the correlation length behave? In zooming out by a factor of λ , it changes by

$$\xi(v) = \lambda \xi(v') = 2\xi(v^2) \quad \implies \quad \xi = -\frac{K}{\log v} \xrightarrow{T \rightarrow 0} \frac{K}{2} e^{2J/T} \quad (3.5)$$

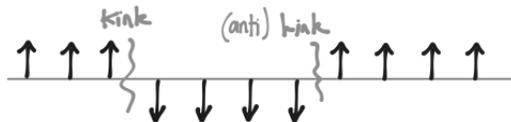
which is finite for $T > 0$.⁵

⁵Preview: near less weird fixed points, the correlation length will diverge like a power law $\xi(T) \sim (T - T_c)^{-\nu}$ instead of this weird non-analytic function.

Why did it happen that there is no critical point at $T > 0$? A point of view which illuminates the distinction between 1d and $d > 1$ (and is due to Peierls and now permeates theoretical condensed matter physics) is to think about the statistical mechanics of *defects* in the ordered configuration.

Consider a favored configuration at low-temperature, where all spins point the same way. Small deviations from this configuration require reversing some of the spins and will cost energy $2J$ above the aligned configuration for each dissatisfied bond, which is called a *kink*. n domains of reversed spins cost energy $4Jn$, since each domain has two boundary links.

In 1d, each region of spins that we reverse has two boundaries, a kink and an antikink.



At $T = 0$, the state minimizes the energy and there is no reason to have any kinks. But at $T > 0$, we care about the free energy $F = E - TS$, and the fact that there are *many* kink configurations matters.

How many are there? If there are n segments of $s = -1$ in a sea of $s = +1$ then we must decide where to place $2n$ endpoints. The number of ways to do this is:

$$\Omega(n) \simeq \binom{N}{2n} = \frac{N!}{(2n)!(N-2n)!} \stackrel{1 \ll n \ll N}{\sim} e^{N \log N - 2n \log 2n - (N-2n) \log(N-2n)}$$

where in the last step we used Stirling's formula. So the free energy for $2n$ kinks is

$$F_n = n \cdot 4J - T \log \Omega(n) \simeq 4Jn - T(N \log N - 2n \log 2n - (N-2n) \log(N-2n)) .$$

In equilibrium, the free energy is minimized with respect to any variational parameters such as n , which happens when

$$-\frac{4J}{T} = 2 \log 2n - 2 \log(N-2n) \implies \frac{n}{N} = \frac{1}{2} \frac{e^{-2J/T}}{1 + e^{-2J/T}} \stackrel{T \lesssim 4J}{\sim} \frac{1}{2} e^{-2J/T} .$$

As a check, the correlation length is approximately the size of the domains, which is approximately the inverse kink density:

$$\xi \sim \left(\frac{n}{N}\right)^{-1} \sim 2e^{2J/T}$$

which again agrees with our RG result (3.5). **[Please beware my factors of two here.]**

The codewords for this phenomenon are: the destruction of long-range order by the proliferation of topological defects. (They are topological, for example, in the sense that a kink must follow an antikink, and the number of kinks on circle must equal the number of antikinks.)

In the special case of 1d, we can be completely explicit and verify the result for the correlation length by calculating the correlation function.

First of all,

$$G(r) = \langle s_{i+r} s_i \rangle \quad (3.6)$$

is independent of i because of translation invariance. The Boltzmann factor can be written as

$$e^{\beta J \sum_{\langle ij \rangle} s_i s_j} = \prod_{\langle ij \rangle} e^{\beta J s_i s_j}$$

Since $s = \pm 1$, we are multiplying

$$e^{\beta J s_i s_j} = \cosh \beta J + s_i s_j \sinh \beta J = \cosh \beta J (1 + v s_i s_j)$$

where $v \equiv \tanh \beta J$ (as above). Think about expanding this product over links into a sum. Each term in the sum gets either a 1 or a $v s_i s_j$ from each link. Any term in the sum can be visualized by coloring the links which contribute a $v s_i s_j$.

When we multiply this out, the dependence on any one of the spins s_i can be only two things: 1 if the term has an even number of factors s_i or s_i if it has an odd number. Here's the Ising model integration table:

$$\sum_{s_i} 1 = 1 + 1 = 2, \quad \sum_{s_i} s_i = 1 - 1 = 0. \quad (3.7)$$

In the last two paragraphs, we haven't used the restriction to 1d at all. (This will be useful in §3.2.) Consider a single spin of an infinite 1d chain; if it is not one of the two sites i or $i+r$ in (3.6) the factors which matter to it are⁶:

$$\sum_{s_2} (1 + v s_1 s_2) (1 + v s_2 s_3) \stackrel{\text{FOIL!}}{=} \sum_{s_2} (1 + v s_2 (s_1 + s_3) + v^2 s_1 s_3) = 2 (1 + v^2 s_1 s_3).$$

This is just of the same form as if we had a direct link between 1 and 3 with weight v^2 (up to the overall prefactor). Therefore, doing this repeatedly (r times) for the sites in between i and $i+r$,

$$G(r) = \frac{\text{tr} s_i 2^r (1 + v^r s_i s_{i+r}) s_r}{\text{tr} 2^r (1 + v^r s_i s_{i+r})} = v^r$$

(The red factors are the ones that survive the trace.) Therefore

$$\xi^{-1} = -\partial_r \log G(r) = -\log v$$

which agrees with (3.5) with $K = 1$.

[End of Lecture 6]

⁶Note that this expression gives a very simple derivation of (3.4).

The thing that's special about 1d here is that only a single term in the expansion of the product survives the sum. This is because there is only one *path* between the two sites i and $i+r$. If we had taken N finite and periodic boundary conditions, there would be another path (around the back of the circle) and hence another term in the answer

$$G(r) = v^r + v^{N-r} \stackrel{N \gg r}{\approx} v^r.$$

In $d > 1$ there are many paths and the answer is more interesting, as we'll see below.

3.2 High-temperature expansion

Return now to the moment at (3.7), right before we restricted our discussion to one dimension. We had written the partition function of the nearest-neighbor Ising model (on any graph) as a product over links

$$Z = \cosh^{N_\Lambda}(\beta J) \sum_s \prod_{\langle ij \rangle} (1 + v s_i s_j) \quad (3.8)$$

and argued that expanding this binomial gives a sum over *paths* in the graph. More explicitly, we think of the two terms in each link factor in (3.8) as a sum over another dynamical variable, $n_{\langle ij \rangle} = 0, 1$:

$$1 + v s_i s_j = \sum_{n_{ij}=0,1} (v s_i s_j)^{n_{ij}}.$$

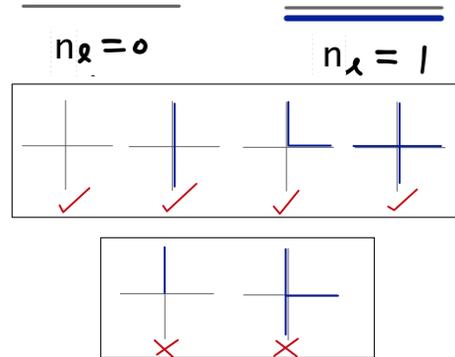
So we can write the Ising partition function as

$$Z = \cosh^{N_\Lambda}(\beta J) \prod_\ell \sum_{\{n_\ell=0,1\}} \sum_{\{s\}} \prod_{\langle ij \rangle} (v s_i s_j)^{n_{ij}}.$$

Now we can do the sums over the spins using our 'integration table' above (3.7). For each spin, the sum is

$$\sum_{s_i=\pm 1} s_i^{\sum_{\langle ij \rangle} n_{ij}} = \delta \left(\sum_{\langle ij \rangle} n_{ij} \equiv 0 \pmod{2} \right)$$

Here we've defined the notation ' $\langle ij \rangle$ ' to mean 'neighbors j of a fixed site i '. That is: the sum is only nonzero if an even number of the links ending at site i have $n_{ij} = 1$. If we represent $n_l = 1$ by coloring the link l , the configurations which survive this constraint are made of closed loops.



$$Z = \cosh^{N_\Lambda}(\beta J) \prod_{\langle ij \rangle} (1 + v s_i s_j) = \cosh^{N_\Lambda}(\beta J) \sum_C v^{\sum_l n_l(C)} \quad (3.9)$$

where we are summing over configurations of binary numbers $n_l = 0, 1$ on the links that are *closed* in the sense that

$$\sum_{\langle ij \rangle} n_{\langle ij \rangle} \equiv_2 0 \quad \forall i.$$

That is: we sum over lattice curves which have an even number of links going into each site.

This rewriting of the Ising partition sum will be useful below.

3.3 RG evaluation of physical quantities

Behavior of the correlation length under RG. We've defined the correlation length using the spin-spin correlator $G(r)$, in terms of its rate of falloff for large r . Let us use this to examine its behavior under the RG more directly. To do this, denote more explicitly

$$G_H(r) \equiv \frac{\text{tr} s_i s_{i+r} e^{-H}}{\text{tr} e^{-H}}.$$

Now suppose that i and $i+r$ are both odd sites (so that they survive our decimation); in that case we can still do all the decimation as in the partition function :

$$G_H(r) \equiv \frac{\text{tr}_{e,o} s_i s_{i+r} e^{-H(s_e, s_o)}}{\text{tr} e^{-H(s_e, s_o)}} = \frac{\text{tr}_o s_i s_{i+r} \text{tr}_e e^{-H(s_e, s_o)}}{\text{tr}_o \text{tr}_e e^{-H(s_e, s_o)}}$$

But recall that $e^{-H'(s_o)} \propto \text{tr}_e e^{-H(s_e, s_o)}$ defines the effective Hamiltonian for the remaining odd sites, so this is precisely

$$G_{H'}(r) \equiv \frac{\text{tr}_o s_i s_{i+r} e^{-H'(s_o)}}{\text{tr} e^{-H'(s_o)}}.$$

Under this RG, we are zooming out by a factor of 2, so altogether $G_{H'}(r'/2) = G_{H'}(r) = G_H(r)$ (there are only half as many sites in between in the new lattice spacing). Combining this with the definition of ξ , we have $e^{-r/\xi} = e^{-\frac{1}{2}r'/\xi'}$ which means

$$\xi' = \frac{1}{2}\xi \quad (3.10)$$

(as we said earlier).

The correlation length is completely determined by the Hamiltonian (I am assuming thermal equilibrium here). At a fixed point, the Hamiltonian does not change under

the RG, so the correlation length can't either. This can be consistent with (3.10) in one of two ways

$$\xi_\star = 0 \quad \text{or} \quad \xi_\star = \infty.$$

The first case means that spins at different sites do not care about each other, as at $T = \infty$. I've already disparaged this case as boring. The second case of a divergent correlation length characterizes critical behavior and we define it to be *interesting*.

[Domany, RG notes, chapter 1] **Free energy density.** Next I want to show how to calculate the free energy from an 'RG trajectory sum'. It is a reason to care about the constants in the effective hamiltonian, as in a' in $H'(s') = \sum J' s' s' + a'$. In the example above, we found $a' = \frac{1}{2} \log \Delta$, where Δ was some function of the microscopic J .

Let the free energy density (free energy per site) be

$$f \equiv -\frac{T}{N} \log Z_N(K).$$

Here I am denoting by K the collection of all couplings, and labelling the partition function Z_N by the number of sites. More explicitly,

$$Z_N \equiv \sum_{\{s\}} e^{-\beta H(s)} \equiv \sum_{\{s\}} e^{-\beta(C + \tilde{H}(s))} \equiv e^{-N\beta C} \tilde{Z}_N.$$

Here \tilde{H} is a Hamiltonian modified by subtracting a constant so that

$$\sum_{\{s\}} \tilde{H}(s) = 0$$

so that it has no constant piece (for quantum mechanical folks: it is like a 'normal-ordered' Hamiltonian). And $\tilde{Z} \equiv \sum_s e^{-\beta \tilde{H}(s)}$, and naturally we'll denote

$$\tilde{f} \equiv -\frac{T}{N} \log \tilde{Z}_N.$$

This last expression is a little less innocent than it seems: I am claiming here that the free energy is extensive – has a leading piece at large N that grows like N , $F \stackrel{N \gg 1}{\sim} Nf + \mathcal{O}(N^0)$ – so that \tilde{f} is independent of N in the thermodynamic limit. Then $f(K) = C + \tilde{f}(K)$.

Now some RG content: the partition function is invariant under the RG:

$$\begin{aligned} Z_N &= e^{-\frac{NC}{T}} \tilde{Z}_N = e^{-\frac{NC}{T}} e^{-\frac{N'a'}{T}} \tilde{Z}_{N/b}(K') \\ &= e^{-\beta(NC + N'a' + N^{(2)}a^{(2)})} \tilde{Z}_{N/b^2}(K^{(2)}) \\ &= e^{-\beta(NC + N'a' + \dots + N^{(n)}a^{(n)})} \tilde{Z}_{N/b^n}(K^{(n)}). \end{aligned} \quad (3.11)$$

Here we've defined $N^{(n)}$ to be the number of sites decimated at step n , and N/b^n is the number of sites remaining. For the example above, these are the same, and $b = 2$: $N^{(n)} = N/2^n$. As above $K^{(n)} = \mathcal{R}^n(K)$ is the image of the couplings under n -times repeated RG transformation. (Notice that if we were in d dimensions, we would have $b = \lambda^d$, where λ is the linear zoom factor, and the number of sites decimated would not equal the number remaining even for $\lambda = 2$.) Taking logs of the BHS of the previous equation

$$f(K) = C + \sum_{k=1}^n \frac{N^{(k)}}{N} a^{(k)} + \frac{N}{b^n} \tilde{f}(K^{(n)}).$$

If the RG flow If we iterate enough times, and $\tilde{f}^{(n)}$ is finite, its contribution is suppressed by $b^{-n} \rightarrow 0$.

Magnetization. The magnetization can be calculated by taking derivatives of the previous result:

$$M = \partial_h f = \langle s_i \rangle$$

but here is some cleverness. By translation invariance the BHS is independent of i . Therefore, we can choose i to be a site that survives all the decimation. Then

$$\langle s_i \rangle_H = \frac{\sum_s s_i e^{-H}}{\sum_s e^{-H}} = \frac{\sum_{s_o} s_i \sum_{s_e} e^{-H(s_o, s_e)}}{\sum_{s_o} \underbrace{\sum_{s_e} e^{-H(s_o, s_e)}}_{=e^{-H'(s_o)}}} = \frac{\sum_{s_o} s_i e^{-H'(s_o)}}{\sum_{s_o} e^{-H'(s_o)}} = \langle s_i \rangle_{H'}.$$

We have just shown that the magnetization is an RG invariant. This result required that we are using a *decimation* scheme, where the spins surviving the RG are a subset of the initial spins. I will come back to alternatives soon, and we will see why we need them. This means we can compute the magnetization for a macroscopic system just by following the flow to the end:

$$\langle s_i \rangle = \frac{\sum_{s_i=\pm 1} s_i e^{-H^\infty(s_i)}}{\sum_{s_i} e^{-H^\infty(s_i)}}$$

but $H^\infty(s_i) = a^\infty + h^\infty s_i$ (these are the only two possible terms) and h^∞ is the fixed-point value of the Zeeman field. So

$$\langle s_i \rangle = \frac{\sum_{s_i=\pm 1} s_i e^{-h^\infty s_i}}{\sum_{s_i} e^{-h^\infty s_i}} = \frac{-e^{+h^\infty} + e^{-h^\infty}}{e^{+h^\infty} + e^{-h^\infty}} = -\tanh h^\infty.$$

3.4 Need for other schemes

Let's think about decimation of the Ising model on the square lattice. Again it is bipartite, and we can do the sum of each spin one of the sublattices fixing the spins on

the other, one at a time:

$$\sum_{s_x} e^{J s_x (s_1 + s_2 + s_3 + s_4)} \equiv \psi(s_1 + s_2 + s_3 + s_4).$$

The argument of the function ψ defined by this equation only takes the values $0, \pm 2, \pm 4$. We've set the Zeeman field $h = 0$, so it is even $\psi(-x) = \psi(x)$, so there are only three values we are about. For these values, it can be written as

$$\psi(s_1 + s_2 + s_3 + s_4) = e^{a' + J'(s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_1 + s_1 s_3 + s_2 s_4) + M' s_1 s_2 s_3 s_4}$$

with values of a', J', M' determined by J which you can figure out. The first term a' is just a constant. The first four terms multiplied by J' are nearest-neighbor interactions on the new (square) lattice with lattice spacing $\sqrt{2}a$ (rotated by $\pi/4$). This means $\lambda = \sqrt{2}$; the number of remaining spins is $N/2$, so $b = \lambda^{d-2} = 2$ as expected in two dimensions. The next two terms are next-nearest-neighbor exchange couplings (s_1 and s_3 are separated by $2a$) of the same size. Finally, M' multiplies a qualitatively-new 4-spin interaction. Ick!

This isn't so bad if we think of the initial Hamiltonian as sitting in a special corner of the large and high-dimensional space of possible couplings, and the RG just moves us to a more generic point:

$$(J, 0, 0, \dots) \xrightarrow{\mathcal{R}} (J', K', M' \dots).$$

That's just a little ugly. But there's a reason why it's objectively bad: we can't repeat this RG step. After the first iteration, we generate couplings between spins of the same sublattice of the remaining square lattice. This means we can't just sum them independently anymore. We could do some uncontrolled truncation, or we can find a better scheme. There are 2d lattices for which a decimation scheme can work (*i.e.* can be iterated).

[End of Lecture 7]

3.5 Low-temperature expansion, and existence of phase transition in $d > 1$

Maybe you still don't believe me that there has to be a phase transition in the nearest-neighbor Ising model, even in $d = 2$. At arbitrarily high temperatures, there is definitely no spontaneous symmetry breaking, since each spin is just looking out for itself and there can be no collective behavior, and $\langle s \rangle = m = 0$. At $T = 0$, the spins all align (as they do in $d = 1$, too). Here is an argument (due the Peierls, still) that the ordered state survives to some finite temperature.

A configuration of lowest energy, say all $s_i = +$, has energy $E_0 = -JN_l$, where N_l is the number of links of the graph (this is $2N$ for the square lattice since there are two links in each unit cell, one up and one right). The minimal excitation above the ordered configuration flips one spin and has energy $E_0 + 2zJ$ where z is the number of neighbors of the flipped spin. We can estimate the entropy of a dilute gas of n such flipped spins, with energy $E(n) \sim E_0 + 2Jzn$; the number of configurations is again approximately $\Omega(n) = \binom{N}{n}$, and so their free energy is

$$F \stackrel{\text{Stirling}}{\sim} nz2J - T(N \log N - (N - n) \log(N - n) - n \log n).$$

(Actually, the flipped spins have a short-ranged attraction because if they are adjacent they share a happy bond. We ignore this; think about why we can get away with it.) This is minimized by an equilibrium density of flipped spins

$$\frac{n_{\text{eq}}}{N} \simeq e^{-2zJ/T}.$$

All this so far is just like in the $1d$ argument, except we replaced 2 neighbors with z neighbors, and counted spin flips rather than domain walls.⁷

Here's the catch: The magnetization is not so strongly affected by a flipped spin. It is only decreased from the maximum ($m = 1$) to

$$m = 1 - 2\frac{n_{\text{eq}}}{N} \simeq 1 - 2e^{-2zJ/T} \simeq 1 \quad \text{if } T \ll zJ.$$

So this means that at low (but nonzero) temperature, the magnetization survives. And something interesting has to happen at some intermediate temperature.

⁷Why did we count domain walls in $d = 1$? Because in $d = 1$, the energy of a row of k flipped spins in a row is the same for any k . (Thanks to S. Ortoleva for reminding me of this during lecture.) The elementary dynamical object is really the kink itself in $d = 1$. This is the tip of an iceberg called 'fractionalization'.

3.6 Duality

[Feynman, *Statistical Mechanics* §5.4, Parisi, §4.6]

Let's set $J = 1$ for a bit, since only the combination βJ appears in this discussion.

Now consider again the low-temperature expansion: previously we thought about flipping some dilute collection of isolated spins, each of which costs a (Boltzmann) factor of $e^{-2\beta Jz}$ in the partition function. More accurate is to think of this as a sum over *islands* of reversed spins. If we were speaking about a 3d lattice, they would be regions, with two-dimensional boundaries. Let's focus on the case of a 2d lattice, so that the boundaries of the islands are one-dimensional curves.

If we keep track of the boundaries between these regions, we have complete information about the spin configuration, up to an overall reversal. The weight for a given configuration C is $e^{-2\beta\ell(C)}$ where $\ell(C)$ is the total length of boundary in that configuration. We could include a field h in this description, that multiplies the weight by

$$e^{-\beta h(A_{\text{in}}(C) - A_{\text{out}}(C))}$$

where $A_{\text{in/out}}(C)$ is the number of sites inside/outside the chosen configuration of boundaries.

Can we represent the entire partition sum as a sum over these boundaries (which are called *domain walls*)? It is not just a sum over curves. Notice that the boundaries are always closed curves – it is a deep topological fact that a boundary has no boundary. Furthermore, the boundary curve is always *contractible* in the sense that by flipping back some of the spins in the region to agree with the rest, we can gradually shrink it away to nothing. Here's an example of some curves that are not contractible, on the

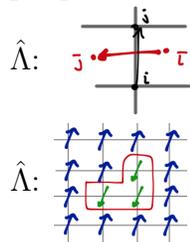


surface of a complicated pastry:

The curves A and B are not

the boundary of any 2d region on the surface of the pastry. Let us restrict ourselves to lattices which do not contain such curves (they are called *simply-connected*).

It is useful to introduce at this point the *dual lattice*: for a 2d lattice Λ , this is a lattice $\hat{\Lambda}$ whose sites correspond to the plaquettes of Λ . A link of Λ separates two plaquettes of Λ ; it corresponds to a link of $\hat{\Lambda}$ connecting the two corresponding sites of



The domain walls of a spin configuration on $\hat{\Lambda}$ cover a set of links of

But our description of the low-temperature expansion on Λ as

$$Z_\Lambda(T) = \sum_C e^{-2\beta\ell(C)}$$

has exactly the same form as our high-temperature expansion (3.9) if we identify

$$e^{-2\beta} = \hat{v} \equiv \tanh \hat{\beta} .$$

This equation relates high temperature (small J) on Λ to low temperature (large J) on $\hat{\Lambda}$. It is due to Kramers and Wannier.

For the square lattice, the dual lattice is the square lattice again! This means that if there is only *one* critical point, it must be the fixed point of the duality transformation:

$$e^{-2\beta J} = \hat{v} \equiv \tanh \beta J .$$

The dual of the honeycomb lattice is the triangular lattice (and vice versa). To learn their critical temperature, we add one more maneuver, called *star-triangle transformation*: The honeycomb lattice is bipartite, and the two sublattices are triangular lattices. By decimating one of the two sublattices, we can relate

$$Z_{\square}^N(J) = \Delta^{N/2} Z_{\triangle}^{N/2}(K)$$

where Δ and K are determined from J by:

$$\sum_{s_x=\pm 1} e^{J s_x (s_1 + s_2 + s_3)} = \cosh^3 J (1 + \tanh^2 J \cdot (s_1 s_2 + s_2 s_3 + s_3 s_1)) \equiv \Delta e^{K(s_1 s_2 + s_2 s_3 + s_3 s_1)} .$$

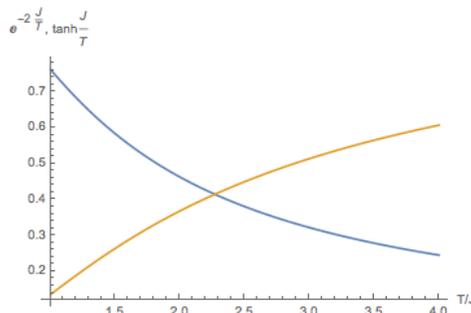
Combining this with the duality relation we can relate the critical temperature of the Ising model on the triangular lattice to itself.

Here is a table of the critical values of $\frac{1}{\beta J}$ for various lattices. z is the coordination number, the number of neighbors of each site.

Λ	z	T_c/J
—	2	0
\diamond	3	1.52
\square	4	2.27
\triangle	6	3.64

The first entry is the 1d chain. You can see that the critical temperature rises with coordination number.

There is more to be said about this sum over curves. They can be used to solve the Ising model exactly.



3.7 A word from our sponsor

We've been spending a lot of time talking about Ising models. Let's take a break and talk about another role it plays in physics.

Lattice gas. Suppose our dynamical variables are the locations $r_1..r_N$ of a collection of point particles. The grand canonical partition function is

$$\Xi(\zeta) = \sum_N \frac{\zeta^N}{N!} \int d^d r_1 \cdots d^d r_N e^{-\beta \sum_{i<j} V(r_i - r_j)} \quad (3.12)$$

where ζ is a fugacity for particle number, and $V(r)$ is an interparticle potential, which usually has a short-range repulsion and long-range attraction (most kinds of particles find each other vaguely attractive from far away...). The kinetic energy was $\sum_i \frac{p_i^2}{2m}$, but we did the p integrals already: $\int d^d p e^{-\beta \frac{p^2}{2m}} = (\pi m T)^{d/2}$.

These integrals in (3.12) are hard. If our interest is in critical behavior, we can zoom out, and take the particles to live at the sites of a lattice $r \in \Lambda$, so our dynamical variables are instead the number of particles at site r , $n(r)$. To implement the short-range repulsion, we take $n(r) = 0, 1$. Then we study

$$\Xi_\Lambda(\zeta) = \sum_{\{n(r)=0,1\}} \zeta^{\sum_r n(r)} e^{-\frac{1}{2}\beta \sum_{r,r'} J_{r,r'} n(r)n(r')}$$

where $J(r - r')$ implements the attractive part of the potential. If we change variables to $s(r) \equiv 2n(r) - 1 = \pm 1$, we have

$$H(s) = -\frac{1}{2}\beta \sum_{r,r'} J_{r,r'} s_r s_{r'} - \beta \sum_r h_r s_r + \text{const}$$

with $\beta h_r = \frac{1}{2} \log \zeta + \beta \sum_{r'} J_{r,r'}$. This is an Ising model. The ferromagnetic transition is the liquid-gas transition! The $s \rightarrow -s$ symmetry of the Ising model is only a symmetry of the lattice gas near the critical point – it is an ‘emergent symmetry’.

Another useful interpretation of the same model is as a ‘binary fluid’, where $n = 0, 1$ represent occupation by two kinds of fluid elements.

3.8 Block spins

Here we introduce a more general class of coarse-graining transformations, called *blocking*. The essential rule is that the partition function is an RG invariant:

$$Z = \sum_s e^{-H(s)} \stackrel{!}{=} \sum_{s'} e^{-H'(s')}. \quad (3.13)$$

Previously, in the decimation schemes, the coarse-grained variables $\{s'\} \subset \{s\}$ were a subset of the microscopic variables. This is a special case of the more general blocking rule

$$e^{-H'(s')} \equiv \sum_s \prod_{\text{blocks}, b} T(s'; s_{i \in b}) e^{-H(s)}$$

where T is a function which decides how the *block spin* s' depends on the spins s_i in the block. Decimation is the special case where we weight the opinion of one of the spins over all the others:

$$T_{\text{decimate}}(s'; s_{i \in b}) = \delta_{s', s_2}.$$

Another option is majority rule:

$$T(s'_b; s_{i \in b}) = \begin{cases} 1, & \text{if } s'_b \sum_{i \in b} s_i > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Notice that for each block, $\sum_{s'=\pm 1} T(s'; s) = 1$ guarantees (3.13). Furthermore, it is useful if $T(s'; s) \geq 0$, so that everything is a probability. Also, it is best if T preserves the symmetries of the system.

4 Mean Field Theory

Mean field theory (MFT) is always simple and sometimes right, and it is all around us in physics departments, so we must understand well when to believe it. We will see that it goes bad near critical points, and the RG will come to our rescue. It is great for getting a big picture of the phase diagram.

We'll give three roads toward MFT, in order of decreasing squiggleness. For definiteness, consider the Ising model, on any graph Λ :

$$Z = \sum_s e^{-H(s)}, \quad H(s) = -\frac{1}{2} \sum_{i,j} J_{ij} s_i s_j - h \sum_i s_i.$$

(I've put the $\frac{1}{2}$ because the sum is over all i, j .) Mean field theory is an attempt to fulfill the urge everyone has to be able to do the sums over the spins one at a time. If only J were zero, we could do this, for example to compute the magnetization:

$$m = \langle s \rangle = \frac{\sum_{s=\pm 1} s e^{\beta h s}}{\sum_{s=\pm 1} e^{\beta h s}} = \tanh \beta h. \quad (4.1)$$

But $J \neq 0$ is much more interesting. So what to do?

Our first approach to MFT is via political science. Mean field theory is the physics realization of libertarian political philosophy⁸. This has two ingredients.

(1) No one cares about anyone else. What I mean by this is: put yourself in the position of one of the spins in the Ising model. How does it even know about its neighbors? Its role in the hamiltonian is

$$H(s_i) = s_i \left(-\frac{1}{2} \sum_j J_{ij} s_j - h \right).$$

From its point of view, this is just like some external magnetic field depending on what its neighbors are doing. What's s_j ? Well, it's probably equal to its average value, $\langle s_j \rangle = m$. So let's just forget everyone else, and assume they are average and incorporate them into an effective magnetic field:

$$h_{\text{eff}} \equiv \frac{1}{2} \sum_j J_{ij} m + h.$$

The second tenet is

(2) Everyone's the same (and I am just like everyone). That is: if there is only one spin in the world, and this is the field it sees, then we can compute m using (4.1):

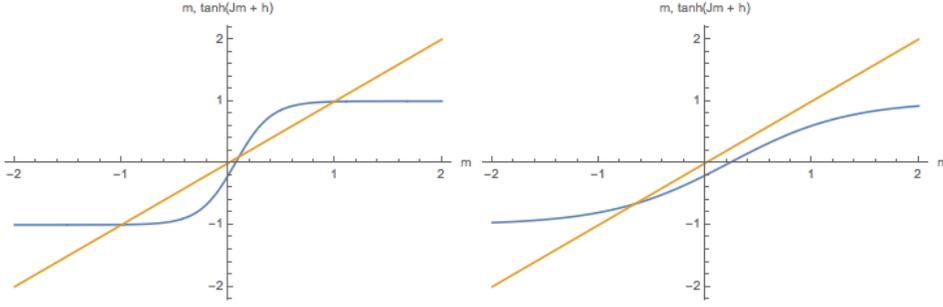
$$m \stackrel{(4.1)}{=} \tanh \beta h_{\text{eff}} = \tanh (Jm + h).$$

⁸Disclaimer: I actually don't know anything at all about political philosophy and made all this up during the lecture today.

Here I defined $J \equiv \frac{1}{2} \sum_j J_{ij}$. This is an equation for m ! We can solve it!

[End of Lecture 8]

At least graphically or numerically we can solve it. Here is m (yellow) and $\tanh(Jm+h)$ (blue) plotted versus m for two values of J (large and small compared to T , with some small h)



Here's our second approach to MFT. Basically, here we will be more explicit about what we're leaving out (but it is the same as the previous discussion). We rewrite the interaction term in the Ising hamiltonian as

$$\begin{aligned} s_i s_j &= (m + (s_i - m))(m + (s_j - m)) \equiv (m + \delta s_i)(m + \delta s_j) \\ &= m^2 + m(\delta s_i + \delta s_j) + \mathcal{O}(\delta s)^2 = -m^2 + m(s_i + s_j) + \mathcal{O}(\delta s)^2. \end{aligned} \quad (4.2)$$

We are going to treat the fluctuation about the mean δs as small. Then

$$\begin{aligned} -H &= \frac{1}{2} \sum_{ij} J_{ij} (m(s_i + s_j) - m^2) + h \sum_i s_i + \mathcal{O}(\delta s)^2 \\ &= -\frac{1}{2} N J m^2 + (Jm + h) \sum_i s_i + \mathcal{O}(\delta s)^2. \end{aligned} \quad (4.3)$$

N is the number of sites, and $J \equiv \sum_j J_{ij}$. The contribution Jm to the external field from the neighbors is sometimes called the 'molecular field'. What we are neglecting here (when we drop the $\mathcal{O}(\delta s)^2$ in a moment) is the correlations between the spins at different sites i, j . This is not small if $|r_i - r_j| < \xi$, by definition of the correlation length ξ . Brutally ignoring the correlations, then, we can do all the sums have

$$Z \simeq e^{-\frac{1}{2} N \beta J m^2} (2 \cosh \beta (Jm + h))^N \equiv Z_{\text{MFT}}$$

So in this approximation, the free energy density is

$$f_{\text{MFT}}(m) \equiv -\frac{T}{N} \log Z_{\text{MFT}} = \frac{1}{2} J m^2 - T \log \cosh \beta (Jm + h) + \text{const.}$$

I claim, and will prove next, that $f_{\text{MFT}}(m) \leq f$ is a lower bound on the correct free energy. This is true for every m , and so the best bound comes from minimizing over

m . That condition gives back the equation for m (namely $m = \tanh \beta(Jm + h)$) that we got from self-consistency above.

Our third approach is the *variational method*. [There is a good discussion of this in Parisi's book.] It will give our proof that $f_{\text{MFT}}(m)$ lower bounds f . The idea can be found from a Bayesian viewpoint on statistical mechanics. Let's put this in a box:

Bayesian viewpoint on statistical mechanics. Suppose we are given a (classical) physical system, defined by a configuration space (*e.g.* the values of a bunch of spins $\{s\}$) and a Hamiltonian ($H(s)$). Further suppose that the only thing we know about the state of the system is the average energy E . What probability distribution $P(s)$ should we use to make predictions? We don't want to add unjustified information. One way is to find the distribution P_\star which maximizes the (Shannon) entropy

$$S[P] \equiv - \langle \log P \rangle_P = - \sum_s P(s) \log P(s),$$

⁹ to the constraint that $E = \langle H \rangle_{P_\star} \equiv E[P_\star]$. The distribution should also be normalized $\sum_s P(s) = 1$. We can impose these conditions with lagrange multipliers:

$$\Phi[P] \equiv S[P] + b(E[P] - E) + a(\sum_s P(s) - 1) = - \sum_s P(s) \log(P(s)) + \sum_s (bH(s) + a)P(s) - bE - a$$

$$\begin{aligned} \frac{\delta \Phi[P]}{\delta P(s)} &= - \log P(s) - 1 + bH(s) + a \\ \implies P_\star(s) &= e^{bH(s) + a - 1} \end{aligned}$$

where a, b must be determined to satisfy the two constraints.

If instead of fixing the average energy, we want to fix the temperature $1/\beta$, what do we do? We should instead find the distribution $P_\star(s)$ which *minimizes* the free energy

$$F[P] = E[P] - S[P]/\beta$$

as a functional of P . It is still normalized, so we need to use a lagrange multiplier again, and minimize

$$F_\lambda[P] \equiv F[P] + \lambda(\sum_s P(s) - 1)$$

⁹A useful way to think about this quantity is the following. Given a distribution $P(s)$, the quantity $-\log P(s)$ is called the *surprise* of the configuration s – the bigger it is, the more surprised you should be if s actually obtains. So the Shannon entropy is simply the *average surprise* (or maybe the *expected surprise*). Clearly, all else being equal, we will make the best predictions using the distribution that minimizes the expected surprise. If you like this perspective on the world, the place to get more is E. T. Jaynes, *Probability Theory: The Logic of Science*.

which is extremized when

$$0 = H(s) + T + \lambda + T \log P(s)$$

from which we again recover the Boltzmann distribution, $P(s) = e^{-\beta H(s)}/Z$ (the multiplier λ is eliminated in favor of Z by normalizing).

This derivation is useful philosophically (for example, it evades all the vexing questions about ergodicity), and it also implies a variational bound on the free energy F . That is, if we pick some arbitrary *other* distribution $P_{\text{off-the-street}}(s)$, then we know that its free energy is bigger than the correct equilibrium free energy:

$$F[P_{\text{off-the-street}}] \geq F[e^{-\beta H}/Z].$$

So: to recover mean field theory, we choose a distribution which we like because we know how to calculate its averages, that is, one which *factorizes*:

$$P_{\text{MFT}}(s) = \prod_i p_i(s_i)$$

where we can parametrize the individual factors as

$$p_i(s) = \frac{1 + m_i}{2} \delta_{s,1} + \frac{1 - m_i}{2} \delta_{s,-1}.$$

It is normalized since each factor is. Then with this distribution all the moments factorize:

$$\langle g_1(s_1)g_2(s_2) \rangle_P = \langle g_1(s_1) \rangle_{p_1} \langle g_2(s_2) \rangle_{p_2}$$

and

$$\langle g(s_1) \rangle_{p_1} = \frac{1 + m_i}{2} g(1) + \frac{1 - m_i}{2} g(-1).$$

And in particular,

$$\langle s_i \rangle = m_i, \quad \langle H \rangle = - \sum_{ij} J_{ij} m_i m_j - \sum_i h m_i$$

and its entropy is

$$S[P] = - \langle \log P \rangle_P = \sum_i s(m_i), \quad s(x) \equiv - \frac{1+x}{2} \log \frac{1+x}{2} - \frac{1-x}{2} \log \frac{1-x}{2}.$$

Now we apply the variational bound. The free energy $F(m_i) = F[P_{\text{MFT}}^{(m_i)}] \geq F$ upper bounds the true free energy for any m , so we do best by minimizing it:

$$0 = \partial_{m_i} F = - \sum_j J_{ij} m_j - h_i + T \arctanh m_i$$

which gives back

$$m_i = \tanh \beta \left(\sum_j J_{ij} m_j + h_i \right)$$

the mean field equation.

If we set $h = 0$ and study small m , we find

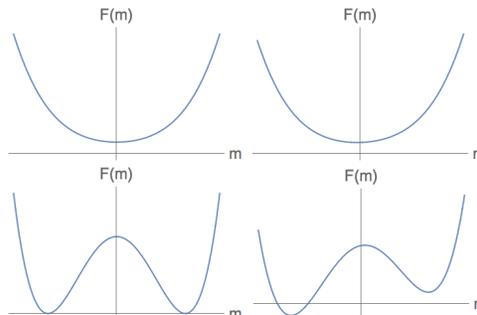
$$f_{\text{MFT}}(m) \simeq a + \frac{1}{2} B m^2 + c m^4 + \dots \quad (4.4)$$

where a, c are constants. The coefficient B is

$$B = (1 - \beta J) \equiv b t,$$

where $t \equiv \frac{T - T_c}{T_c}$ is the “reduced” temperature. If

$c > 0$, this function looks like one of the figures at right, where the top left figure is for $T > T_c^{\text{MF}} = J$ and the bottom left figure $T < T_c$ the critical temperature. If $c < 0$, then we have to keep more terms in the expansion to know what happens. (The right column is with $h < 0$.) So you can see that the minimum of f occurs at $m = 0$ for $T > T_c$ (disordered phase) and $m \neq 0$ for $T < T_c$ (ordered phase). This figure makes it clear that the third solution of the MF equations (at $m = 0$) that exists for $T < T_c$ is a *maximum* of the free energy – it is unstable.



4.1 Landau-Ginzburg theory

[Parisi §4.3, 5.2] Before drawing any further physical conclusions from the MFT free energy we just derived, let me say some words in defense of this form of the free energy (4.4). These are the words (the idea is due to Landau; this is a paraphrase):

What else could it be?

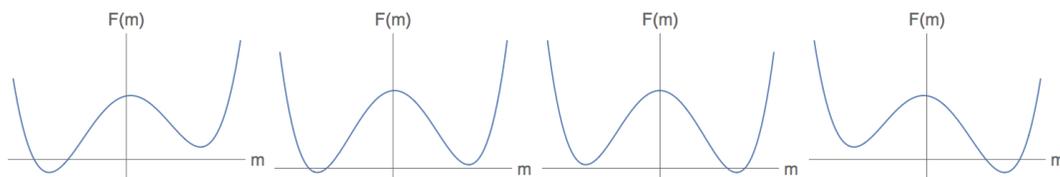
If the free energy is analytic near $m = 0$, it looks like this. So all that song and dance about justifying mean field theory is really irrelevant to the conclusions we draw about the phase transition from $m = 0$ (at $T > T_c$) to $m \neq 0$ (at $T < T_c$). The dependence of B on $T - T_c$ follows from (4.4) itself! With this assumption, $f_{\text{MFT}}(m)$ is the most

general answer, consistent with the symmetry under $\begin{cases} m & \mapsto -m \\ h & \mapsto -h \end{cases}$ (at the same time).

So: the only real assumption leading to (4.4) is the analyticity of $f(m)$. Some points: (1) we will see immediately below that analytic $f(m)$ does *not* mean that the physics is analytic in external parameters – we *can* get critical behavior from this framework. (2) When we find out that MFT gives wrong predictions for critical exponents, we will

have to find out how and why we get an $f(m)$ which is not analytic. (3) The fact that the coefficient of m^2 is proportional to the deviation from the critical temperature follows from our analysis of (4.4). The only input from the microscopic calculation (with all the approximations above) is how do the coefficients a, b, c, d depend on the microscopic couplings. Notice that the actual magnetization $m = N^{-1} \sum_{i=1}^N \langle s_i \rangle$ is an average of numbers each ± 1 , and therefore lies between these two numbers. The minimum of $f(m)$ will not satisfy this constraint for all values of a, b, c, d, \dots consistent with the input above: this is a “UV constraint on IR physics” of the kind that the string theorists dream about.

Types of phase transitions. A *first order phase transition* is one where the minimum of the free energy *jumps* from one value to another, distant value, like if the potential evolves as in this comic strip as a function of the parameter in question:

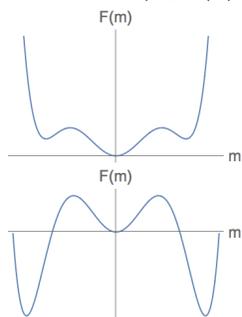


The two configurations need have nothing to do with each other, and there is no notion of universal properties of such a transition. The correlation length need not grow. This is what happens when we vary h from positive to negative, at nonzero t . The correlation length stays fixed, but the minimum jumps from $-m_0$ to $+m_0$ as h goes through zero (as in the comic strip above).

The alternative is a *continuous phase transition* which is more interesting. (Sometimes, one hears about ‘ n th-order’ phase transitions for various $n \geq 2$, but I haven’t found the need to distinguish between these.)

Notice that when we say that ‘a transition is continuous’ it depends on what parameter we are varying: at $T < T_c$, as a function of the magnetic field, the transition from one minimum to the other of the Ising model is first order. (this is what’s illustrated in the comic above).

Here are some simple examples of the power of the LG point of view: If we break the Ising symmetry the transition should generically be first order. This allows a cubic term in the potential, and it means that as we cool from high temperatures, one of the two minima at $m \neq 0$ will have $f(m) < f(0)$ before (at a higher temperature than the one where) $f''(0)$ becomes negative.



A continuous transition is, however, not a consequence of Ising symmetry: if $c < 0$, then we must consider the m^6 term. Depending on the signs, there is a regime where the minima at $m \neq 0$ descend

before $f''(0)$ goes negative.

Usually (but not always) $T_c^{\text{MF}} > T_c$, since the fluctuations we are ignoring disfavor the ordered state. (Sometimes in fact $T_c \leq 0$.)

[End of Lecture 9]

Mean field critical exponents. The very fact that there is a notion of T_c in MFT is worth remarking on. Lots of stuff is non-analytic at T_c !

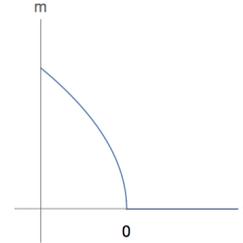
Near T_c , we can expand

$$f_{\text{MF}}(m) = a + btm^2 + cm^4 + \mu hm + \dots$$

where $T \equiv \frac{T-T_c}{T_c}$ is the non-dimensionalized deviation from the critical temperature. Notice that a, b, c, μ really do depend on T , but only weakly.

When $h = 0$, the free energy is minimized when :

$$m = \begin{cases} 0, & t > 0 \\ \pm \sqrt{\frac{b}{2c}} \sqrt{-t}, & t < 0 \end{cases}$$



The magnetization critical exponent is β and $\beta_{\text{MFT}} = \frac{1}{2}$.

When $h \neq 0$ (and small) and $t > 0$, we can ignore everything but $f \sim a + btm^2 + \mu hm$ (beware typos in Cardy's book here) to find

$$m \sim \frac{\mu h}{bt}.$$

This exponent is called γ , and $\gamma_{\text{MFT}} = 1$.

Right at the transition, $t = 0$, we must keep the quartic term and we find

$$m \sim \left(\frac{\mu h}{c} \right)^{1/3}.$$

This exponent is called δ and $\delta_{\text{MFT}} = \frac{1}{3}$. (I'm mentioning this botany of greek letters because there are people for whom these letters are close friends.)

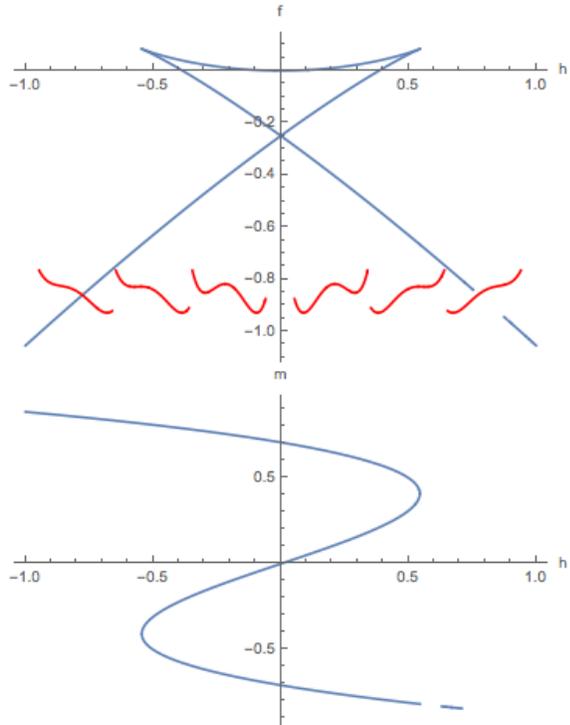
Finally, the free energy density evaluated at the minimum, at $h = 0$, is

$$f(t) = \begin{cases} a, & t > 0 \\ a - \frac{(bt)^2}{c}, & t < 0 \end{cases}$$

which means that $\partial_t^2 f$ jumps at the transition; this jump is actually an artifact of MFT.

Otherwise, the behavior in general predicted by MFT is good, but we'll see that the values of these exponents aren't always right.

It is worth thinking about what the extrema of this potential do as we vary the parameters. At right is a plot of the free energy evaluated at all of the critical points of $f(m)$ as h varies (the other couplings are fixed to $T < T_c$). (This picture is sometimes called a ‘swallowtail’.) Inset in red is the shape of the potential at the corresponding value of h . Plotted below is the corresponding magnetization. Notice that the number of (real) critical points goes from 1 to 3 as $|h|$ is decreased below some value; the two new extrema are pair-produced from the complex plane, that is, the new extrema come in pairs and have a *larger* free energy. Notice further that $\partial_h^2 f > 0$ along the top trajectory – this is the maximum near the origin. The other one is actually a local minimum – a *metastable state*. More on the physics of this in §5.5.



LG Theory for other symmetries. Here is another illustration of the Power of Landau. We've been studying models with a \mathbb{Z}_2 symmetry acting by $\begin{cases} m & \mapsto -m \\ h & \mapsto -h \end{cases}$. Suppose instead of this, we made the replacement $\mathbb{Z}_2 \rightsquigarrow \text{O}(n)$ rotation symmetry acting on a generalization of the magnetization with n components, $m \rightsquigarrow m_a$, in that case the external field would be $h \rightsquigarrow h^a$, and the transformation rule would be

$$\begin{cases} m_a & \mapsto R_a^b m_b \\ h^a & \mapsto h^a R_a^b \end{cases}$$

(where R is an orthogonal matrix ($R^t R = 1$) so that $m_a h^a$ is invariant and can be added to the hamiltonian). Incidentally, the field m_a playing the role of the magnetization, the thing that orders at low temperatures, is called the *order parameter*.

What's the free energy in this case? If it's analytic in m , for small m it must be of the form

$$f_{\text{LG}}(m_a) = a + bt \sum_a m_a^2 + c \left(\sum_a m_a^2 \right)^2 + \dots + \mu h^a m_a$$

For $t < 0$, a minimum occurs at $m_a = (m_0, 0, 0\dots)_a$ with $m_0 = \sqrt{\frac{-bt}{2c}}$, as well as at all its images under an arbitrary $\text{O}(n)$ rotation. The $\text{O}(n)$ symmetry is broken in the sense that the free energy minima form a nontrivial orbit of the symmetry (and furthermore, the free energy will be non-analytic in h near $h = 0$). This degeneracy has the following consequence. If we expand $m_a = (m_0, 0, 0\dots)_a + \delta m_a$ about the minimum, we find

$$f_{\text{LG}} = c + 2b|t|\delta m_1^2 + \mathcal{O}(\delta m^4)$$

– the quadratic terms are completely independent of the other N components of the fluctuations $\delta m_2 \dots \delta m_N$! We'll see in a moment that this absence of a restoring force means that those degrees of freedom have infinite correlation length, even in the ordered phase. They are called *Goldstone modes*.

‘Microscopic’ Landau-Ginzburg Theory. In our variational derivation of mean field theory, we actually derived a stronger bound, since we allowed for spatially-varying magnetization. Let’s combine the Landau point of view with the knowledge that the free energy is extensive¹⁰ to learn the answer without doing any work. Because F is extensive, we can write the free energy as a sum over a contribution associated to each lattice site, or patch of the lattice $F = \sum_i f_i$, where f_i depends on the magnetization m_i at site i and nearby sites. (Think about assembling the system from big enough chunks.) If the correlation length is not so small, f_i will vary smoothly and we can approximate this as an integral: $\sum_i f(x_i) \simeq a^{-d} \int d^d x f(x)$ The integrand in turn, depends locally on the field and its derivatives. Translation invariance forbids any explicit dependence on x :

$$F[m] = \int d^d x f(m(x), \vec{\nabla} m(x), \nabla^2 m(x) \dots).$$

Symmetries further constrain the form of f : \mathbb{Z}_2 symmetry forbids terms odd in m and h , parity symmetry forbids terms with an odd number of derivatives, rotation invariance requires us to dot all the vector indices together. So, under the crucial analyticity assumption, we have

$$f_{\text{LG}} = V(m) + \gamma \vec{\nabla} m \cdot \vec{\nabla} m + \gamma' (\nabla^2 m)^2 + \dots \quad (4.5)$$

where $V(m) = a + Bm^2 + cm^4 + dm^6 + \dots$ is the value when m is constant in space – it contains all the information about the mean field treatment of phase transitions, some of which we discussed above.

We will have a lot more to say about how to organize this expansion. So far it is an expansion in powers of m (since know that in the neighborhood of the critical point m is small). It is also an expansion in the number of derivatives, something like the dimensionless quantity $a \nabla m$, where a is the lattice spacing. If this quantity is not small then we are asking the wrong question, because the ‘field’ we are treating as continuous is varying rapidly on the scale of the lattice spacing a . The RG will give us a better understanding of this expansion: we’ll see that operators with more derivatives are more irrelevant (near any of fixed points under discussion here).

The equation (4.5) contains an enormous amount of information. To better appreciate it, let’s first discuss the mean-field treatment of the correlation function.

4.2 Correlations; Ginzburg criterion for MFT breakdown

You might think that the spirit of mean field theory is antithetical to obtaining information about correlations between the spins, since after all that was precisely what we

¹⁰I owe you some discussion of why this is the case. This happens in §5.1.

ignored to do the sums. Not so!

Here's a first pass. The connected correlator (assume translation invariance) is

$$G(r_i - r_j) \equiv \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle.$$

The magnetization is related to $\langle s_i \rangle$ by

$$m = \frac{1}{N} \sum_i \langle s_i \rangle = \frac{1}{\beta N} \partial_h \log Z$$

and the (isothermal magnetic) susceptibility is

$$\chi_T = \partial_h m = \frac{1}{N\beta} \partial_h^2 \log Z = \frac{1}{NT} \sum_{ij} G(r_i - r_j) = \frac{1}{T} \sum_i G(r_i). \quad (4.6)$$

If the correlation length is big enough, $\xi > a$, then we can approximate the integral by a sum

$$\chi_T = \frac{1}{T a^d} \int d^d r G(r).$$

This is called the static susceptibility sum rule. It relates a thermodynamic quantity χ_T to a (integrated) correlation function.

Is the integral well-defined? The lower limit of integration, the UV, is fine because we are talking about a lattice model. When ξ is finite, the fact that the correlations fall off rapidly $G(r) \stackrel{r \gg a}{\sim} e^{-r/\xi}$ means that the integral converges in the IR (the upper limit of integration) as well.

But: $\chi_T \rightarrow \infty$ at the critical point, in fact we saw above that $\chi_T \stackrel{\text{MFT}}{\sim} \frac{1}{T-T_c} + \text{regular terms}^{11}$ as $T \rightarrow T_c$. The only way this can happen consistently with the susceptibility sum rule is if $\xi \rightarrow \infty$ as well at the transition. We'll see in a moment with what power it diverges.

MFT for $G(r)$. We can actually do better and find the form of $G(r)$ *within* the mean field approximation. This is because $G(r)$ is a *response function*. Here's what this means.

When $h = 0$, the correlation function is

$$\langle s_r s_0 \rangle = \frac{\sum_s s_r s_0 e^{-H(s)} \cdot 1}{\sum_s e^{-H(s)} \cdot 1}$$

where we can write 1 cleverly as

$$1 = \delta_{s_0, 1} + \delta_{s_0, -1}.$$

¹¹If I keep chanting 'γ = 1' maybe I will remember these letters someday.

Using the fact that $H(-s) = H(s)$ (for $h = 0$), we have

$$\langle s_r s_0 \rangle = \frac{\sum' s_r e^{-H(s)}}{\sum' e^{-H(s)}} \equiv s_r'$$

where \sum' means we sum over all the spins but the one at 0, and we fix $s_0 = +1$, and $\langle \dots \rangle'$ denotes expectation in this ensemble. So the correlation function $\langle s_r s_0 \rangle$ is just the magnetization at r , $m(r)$ in response to an (infinite) applied field (completely) localized to $r = 0$. In the presence of this localized source, $m(r)$ will certainly depend on its distance from the source. But the mean field equation (for $r \neq 0$) still takes the form

$$m(r) = \tanh \beta \left(\sum_{r'} J_{rr'} m(r') \right) \\ \stackrel{m \ll 1}{\simeq} \beta \sum_{r'} J_{rr'} m(r') \quad (r \neq 0)$$

In the second line, we retreated to small m , which is useful for $T > J$. (Otherwise maybe we need some numerics.) We can do better and include the corrections at the origin, by include a source:

$$m(r) = \beta \sum_{r'} J_{rr'} m(r') + A \delta(r).$$

[End of Lecture 10]

This (linear!) equation

$$\sum_{r'} (\delta_{rr'} - \beta J_{rr'}) m(r') = A \delta(r).$$

can be solved by Fourier transform. That is, the matrix we need to invert is diagonal in momentum space. That is, take $\sum_r e^{i\vec{k}\cdot\vec{r}}$ (BHS) to get:

$$(1 - \beta \tilde{J}(k)) \tilde{m}_k = +A,$$

where

$$\tilde{m}_k \equiv \sum_{r \in \Lambda} e^{i\vec{k}\cdot\vec{r}}, \quad m(r) = \int_{\text{BZ}} d^d k e^{-i\vec{k}\cdot\vec{r}} \tilde{m}_k.$$

In the inversion formula, the integral is over the Brillouin zone of the lattice Λ ; for a cubic lattice, this just means $k \in (-\pi/a, \pi/a]$. The Fourier transform of the coupling is

$$\tilde{J}(k) \equiv \sum_r e^{i\vec{k}\cdot\vec{r}} J_{r,0}.$$

For example, for a cubic lattice, this is $\tilde{J}_{\text{cubic}}(k) = \sum_{\mu=x,y,z} 2 \cos k_{\mu} a$, where a is the lattice spacing. For small k , the general case is

$$\tilde{J}(k) = J(1 - R^2 k^2) + \mathcal{O}(k^4)$$

where $R^2 \equiv \frac{\sum_r r^2 J_{r,0}}{\sum_r J_{r,0}}$ is the *range-squared* of the coupling. In the last expression we assumed the lattice had an inversion symmetry so that there are no terms with odd powers of k . We'll be interested in small k because it determines the long-distance behavior of $G(r)$.

Therefore,

$$\tilde{m}_k \simeq \frac{A}{1 - \beta J(1 - R^2 k^2)}$$

and

$$\tilde{G}_k = \frac{A/R^2}{k^2 + \xi_{\text{MF}}^{-2}}, \quad \xi_{\text{MF}} = \frac{R}{\sqrt{t}}.$$

You can check that $\int d^d r G(r) = \tilde{G}_{k=0} = \chi_T = \frac{\mu}{bt}$, independent of R as we found above, consistent with the susceptibility sum rule.

Why is ξ in this formula the correlation length? Fourier transform back:

$$G(r) \sim \frac{e^{-r/\xi_{\text{MF}}}}{r^{\frac{d-1}{2}}}$$

which is a formula named after Ornstein-Zernicke, I don't know why. So we've found the rate at which the correlation length diverges as we approach the critical temperature from above (in the mean field approximation) $\xi_{\text{MF}} \sim \frac{1}{\sqrt{t}}$ which says $\nu_{\text{MF}} = \frac{1}{2}$.

Right at $t = 0$, we have

$$G(r) = \int d^d r \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2} \sim r^{-d+2}$$

which says $\eta_{\text{MF}} = 0$.

Ginzburg criterion for breakdown of MFT. So, is mean field theory right? To get it, we had to ignore the following term in the hamiltonian

$$\Delta H = \sum_{r,r'} J_{rr'} \delta s_r \delta s_{r'}.$$

A necessary condition for its self-consistency is that the expected value of this term, calculated within MFT, is small compared to the MFT energy:

$$\langle \Delta H \rangle_{\text{MFT}} \stackrel{!}{\ll} E_{\text{MF}}.$$

The right hand side divided by N is

$$\frac{1}{N} E_{\text{MF}} = -\partial_\beta (\beta f_{\text{MF}}) \sim Jt,$$

where we used $f_{\text{MF}} \stackrel{t \ll 1}{\approx} J(1 - \beta J)^2$. The LHS/ N is

$$\frac{1}{N} \langle \Delta H \rangle_{\text{MFT}} = \frac{1}{N} \sum_{rr'} J_{rr'} \langle \delta s_r \delta s_{r'} \rangle_{\text{MF}} = \sum_{r'} J_{rr'} G_{rr'}$$

We assume that $J_{rr'}$ has a smaller range than $G_{rr'}$ (*i.e.* $R < \xi$), so that we may approximate the RHS as

$$JG(0) = \frac{AJ}{R^2} \int \frac{d^d k}{k^2 + \xi^{-2}}.$$

In a lattice model, the integral is over the Brillouin zone. The dangerous bit, where the RHS can become big, though, comes from $k \rightarrow 0$, which doesn't care about your lattice details. Actually, we already used this in replacing \tilde{G}_k with its long-wavelength approximation.

To separate out the UV physics ($k \sim \frac{2\pi}{a}$) from the IR physics ($k \sim \frac{2\pi}{L}$), let's use the partial-fractions trick familiar from calculus:

$$\frac{1}{k^2 + \xi^{-2}} = \frac{1}{k^2} - \frac{\xi^{-2}}{k^2(k^2 + \xi^{-2})}$$

so that

$$I \equiv \int_{|k| > a^{-1}} \frac{d^d k}{k^2 + \xi^{-2}} = \underbrace{\int_{|k| < a^{-1}} \frac{d^d k}{k^2}}_{\text{ind. of } T} - \xi^{-2} \int_{|k| < a^{-1}} \frac{d^d k}{k^2(k^2 + \xi^{-2})}$$

The first term is a (possibly big, honking) constant, which doesn't care about the temperature or the correlation length. The second term is finite as $a \rightarrow 0$ if $d < 4$. (Note that it's finite as $L \rightarrow \infty$ if $d > 2$.) When it is finite, we can scale out the dependence on ξ :

$$I \stackrel{\xi \gg a}{\approx} \text{const} + \xi^{2-d} K_d \int_0^\infty \frac{x^{d-3} dx}{x^2 + 1}$$

where

$$K_d \equiv \frac{\Omega_d}{(2\pi)^d}$$

is a ubiquitous combination of angular factors; Ω_d is the volume of the unit d -sphere.

So: Small corrections to the MFT energy *computed within MFT* requires

$$\frac{J\xi^{2-d}}{R^2} \ll Jt$$

Remembering that we derived $\xi_{MF} = Rt^{-1/2}$, we can write this condition purely in terms of the mean field correlation length. If the condition

$$\xi^{4-d} \ll R^4$$

the mean field theory is wrong. And for sure it is violated if ever $\xi \rightarrow \infty$ in $d < 4$. (Remember that d is the number of space dimensions.) Note that the condition depends on the range R of the interactions: MFT works better for longer-range interactions, and in more dimensions.

Continuum field theory

Along the way in the preceding discussion of correlation functions in mean field theory, we showed the following, which is a useful summary of the whole discussion, and makes contact with the microscopic Landau-Ginzburg theory. Consider the simple case where

$$J_{ij} = \begin{cases} J, & r_{ij} \leq R \\ 0, & r_{ij} > R \end{cases}.$$

Then we showed that the contribution to the mean-field free energy from the interaction term is

$$\begin{aligned} \Delta f_{MF}[m] &= \sum_{ij} J_{ij} m_i m_j \\ &= -J \frac{a^2}{4} \sum_i \sum_{|\delta| \leq R} \left(\left(\frac{m_{i+\delta} - m_i}{a} \right)^2 - \left(\frac{m_{i+\delta} + m_i}{a} \right)^2 \right) \\ &= -J \frac{a^2}{4} \sum_i \sum_{|\delta| \leq R} \left(\frac{m(r_i + \delta) - m(r_i)}{a} \right)^2 + \underbrace{\mathcal{O}(m^2)}_{\text{correction to } V(m)} \\ &\simeq -J \frac{a^2}{4} \sum_i \sum_{|\delta| \leq R} \left(\frac{\vec{\delta} \cdot \vec{\nabla} m(r_i)}{a} \right)^2 + \mathcal{O}(m^2) \\ &\simeq -\frac{zJR^2}{4} \int \frac{d^d r}{a^d} (\vec{\nabla} m)^2 + \mathcal{O}(m^2) \end{aligned}$$

where z is the coordination number of the lattice. Comparing this to our ‘local’ Landau-Ginzburg expression (4.5), we’ve learned that the constant in front is

$$\gamma \simeq R^2 \left(\frac{Jz}{4a^d} \right) = \frac{R^2 T_c^{MF}}{a^d}.$$

(When we derived the mean-field critical temperature, I defined $J_{\text{prev}} \equiv \sum_{ij} J_j = zJ_{\text{here}}$.)

The equations of motion for m coming from extremizing f_{LG} , in this continuum approximation, are¹²

$$0 = \frac{\delta F_{\text{LG}}}{\delta m(x)} = -2\gamma \nabla^2 m + \frac{\partial V}{\partial m} \Big|_{m=m(x)} + \dots \quad (4.7)$$

If V contains a source term $\int dx h(x)m(x)$, then this is

$$0 = -2\gamma \nabla^2 m(x) + h(x) + 2btm(x) + \mathcal{O}(m^2).$$

For the case of a localized source, $h(x) = \delta(x)$, the solution in fourier space

$$\tilde{m}_k = \frac{(2\gamma)^{-1}}{k^2 + bt/\gamma}$$

gives back $\xi^{-1} = \sqrt{bt/\gamma}$.

In case you're not comfortable with this derivation of the continuum field theory description of Ising models with large correlation length, another approach is outlined on problem set 4.

¹²For those of you who are unfamiliar with variational calculus, please see the sidebar on the subject at §4.2.1.

4.2.1 Sidebar on Calculus of Variations

Let us spend a moment thinking about functionals – a functional is a monster that eats a function and returns a number – and how they vary as we vary their arguments. I'll describe this in the context where the function in question is $x(t)$, the trajectory of a particle in time, but you can substitute $m(x)$.

The basic equation of the calculus of variations is:

$$\frac{\delta x(t)}{\delta x(s)} = \delta(t - s).$$

This the statement that $x(t)$ and $x(s)$ for $t \neq s$ are independent. From this rule and integration by parts we can get everything we need. For example, let's ask how does the potential term in the action $S_V[x] = \int dt V(x(t))$ vary if we vary the path of the particle. Using the chain rule, we have:

$$\delta S_V = \int ds \delta x(s) \frac{\delta \int dt V(x(t))}{\delta x(s)} = \int ds \delta x(s) \int dt \partial_x V(x(t)) \delta(t-s) = \int dt \delta x(t) \partial_x V(x(t)).$$

We could rewrite this information as:

$$\frac{\delta \int dt V(x(t))}{\delta x(s)} = V'(x(s)).$$

¹³ What about the kinetic term $S_T[x] \equiv \int dt \frac{1}{2} m \dot{x}^2$? Here we need integration by parts:

$$\frac{\delta}{\delta x(s)} S_T[x] = \frac{2}{2} m \int dt \dot{x}(t) \partial_t \frac{\delta x(t)}{\delta x(s)} = m \int dt \dot{x}(t) \partial_t \delta(t-s) = -m \int dt \ddot{x}(t) \delta(t-s) = -m \ddot{x}(s).$$

If we set the total variation to zero, we get Newton's equation:

$$\frac{\delta}{\delta x(s)} \left(\int dt \left(\frac{1}{2} m \dot{x}^2 - V(x(t)) \right) \right) = -m \ddot{x}(s) + V'(x(s))$$

¹³If you are unhappy with thinking of what we just did as a use of the chain rule, think of time as taking on a discrete set of values t_i (this is what you have to do to define calculus anyway) and let $x(t_i) \equiv x_i$. Now instead of a functional $S_V[x(t)]$ we just have a function of several variables $S_V(x_i) = \sum_i V(x_i)$. The basic equation of calculus of variations is even more obvious now:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

and the manipulation we did above is

$$\delta S_V = \sum_j \delta x_j \partial_{x_j} S_V = \sum_j \delta x_j \partial_{x_j} \sum_i V(x_i) = \sum_j \sum_i \delta x_j V'(x_i) \delta_{ij} = \sum_i \delta x_i V'(x_i).$$

5 Festival of rigor

I would like to collect some Facts that we know for sure about the free energy of short-ranged lattice models. As with any rigorous, formal results in physics, it will be crucial to understand the hypotheses.

5.1 Extensivity of the free energy

[Parisi pp. 41-42]

The Ising model free energy is extensive, $F/N = f +$ terms which go to zero as the number of sites $N \rightarrow \infty$. In particular, in the thermodynamic limit, the bulk free energy density f doesn't care about boundary conditions. This assumes that J is short-ranged: $J_{r,0}$ is either of finite support (system-size-independent range), or falling off sufficiently rapidly in r .

Here is an RG-inspired proof of this result. We begin with a finite system, with N sites.

First, notice that the hamiltonian $H(s)$ is bounded

$$-ND < H(s) < ND$$

for some constant D (for the near-neighbor Ising model on a cubic lattice it's J for each link, so $D = dJ$).

We can bound the *free* energy, too, by realizing that the number of configurations is finite – for a finite lattice with N sites, there are only 2^N of them. Each one contributes an energy above the minimum value, and below the maximum value, so

$$2^N e^{-\beta ND} \leq Z_N \leq 2^N e^{\beta ND}.$$

Taking log of the BHS gives

$$-\infty < -D - \frac{\log 2}{\beta} \leq f_N \leq D + \frac{\log 2}{\beta} < \infty$$

the important thing being that the free energy density is [bounded on either side](#), independently of N .

Now here comes the RG bit. For definiteness, take free boundary conditions on an $L \times L \times \dots \times L$ chunk of cubic lattice. (Free boundary conditions means that we just stop writing terms in the hamiltonian when the sites that would participate in them don't exist.) Take $L \gg R$, the range of the interactions. Let Z_L^F be the partition function for this chunk.

Now we try to double the (linear) size of the system, by gluing together the right number (2^d) of smaller chunks of size L . Gluing just means that we add the terms in the hamiltonian which couple the sites across the interface. The number of terms we have to add is $L^{d-1}R$ for each interface (each pair of chunks) we glue, and we have to glue 2^d interfaces. The magnitude of the contribution of each term is bounded by D . Therefore

$$(Z_L^F)^{2^d} e^{-\beta D 2^d L^{d-1} R} \leq Z_{2L}^F \leq (Z_L^F)^{2^d} e^{+\beta D 2^d L^{d-1} R}.$$

Taking the log and dividing by $(2L)^d$ gives

$$f_L + DL^{-1}R \geq f_{2L} \geq f_L - DL^{-1}R$$

or

$$|f_{2L} - f_L| \geq \frac{c}{L}$$

for some positive number c which does not depend on L .

[End of Lecture 11]

This means that the sequence $\{f_{2^n L}\}_n$ converges as $n \rightarrow \infty$ (for a big enough initial L). The same argument can be used to show that the effect of changing boundary conditions can be bounded on either side. Suppose we change terms in the hamiltonian in a way which is localized near the boundary and where the magnitude of the change of each term is bounded by some Δ . Then if Z^B is the resulting partition function,

$$Z_L^B e^{-\beta L^{d-1} \Delta} \leq Z_L^F \leq Z_L^B e^{\beta L^{d-1} \Delta}.$$

Again when we take the log and divide by the volume L^d , the terms proportional to Δ are suppressed by a factor of L .

Thermodynamic limit

We conclude that in a system in d dimensions of linear size L , with short-range interactions, the free energy takes the form:

$$F = L^d f_b + L^{d-1} f_\partial + \mathcal{O}(L^{d-2})$$

$$f_b = \lim_{L \rightarrow \infty} \frac{F}{L^d}, \quad f_\partial = \lim_{L \rightarrow \infty} \frac{F - L^d f_b}{L^{d-1}}.$$

f_∂ is a *boundary free energy density*.

Two questions to ponder:

1. What should we hold fixed in the limit $L \rightarrow \infty$? In a fluid, we might want to fix the density of particles, $\rho = N_{\text{particles}}/L$. If we instead fix $N_{\text{particles}}$, we get a boring fluid.
2. How can the thermodynamic limit fail to exist? We consider a class of examples where it might fail next.

5.2 Long-range interactions

Charged system. Consider a bunch of stuff with a net electric charge, at $T = 0$. Imagine we can fix the charge density ρ , and take $d = 3$ so that the inter-charge potential is $U(r) = A/r$. The self-energy of a sphere of this stuff is (integrating the contributions from shells of radius r , which only care about contributions from inside)

$$E(R) = \int_0^R \left(\frac{4}{3} \pi r^2 \rho \right) \frac{A}{r} (4\pi r^2 \rho dr) \sim A\rho^2 R^5.$$

So the ‘bulk free energy density’ is

$$E_b = \frac{E(R)}{V(R)} \sim A\rho^2 R^2 \xrightarrow{R \rightarrow \infty} \infty. \quad (5.1)$$

So a Coulomb-like force is too long-ranged for the thermodynamic limit to exist. More physically, the conclusion (5.1) means (for $A > 0$, repulsive interactions) that the system (with a fixed number of particles or fixed charge) can lower its energy by expanding into a larger volume – it explodes.

But wait: there are Coulomb-like forces in the world, and here we are in the thermodynamic limit. A fatal assumption above is that there was only one sign of the charge. If we allow charge of both signs, we can have the phenomenon of *screening*. Screening makes a microscopically long-range force short-ranged. That last sentence has a lot of RG physics in it, and it’s worth absorbing more. This is an opportunity to say something about “running couplings”.

Screening: mean field theory of Debye and Huckel.

[McComb] We take a uniform background of $+$ charges, fixed in place. (This is sometimes called ‘jellium’.) Their number density is n_∞ . We add to this *mobile* $-$ charges (‘electrons’), with equal average number density.

Suppose we stick in a test $(-)$ charge at the origin. At temperature T , what is the probability of finding an electron a distance r from the test charge? If we knew the electrostatic potential $\phi(r)$, the classical equilibrium probability would be

$$p(r) = Z^{-1} e^{-e\phi(r)/T}.$$

In vacuum, $\phi(r)$ would be $\frac{e}{r}$. We will determine it self-consistently. The electron number density is proportional to the probability $p(r)$, and must approach the average density far away (where $\phi \rightarrow 0$), so

$$n(r) = n_\infty e^{-\beta e\phi(r)}.$$

Now we can impose Gauss' law:

$$\begin{aligned}
-\nabla^2\phi(r) &= 4\pi\rho(r) \\
&= 4\pi e(n(r) - n_\infty) \\
&= 4\pi en_\infty (e^{-\beta e\phi(r)} - 1) \\
&\stackrel{T \gg e\phi}{\simeq} -4\pi\beta e^2 n_\infty \phi(r) + \mathcal{O}(e\beta\phi)^2.
\end{aligned} \tag{5.2}$$

This is just the equation we solved in (4.7) to find the correlation function $G(r)$ away from the critical point, at finite $\xi^{-2} = 4\pi\beta e^2 n_\infty$, and the solution is

$$\phi(r) = \frac{e}{r} e^{-r/\ell_D} \equiv \frac{e_{\text{eff}}(r)}{r}. \tag{5.3}$$

The name of the correlation length in this case is

$$\ell_D \equiv \sqrt{\frac{T}{4\pi e^2 n_\infty}},$$

the Debye screening length. In the second equality in (5.3) I introduced a distance-dependent effective charge $e_{\text{eff}}(r)$: how much charge you see depends how closely you look.

The continuum approximation we've used here is consistent with classical corpuscles if the average interparticle distance is small compared to the screening length:

$$n_\infty^{-1/3} \ll \ell_D$$

which is true when $e^3\sqrt{N} \ll T^{3/2}$, *i.e.* at high enough temperature, consistent with our approximation in (5.2).

You might worry that a collection of charges of both signs, once we let them all move around, might either implode or explode. [This paper by Lieb, called *The Stability of Matter*](#), is very interesting and not too forbidding. The early sections are about the stability of matter to *implosion*, which is a short-distance issue (whose resolution crucially involves quantum mechanics and the Pauli principle and hence is off-limits here); but Section V contains a 'rigorous version of screening' which removes the concern that matter should want to explode like in (5.1).

Other power laws. Suppose instead of Coulomb interactions in $d = 3$, we have particles interacting pairwise via a potential $U(r) = \frac{A}{r^\sigma}$ in d dimensions. Then the energy of a collection of particles with density $\rho(r)$, in a ball of radius R , B_R is

$$E(R) = \frac{1}{2} \int_{B_R} d^d r \int_{B_R} d^d r' \rho(r) U(r - r') \rho(r')$$

$$\begin{aligned}
&\stackrel{\text{uniform } \rho}{\simeq} A \frac{\rho^2}{2} \int_{B_R} d^d r \int_{B_R} d^d r' \frac{1}{|r - r'|^\sigma} \\
&= A \frac{\rho^2}{2} R^{2d-\sigma} C(d, \sigma)
\end{aligned} \tag{5.4}$$

where

$$C(d, \sigma) \equiv \int_{B_1} \frac{d^d x d^d y}{|x - y|^\sigma}$$

In the last step we scaled out the system-size dependence of the integral by defining $r \equiv Rx, r' \equiv Ry$. This C is just a dimensionless number – if it's finite. In that case, the ‘bulk energy density’ (free energy density at $T = 0$) is

$$\varepsilon_{\text{bulk}} \equiv \frac{E(R)}{V(R)} = \frac{R^{2d-\sigma} A \rho^2 C/2}{R^d V_d} \sim R^{d-\sigma}$$

which is finite as $R \rightarrow \infty$ (the would-be thermodynamic limit) if $\sigma > d$. (V_d is the volume of the unit d -ball.) So $\sigma > d$ is a sufficiently fast falloff of the interactions to allow for a thermodynamic limit.

When is $C(d, \sigma)$ finite? What I really mean by this is: the power law form of $U(r) \sim r^{-\sigma}$ surely only holds for $r \gg a$, some UV cutoff – for example the size of the particles. The real question is: when can we ignore this approximation for purposes of computing C ? Changing integration variables to $u \equiv x - y, v = x + y$,

$$C = V_d \Omega_{d-1} \int_{a/R}^1 \frac{u^{d-1} du}{u^\sigma} \begin{cases} = \frac{V_d \Omega_{d-1}}{d-\sigma} \left(1 - \left(\frac{a}{R}\right)^{d-\sigma}\right), & d \neq \sigma \\ \propto \log \frac{R}{a}, & d = \sigma \end{cases}.$$

5.3 Convexity of the free energy

We're going to prove some facts about the nearest-neighbor Ising model, with Hamiltonian

$$H(s) = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i. \tag{5.5}$$

Many of them are true more generally.

(1) With the additive normalization in (5.5), the bulk free energy density is negative:

$$f < 0.$$

This statement is sensitive to the additive normalization of H – if we add a big positive constant to H , we can change this fact. The normalization (5.5) is special because there is no constant term, in the sense that

$$\sum_s H(s) = 0 \tag{5.6}$$

– it is normal-ordered.

Proof of (1): Begin with $N < \infty$ sites. The free energy density is $f = F/N = -\frac{T}{N} \log Z$, so the claim $f < 0$ means $Z > 1$. The partition function $Z = \sum_s e^{-\beta H(s)}$ is a sum of 2^N positive terms (for $0 < T < \infty$). And $Z > 1$ because there exists a configuration s^* which by itself contributes a term $e^{-\beta H(s^*)} > 1$. For example, for $J > 0, h > 0$, it happens when $s_i^* = 1, \forall i$. But more generally, it follows from the normal-ordering condition (5.6) since $H(s)$ is not identically zero, so there must be configurations with both signs of $H(s)$, and at least one which has $H(s^*) < 0$. ■

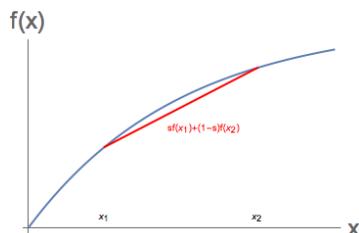
(2) The entropy density is

$$s = -\partial_T f \geq 0.$$

The proof of this statement follows from the identity

$$-\partial_T F = -\sum_s \rho_\beta(s) \log \rho_\beta(s)$$

where $\rho_\beta(s) \equiv e^{-\beta H(s)} Z^{-1}$ is the equilibrium probability distribution at temperature T . Since $0 \leq \rho \leq 1$, this is a sum of positive terms. ■

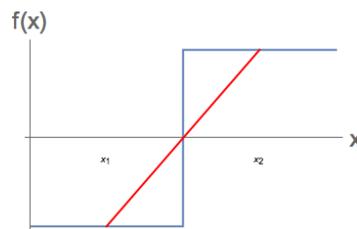


Here is a **Definition**: A function $f(x)$ is convex in x if

$$f(sx_1 + (1-s)x_2) \geq sf(x_1) + (1-s)f(x_2), \quad \forall s \in [0, 1];$$

that is, if the graph of the function is above any chord. If f is convex then I'll call $-f$ anti-convex.

Convex implies continuous, as you can see from the picture at right of a function which jumps. If f is convex, $\partial_x f$ is non-increasing (and in particular the derivative exists almost everywhere). f can have cusps.



(3) $f(\beta, h, J)$ is convex in its arguments.

Proof of (3): The proof relies on a Hölder inequality. Given two sequences $\{g_k\}, \{h_k\}$ of non-negative numbers $g_k, h_k \geq 0$,

$$\sum_k (g_k)^s (h_k)^{1-s} \leq \left(\sum_k g_k \right)^s \left(\sum_k h_k \right)^{1-s}.$$

This (I learned on wikipedia) follows from the convexity of the logarithm¹⁴.

Here's the idea for $f(\beta)$:

$$\begin{aligned} Z(s\beta_1 + (1-s)\beta_2) &= \sum_s e^{-s\beta_1 H(s)} e^{-(1-s)\beta_2 H(s)} \\ &\stackrel{\text{Hölder}}{\leq} \left(\sum_s e^{-\beta_1 H(s)} \right)^s \left(\sum_s e^{-\beta_2 H(s)} \right)^{1-s} \\ &= Z(\beta_1)^s Z(\beta_2)^{1-s}. \end{aligned}$$

Taking $-\frac{T}{N} \log(\text{BHS})$ to get f , the minus sign reverses the inequality:

$$f(s\beta_1 + (1-s)\beta_2) \geq sf(\beta_1) + (1-s)f(\beta_2).$$

The limit as $N \rightarrow \infty$ of a family of convex functions is also convex. ■

[\[End of Lecture 12\]](#)

A useful alternative viewpoint: convexity follows by showing that all second derivatives of f are negative. For example,

$$\partial_\beta^2 f = -\frac{1}{\beta N} \langle (H - \langle H \rangle)^2 \rangle \leq 0$$

is proportional to minus the specific heat, aka the variance of the energy. Similar statements hold for other variations, such as the magnetic susceptibility

$$\partial_h^2 f = -c \langle (s - \langle s \rangle)^2 \rangle \leq 0.$$

So the condition of convexity is related to the stability of the equilibrium.

¹⁴

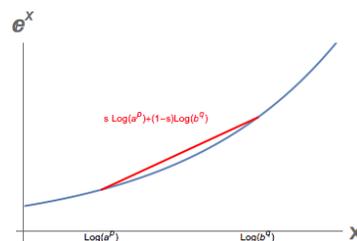
Here's the idea:

$$\begin{aligned} ab &= e^{\log ab} = e^{\frac{1}{p} \log a^p + \frac{1}{q} \log b^q} \\ &\leq \frac{1}{p} e^{\log a^p} + \frac{1}{q} e^{\log b^q} = \frac{a^p}{p} + \frac{b^q}{q} \end{aligned}$$

for $\frac{1}{p} + \frac{1}{q} = 1$, where we used the fact that e^x is anticonvex, as illustrated at right. Apply this inequality with

$$a = \left(\frac{g_k}{\sum g} \right)^{\frac{1}{p} \equiv s}, \quad b = \left(\frac{h_k}{\sum h} \right)^{\frac{1}{q} \equiv 1-s}$$

and sum the BHS over k .



5.4 Spontaneous symmetry breaking

[Goldenfeld p. 56, Parisi p. 15]

Orders of limits. I made a big deal earlier (in §3) about the impossibility of spontaneous symmetry breaking (SSB) in finite volume. There is more to say about this. What does the free energy density (in the thermodynamic limit) look like as a function of h near $h = 0$? It must be

$$f(h) = f(0) - m_s|h| + \mathcal{O}(h^{\sigma>1})$$

so that the magnetization is

$$m = -\partial_h f = \begin{cases} m_s + \mathcal{O}(h^{\sigma-1}), & h > 0, \\ -m_s + \mathcal{O}(h^{\sigma-1}), & h < 0 \end{cases}.$$

(If σ were not larger than one, the magnetization would diverge as $h \rightarrow 0$ and that's not happening, since it's bounded. I also imposed $f(h) = f(-h)$ by Ising symmetry.)

But before the thermodynamic limit, $f(h)$ is a smooth function. This means the two limits $h \rightarrow 0, N \rightarrow \infty$ are clashing violently:

$$\lim_{N \rightarrow \infty} \lim_{h \rightarrow 0} \frac{1}{N} \partial_h F = 0 \quad \text{but} \quad \lim_{h \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \partial_h F = \pm m_s.$$

Yang-Lee singularities. Here is a toy model of how this can come about. Suppose our system of volume V is so tightly bound that only two configurations matter, the one where all N spins point up, $m = +V$, and the one where they all point down, $m = -V$. (All the rest of the configurations have such a large energy that we can ignore their contributions to Z .) So a single spin $s = \pm 1$ determines the whole configuration.

Then, in a field, we have

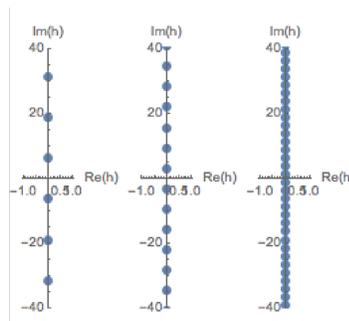
$$Z(h) = \sum_{s=\pm 1} e^{-\beta h V s} = 2 \cosh \beta h V$$

and

$$f(h) = -\frac{T}{V} \log(2 \cosh \beta h V), \quad m(h) = \partial_h f = \tanh \beta h V \xrightarrow{V \rightarrow \infty} m(h) = \text{sign}(h).$$

Where in h is the free energy $F \propto \log(Z)$ singular? When $Z(h) = 0$. For $V < \infty$, the zeros of the partition function happen at pairs of imaginary values of h

$$Z(h_n = 0) \quad \text{at} \quad h_n = \frac{(2n+1)\pi i}{2\beta V}$$



which in the thermodynamic limit $V \rightarrow \infty$ accumulate and pinch the real axis. (They are shown for $\beta V = 1, 2, 5$ in the figure at right.) These zeros are named after Yang and Lee.

Ergodicity breaking. There are many points of view from which SSB seems paradoxical. For example if the equilibrium probability density is

$$p_0(s) = Z^{-1} e^{-\beta H(s)}$$

then the Ising symmetry $H(s) = H(-s)$ implies directly that the magnetization vanishes:

$$m = \langle s \rangle \stackrel{?}{=} \langle s \rangle_0 \equiv \sum_s P_0(s) s = 0.$$

What gives? Consider, at small $h > 0$ and finite N , the ratio of the probabilities of two configurations: a reference configuration s , and the one related to it by a global spin reversal. If $m(s) \equiv \frac{1}{N} \sum_i s_i$ is the magnetization in this configuration, then

$$\frac{p(s)}{p(-s)} = \frac{e^{-\beta(-hNm(s))}}{e^{+\beta(hNm(s))}} = e^{2\beta hNm(s)} \xrightarrow{N \rightarrow \infty, h > 0} \infty.$$

In the thermodynamic limit, if $h \neq 0$ one of these configurations is infinitely more probable than the other! This is true no matter how small h is, even if $h = 0^+$. If we reverse the sign of h , the other configuration wins. We've learned that

$$\lim_{h \rightarrow 0^\pm} \lim_{N \rightarrow \infty} p_0(s) \equiv p_\pm(s)$$

is a different, restricted ensemble compared to the symmetric distribution p_0 . It is restricted in the sense that $p_+(s|m(s) < 0) = 0$ – states with the wrong sign of magnetization have no weight. So in this limit, our distribution only samples a subset of the configuration space – it is not *ergodic*. This is a toy example of ergodicity breaking, a concept which is much more useful in the physics of glassy systems. Very roughly, from the usual point of view of ergodicity as underlying statistical mechanics, in terms of time evolution, the claim is that starting from an initial configuration, the probability of evolving to a configuration with the opposite sign of the magnetization goes like $e^{-\beta \Delta F}$ where the change in free energy is $\Delta F \sim N^{\sigma > 0} \xrightarrow{N \rightarrow \infty} \infty$. So we are also claiming that SSB means that the $N \rightarrow \infty$ limit and the $t \rightarrow \infty$ limit do not commute. (If we take $t \rightarrow \infty$ first, the system will explore all the configurations.)

Cluster decomposition failure. If we prepare the system in an initial configuration with a mixture of \pm (or average over possible initial conditions with the appropriate weight), as

$$p_q(s) = qp_+(s) + (1 - q)p_-(s), \quad q \in [0, 1]$$

then our expectation for the connected correlations are

$$\langle s_i s_j \rangle_q^c \underset{|r_i - r_j| \rightarrow \infty}{\equiv} \langle s_i s_j \rangle_q - \langle s_i \rangle_q \langle s_j \rangle_q = m^2 - ((2q - 1)^2 m^2 = 4q(1 - q)m^2 \neq 0. \quad (5.7)$$

They don't vanish for arbitrarily-separated points!¹⁵ The demand that they should is called the *cluster decomposition principle*; it is an implementation of the notion of 'locality'. SSB means that cluster decomposition fails for the symmetric distribution. Only the (non-symmetric) 'pure states' with $q = 0, 1$ satisfy this demand (this is the definition of 'pure state' in this context).

5.5 Phase coexistence

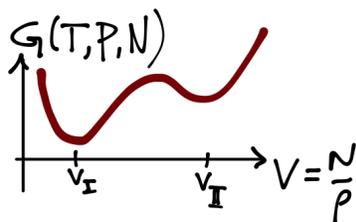
[Goldenfeld, §4] First, let's recall some thermodynamics facts. I will speak in the language of fluids, but with appropriate substitutions of letters, it can be translated into physics of magnets or other examples. At fixed volume, the free energy which is minimized in equilibrium is the Hemholtz one (the one we've been talking about), $F(T, V, N) = E - TS$. If instead we fix the pressure P , the quantity which is minimized in equilibrium is the Legendre transform of F , named for Gibbs:

$$G(T, P, N) = F + PV,$$

in terms of which the first law of thermodynamics is

$$dG = -SdT + VdP + \mu dN.$$

The Gibbs-Duhem relation (basically, integrating the first law) says $E = -PV + TS + \mu N$, so that in fact $G = \mu N$ is just proportional to the chemical potential.



Let's consider a situation at fixed P where there is a first order transition, between two phases I, II (for example, liquid and gas) where the order parameter is the volume, or the density (equivalently at fixed N , since $V = N/\rho$). Along the phase boundary, where they exchange dominance, we must have

$$G_I = G_{II}. \quad (5.8)$$

Hence also $\mu_I = \mu_{II}$; this is a condition for chemical equilibrium of the two phases.

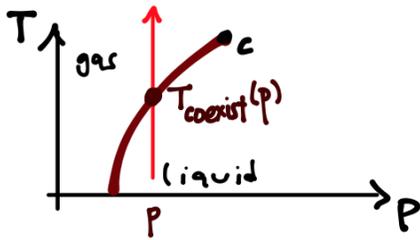
¹⁵For some intuition for the sense in which arbitrarily-separated points are correlated in these ensembles, see the homework.

Moving along the phase boundary, the condition (5.8) says

$$\underbrace{G_I(T + dT, P + dP, N)}_{=dT \underbrace{\partial_T G_I|_P}_{-S_I} + dP \underbrace{\partial_P G_I|_P}_{V_I}} = G_{II}(T + dT, P + dP, N)$$

and therefore we get the Clausius-Clapeyron equation for the slope of the coexistence curve

$$\left. \frac{dP}{dT} \right|_{\text{coexistence}} = \frac{S_I - S_{II}}{V_I - V_{II}}.$$

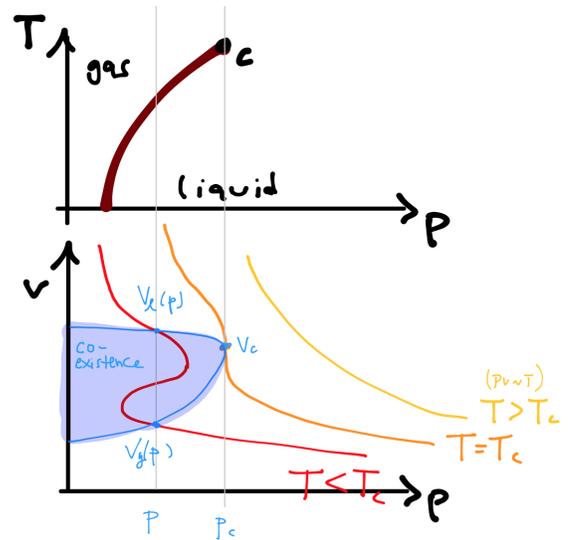


The difference in the numerator is proportional to the latent heat of the transition, $T\Delta S = T(S_I - S_{II})$. If phases I and II are not somehow topologically distinguished (for example, by a different symmetry-breaking pattern), then there can be a *critical endpoint* of the line of first-order transitions, where $\Delta S \rightarrow 0, \Delta V \rightarrow 0$, at some (T_c, P_c) .

The consequence of a first-order transition depends on what is held fixed as the transition is traversed. If we heat a fluid at constant pressure $P < P_c$ (for example atmospheric pressure), starting from $T < T_c$ (moving along the red vertical line in the figure, and doing so slowly enough that we stay in the equilibrium phase diagram at every step) then first the fluid expands and warms up. When it reaches the coexistence curve $T_{\text{coexistence}}(P)$, it *starts* to boil. While this happens, the energy goes into the latent heat converting I into II, and the temperature stays fixed: we are sitting at the point $(T_{\text{coexistence}}(P), P)$ on the coexistence curve in the (P, T) phase diagram, while the fraction x of the fluid which is gas grows:

$$V = xV_l + (1 - x)V_g, \quad x = x(t)$$

is some protocol-dependent function. Although $V_l \neq V_g$, the volume fluid itself does not jump. How do I know this? Bear with me a moment, the proof is at Eq. (5.9).



If instead we compress the fluid at constant T , starting at $T > T_c$ in the gas phase:

$$-\frac{1}{V}\partial_V P|_T \equiv \kappa_T > 0$$

a positive compressibility says that it fights back. It fights back until the volume reaches $V = V_l(T)$, which is when $P = P_{\text{coexistence}}(T)$, beyond which the fluid starts to condense.

What do these isothermal curves look like? Let $v = V/N = 1/\rho$ be the volume fraction per particle. For an ideal gas, recall that $Pv = T$. This is correct at high temperature. For lower temperatures, van der Waals suggests some appealing simple corrections which account for an interparticle interaction described by a potential like we discussed in §3.7:

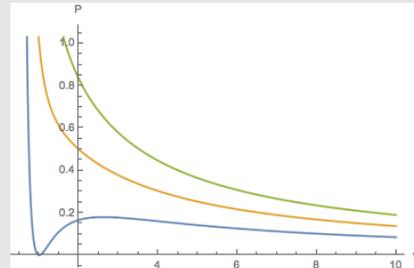
- each particle wants some amount of personal space, and therefore excludes some fixed volume b : $v \rightarrow v - b$.
- the energy per particle is decreased by the long-range attractive part of the potential by an amount proportional to the density:

$$\frac{E}{N} \rightarrow \frac{E}{N} - a\rho \quad \Longrightarrow \quad P = \partial_V F \rightarrow P - \frac{a}{v^2}.$$

So the vdW equation of state is

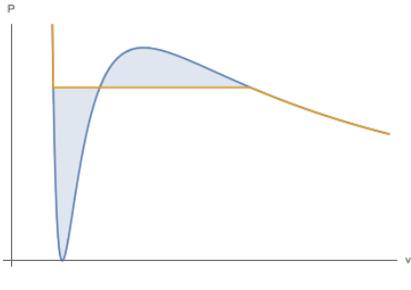
$$P = \frac{T}{v - b} - \frac{a}{v^2}$$

for some constants a, b (in the plot at right we see $a = 2, b = 5$ and $T = 1, 1.5, 2$ from bottom to top). This has two nice new features for our present purposes:



- It has a critical $T = T_c$ below which there is a line of first order phase transitions. The critical point appears when $P(v) = \text{const}$ goes from having one solution ($T > T_c$, like the ideal gas), to having three. When this happens, $\partial_v P = \partial_v^2 P = 0$, so that locally $P \sim (v_c - v)^3$ is locally cubic. In fact, for the vdW equation of state, this condition is exactly a cubic equation for v : $P_0 v^3 - v^2(bP_0 + T) + av - ab = 0$.
- (Relatedly), it has regions where $\kappa_T = -\frac{1}{V}\partial_V P|_T < 0$ which says that if you try to squeeze it, it doesn't fight back, but rather tries to help you squeeze it further. Creepy! (The same thing happened in our study of the Landua-Ginzburg free energy in §4.1 and this led to the picture of the swallowtail.)

- Note by the way that the vdW equation is a masterpiece of estimation: a, b can be determined from high-temperature data and they give a (not bad) estimate of the location of the critical point.



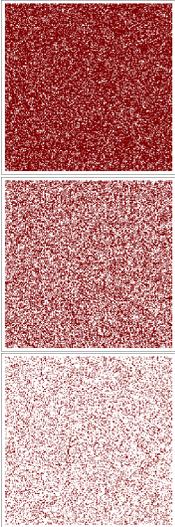
What is the free energy doing while this is going on? At coexistence, in equilibrium, $\mu_l = \mu_g$, and the first law says

$$d\mu = -\frac{S}{N}dT + \frac{V}{N}dP$$

so

$$0 = \mu_l - \mu_g = \int_{\text{gas}}^{\text{liquid}} d\mu \stackrel{\text{isotherm}}{=} \int_{\text{gas}}^{\text{liquid}} \frac{V(P)}{N} dP \quad (5.9)$$

so the area under the $V(P)$ curve is zero (and is the change in the Gibbs free energy), along any path in equilibrium. This is true even for infinitesimal paths. Therefore, the actual equilibrium trajectory of the free energy is a straight line between V_g and V_p . This is the Maxwell construction. It saves the convexity of the free energy.



The creepy self-squeezing regions of the equation-of-state curve are exactly the ones which are removed by the phase-coexistence region.

At left here, I've made some pictures where a decreasing fraction of the dots are colored red, in an attempt to depict the history of the volume fraction of one phase in the other as the coexistence region is traversed. What's wrong with this picture? How could you make it more realistic?

Notice that we are making a strong demand of equilibrium here, effectively taking $t \rightarrow \infty$ before $N \rightarrow \infty$. This failure of commutativity of these limits is the same issue as in our discussion of ergodicity-breaking above.

[End of Lecture 13]

6 Field Theory

Now we are going to try to see where Landau and Ginzburg could possibly have gone wrong near the critical point.

Here is a hint, from experiment. The hard thing about the critical point, which mean field theory misses, is that fluctuations at all scales are important. I know this because I've seen it, *e.g.* [here](#) and (with better soundtrack) [here](#). *Critical opalescence* is a phenomenon whereby a two-fluid mixture which is otherwise transparent becomes opaque at a continuous phase transition. It is explained by the scattering of light by the density fluctuations *at all scales*, at least at all the wavelengths in the visible spectrum. These are the fluctuations we're leaving out in mean field theory.

By the way, what exactly *is* the LG free energy? It is not convex in m , so how can it be the actual free energy?

[Goldenfeld §5.6] The answer to this is that it is the free energy with the constraint that the (coarse-grained) magnetization is fixed to be $m(r)$:

$$e^{-\beta S_{\text{LG}}[m]} \equiv \sum_s e^{-\beta H(s)} \prod_{\text{blocks}, r} \delta \left(\sum_{i \in r} s_i - m(r) N_{\Lambda}(r) \right).$$

Here r denotes a block, and $N_{\Lambda}(r)$ is the number of sites in the block r . This is just like the construction of the block-spin effective Hamiltonian. It is only more ambitious in that we are hoping that $m(r)$ is smoothly varying in r , which will be true if $\xi > a$. So the LG free energy S can be regarded as (a parametrization of) the coarse-grained free energy.

At this point I want to remind you about the derivation of field theory that you made for HW4. There, you studied the *Legendre transform* of the free energy $F[h]$ at fixed field:

$$S[m] = F[h] - \sum_r m_r h_r |_{m = +\partial_h F}.$$

In the thermodynamic limit, I claim that this is the same thing. You get to think about this for homework 5.

It's easy to get confused about Legendre transforms and all that stuff, so it's very helpful to appeal to a simpler narrative of the origin of field theory, by exploiting universality. Recall at the beginning of our discussion of Ising models in §3, I mentioned the many avatars of the Ising model. One I mentioned arose by consider a real-valued variable ϕ_x at each point in space (or on some lattice).

That is: suppose we replace each spin s_x by such a real variable, a factor in whose probability distribution is

$$p_0(\phi_x) \propto e^{-\beta V(\phi_x)} \quad (6.1)$$

where $V(\phi) \sim g(\phi^2 - 1)^2$ for large g . This probability distribution is basically zero unless $\phi = \pm 1$, so this is no change at all if g is big enough. Important piece of foreshadowing: we are going to see that a large g at the lattice scale is not at all the same as a large $g\phi^4$ term in the coarse-grained action.

So we replace

$$\sum_s \dots \equiv \prod_x \sum_{s_x = \pm 1} \dots \rightsquigarrow \int \prod_x d\phi_x p_0(\phi_x) \dots \equiv \int D\phi e^{-\beta \int_x V(\phi(x))} \dots$$

The nearest-neighbor ferromagnetic Ising Hamiltonian becomes (up to an additive constant, using $s^2 = 1$)

$$-J \sum_x \sum_{\mu=1}^d (s_{x+\mu} s_x - 1) = \frac{1}{2} J \sum_x \sum_{\mu=1}^d (s_{x+\mu} - s_x)^2 \rightsquigarrow \frac{1}{2} J \underbrace{\sum_x}_{\simeq a^{-d} \int d^d x} \sum_{\mu=1}^d \underbrace{(\phi_{x+\hat{\mu}} - \phi_x)^2}_{\simeq a^{-1} \partial_\mu \phi}.$$

That is: the ferromagnetic coupling makes the nearby spins want to agree, so it adds a term to the energy which grows when the nearby ϕ_x s disagree.

Altogether, we are going to replace the Ising partition function with

$$Z = \sum_s e^{-\beta H(s)} \rightsquigarrow \int [D\phi] e^{-\int d^D x \mathcal{L}(\phi)}$$

where (I am calling the LG free energy density \mathcal{L} for ‘Landau’ or for ‘Lagrangian’.)

$$\mathcal{L}(\phi) = \frac{1}{2} \gamma (\nabla \phi)^2 + \frac{1}{2} r \phi^2 + \frac{g}{4!} \phi^4 + h \phi + \dots$$

Our hope is that the operation \rightsquigarrow does not take us out of the basin of attraction of the Ising critical point. The constants γ, r, g are related in some way (roughly determinable but not the point here) to the microscopic parameters. For some physical situations (such as high energy particle physics!) this is a better starting point than the lattice model. There is some coarse-graining involved in the \rightsquigarrow operation, and therefore the dependence of γ, r, g on β needn’t be linear, but it should be analytic. After all, the miraculous phenomenon we are trying to understand is how physics can be non-analytic in T at some finite value of T ; we don’t want to assume the answer.

6.1 Beyond mean field theory

[Brezin, §9] So we want to understand the integral

$$Z \equiv \int [D\phi] e^{-S[\phi]}. \quad (6.2)$$

Mean field theory arises by making a saddle point approximation: find m which minimizes $S[\phi]$, $0 = \left. \frac{\delta S}{\delta \phi} \right|_{\phi=m}$, and make a (functional) Taylor of the exponent about the minimum:

$$\begin{aligned} Z &= \int [D\phi] e^{-S[\phi=m+\varphi]} \\ &= \int [D\varphi] e^{-\left(S[m] + \frac{\delta S}{\delta \phi_x} \Big|_{\phi=m} \varphi_x + \frac{1}{2} \frac{\delta^2 S}{\delta \phi_x \delta \phi_y} \Big|_{\phi=m} \varphi_x \varphi_y + \dots \right)} \end{aligned} \quad (6.3)$$

In the second line I used the fact that the change of variables $\phi = m + \varphi$ has unit Jacobian. I also used a matrix notation, where the position indices x, y are repeated indices, and hence are summed. The saddle point condition means that the term in the exponent linear in φ_x vanishes.

The mean field theory answer is just $Z_0 = e^{-S[m]}$. The first correction to mean field theory comes by keeping the quadratic term and doing the gaussian integral:

$$Z_1 = Z_0 \int [D\varphi] e^{-\frac{1}{2} \int_x \int_y K_{xy} \varphi_x \varphi_y}$$

where the quadratic kernel K is

$$K_{xy} \equiv \left. \frac{\delta^2 S}{\delta \phi_x \delta \phi_y} \right|_{\phi=m} = \left(r + \frac{g}{2} m^2 - \nabla^2 \right) \delta^d(x - y).$$

Notice that it depends on the *background field* m .

How do we do the (gaussian!) φ integral?

$$\int \prod d\varphi_x e^{-\frac{1}{2} \varphi_x K_{xy} \varphi_y} = \frac{1}{\sqrt{\det K}} = C \prod_{\lambda} \lambda^{-1/2} = e^{-\frac{1}{2} \sum_{\lambda} \log \frac{\lambda}{\lambda_0}}$$

where λ are the eigenvalues of K :

$$K_{xy} u_y^{(\lambda)} = \lambda u_x^{(\lambda)}. \quad (6.4)$$

I absorbed the constant C into the $-\log \lambda_0$ which we can choose to our advantage. So the leading correction to the mean-field free energy gives

$$F^{(1)}[h] = F_{\text{MF}}[h] + \frac{1}{2} \sum_{\lambda} \log \frac{\lambda}{\lambda_0}.$$

Who are the eigenvalues of the kinetic operator K ? If h and hence m are constant, the problem is translation invariant, and they are plane waves, $u_q(x) = \frac{1}{\sqrt{V}} e^{i\vec{q}\cdot\vec{x}}$ – the eigenvalue equation (6.4) is

$$\int_y \delta(x-y) \left(r + \frac{g}{2} m^2 - \nabla^2 \right) u_q(y) = \underbrace{\left(r + \frac{g}{2} m^2 + q^2 \right)}_{=\lambda_q} u_q(x).$$

Therefore, The free energy is

$$F^{(1)}[h] = F_{\text{MF}}[h] + \frac{1}{2} V \int \mathrm{d}^d q \log \left(\frac{r + \frac{g}{2} m^2(h) + q^2}{r + q^2} \right)$$

where I made a choice of λ_0 to be $\lambda(m=0)$.

Making the Legendre transform (a little tricky, and requiring us to ignore terms of the same size as the corrections to the first order approximation), we have $\Gamma[m] = V\gamma(m)$ with the answer to this order

$$\gamma^{(1)} = \frac{1}{2} r m^2 + \frac{g}{4!} m^4 + \frac{1}{2} \int \mathrm{d}^d q \log \left(\frac{r + \frac{g}{2} m^2 + q^2}{r + q^2} \right). \quad (6.5)$$

So what? First let's use this to recover the Ginzburg criterion. The susceptibility, at $h=0$, for $T > T_c$ is $\chi = \partial_h m|_{h=0}$ which (as you'll remind yourself on the homework) is related to the curvature of the effective potential γ by

$$\frac{1}{\chi}|_{m=0} = \partial_m^2 \gamma|_{m=0} = r + \frac{g}{2} \int \mathrm{d}^d q \frac{1}{q^2 + r}.$$

The phase transition happens when the correlation length goes to infinity; we showed by the susceptibility sum rule (4.6) that $\xi \rightarrow \infty$ is required by $\chi \rightarrow \infty$. So, while in mean field theory the critical point occurs when $r \rightarrow 0$, the fluctuation corrections we've just derived shift the location of the critical point to

$$0 \stackrel{!}{=} \chi^{-1}(T_c) = r(T_c) + \frac{g}{2} \int \mathrm{d}^d q \frac{1}{q^2 + r(T_c)}.$$

You'll show on the homework that we can eliminate the (annoying, non-universal anyway) parameter r from the discussion and relate the susceptibility near the transition to the non-dimensionalized temperature $t = \frac{T-T_c}{T_c}$:

$$\frac{1}{\chi} = c_1 t \left(1 - \frac{g}{4} \int \mathrm{d}^d q \frac{1}{q^2 (q^2 + r)} \right).$$

for some constant c_1 . Everywhere here we are ignoring terms which are as small as the corrections to the gaussian approximation. Since if g were zero, the integral would

be exactly gaussian (ignoring even higher order terms for now), the corrections must come with powers of g .

When is the correction to MFT actually small? The shift in the critical point is of order $gG(0) = g \int d^d q \frac{1}{q^2(q^2+t)} + \text{const}$, which is the same quantity we found in our earlier discussion of the Ginzburg criterion for short-ranged interactions. As $t \rightarrow 0$, the integral (the $q \rightarrow 0$ limit of the integral) is finite for $d > 4$, but for $d \leq 4$ it blows up at $t \rightarrow 0$. More specifically, the corrections to MFT are small when $gt^{\frac{d-4}{2}}$ is small. This determines the *size of the critical region*.

Now wait a minute: when we introduced the coupling g (at (6.1)) we said it had to be big to give a good approximation to the Ising spins, but now I'm using an approximation relying on small g . What gives? The answer is that coarse-graining makes g shrink. Here we go:

6.2 Momentum shells

[Zee, §VI.8 (page 362 of 2d Ed.) Some of you will notice that this discussion has some overlap with §3.2 of [my Physics 215C lecture notes](#). One tip-off is that I am calling the magnetization field ϕ .]

The continuum functional integral I've written in (6.2) is defined (to the extent that it is) by taking a limit where the lattice spacing goes to zero as the number of sites goes to infinity. This limit is dangerous and is the origin of the bad reputation of the subject of quantum field theory. In its application to the lattice Ising model, this isn't a real problem, because the lattice spacing is a real thing. It provides an *ultraviolet (UV) cutoff* on our field theory. To remind ourselves of this let me decorate our expressions a little bit:

$$Z_\Lambda \equiv \int_\Lambda [D\phi] e^{-\int d^d x \mathcal{L}(\phi)}. \quad (6.6)$$

Here the specification \int_Λ says that we integrate over field configurations $\phi(x) = \int d^d k e^{ikx} \phi_k$ such that $\phi_k = 0$ for $|k| \equiv \sqrt{\sum_{i=1}^D k_i^2} > \Lambda$. Think of $2\pi/\Lambda$ as the lattice spacing¹⁶ – there just aren't modes of shorter wavelength.

We want to understand (6.6) by some coarse-graining procedure. Let us imitate the block spin procedure. Field variations *within* blocks of space of linear size na have wavenumbers greater than $\frac{2\pi}{na}$. (These modes average to zero on larger blocks; modes with larger wavenumber encode the variation *between* these blocks.) So the analog

¹⁶This cutoff is not precisely the same as have a lattice; with a lattice, the momentum space is periodic: $e^{ikx_n} = e^{ik(na)} = e^{i(k+\frac{2\pi}{a})(na)}$ for $n \in \mathbb{Z}$. Morally it is the same.

of the partition function after a single blocking step is the following: Break up the configurations into pieces:

$$\phi(x) = \int \mathrm{d}k e^{ikx} \phi_k \equiv \phi^< + \phi^> .$$

Here $\phi^<$ has nonzero fourier components only for $|k| \leq \Lambda/b$ for some $b > 1$ and $\phi^>$ has nonzero fourier components only in the shell $\Lambda/b \leq |k| \leq \Lambda$. These two parts of the field could be called respectively ‘slow’ and ‘fast’, or ‘light’ and ‘heavy’, or ‘smooth’ and ‘wiggly’. We want to do the integral over the heavy/wiggly/fast modes to develop an effective action for the light/smooth/slow modes:

$$Z_\Lambda = \int_{\Lambda/b} [D\phi^<] e^{-\int d^D x \mathcal{L}(\phi^<)} \int [D\phi^>] e^{-\int d^d x \mathcal{L}_1(\phi^<, \phi^>)}$$

where \mathcal{L}_1 contains all the dependence on $\phi^>$ (and no other terms).

[End of Lecture 14]

6.3 Gaussian fixed point

In the special case where the action is quadratic in ϕ , not only can we do the integrals, but the quadratic action is form-invariant under our coarse-graining procedure.

Consider

$$S_0[\phi] = \int dx \frac{1}{2} \phi(x) (r_0 - r_2 \partial^2) \phi(x) = \int_0^\Lambda \mathrm{d}^d k \phi(k) \phi(-k) \frac{1}{2} (r_0 + r_2 k^2) .$$

The coefficient r_2 of the kinetic term (I called it γ earlier) is a book-keeping device that we may set to 1 by rescaling the field variable ϕ if we choose. Why set this particular to one? One good reason is that then our coarse-graining scheme will map Ising models to Ising models, in the sense that the kinetic term is the continuum representation of the near-neighbor Ising interaction $J \sum_{\langle ij \rangle} s_i s_j$.

We can add a source $\sum_q^\Lambda h_q \phi_q$ to compute

$$Z[h] = \left\langle e^{-\sum_q^\Lambda h_q \phi_q} \right\rangle = Z[0] e^{-\frac{1}{2} \sum_q \frac{\eta_q \eta_{-q}}{q^2 + r}}$$

and

$$\langle \phi_q \phi_{q'} \rangle = \frac{1}{Z} \frac{\partial}{\partial h_{-q}} \frac{\partial}{\partial h_{-q'}} Z[h] |_{h=0} = \frac{1}{q^2 + r} \delta_{q+q'} \equiv G(q) \delta_{q+q'} .$$

We can relate the parameter r to a physical quantity by our friend the susceptibility sum rule:

$$\chi = \int d^d x G(x) = G(q=0) = \frac{1}{r}.$$

Here's what I mean by form invariant: because S_0 does not mix modes of different wavenumber, the integrals over the fast and slow modes simply factorize:

$$e^{-S_{\text{eff}}[\phi^<]} = \int [D\phi^>] e^{-S_0[\phi^>] - S_0[\phi^<]} = Z_{>} e^{-S_0[\phi^<]}$$

– the effective action for the slow modes doesn't change at all, except that the cutoff changes by $\Lambda \rightarrow \Lambda/b$. To make the two systems comparable, we do a change of rulers:

$$\Lambda' \equiv b\Lambda, \phi'_q \equiv b^{\frac{d-2}{2}} \phi_{bq}$$

so that

$$S_{\text{eff}} = \int_0^\Lambda dq \phi'_q \phi'_{-q} \frac{1}{2} (q^2 + r')$$

where $r' = b^2 r$.

What we just showed is that this RG we've constructed maps the quadratic action to itself. There are two fixed points, $r_0 = \infty$ and $r_0 = 0$. The former is the high-temperature disordered state. The parameter r_0 is relevant and grows as we iterate the RG. No other terms (besides a constant) are generated.

This is the same calculation we did of the random walk, the very first calculation we did, with a lot more labels! The linear term in ϕ (the external magnetic field here) would be relevant, just like the bias term in the random walk that we introduced in §2.1. It is forbidden by the Ising symmetry.

Following the general RG strategy, once we find a fixed point, we must study the neighborhood of the fixed point.

6.4 Perturbations of the Gaussian model

Just as with the spin sums, the integrals are hard to actually do, except in a gaussian theory. But again we don't need to do them to understand the form of the result. We use it to make an RG. As usual there are two steps: coarse-graining and rescaling.

First give it a name:

$$e^{-\int d^D x \delta L(\phi^<)} \equiv \int [D\phi^>] e^{-\int d^d x \mathcal{L}_1(\phi^<, \phi^>)} \quad (6.7)$$

so once we've done the integral we'll find

$$Z_\Lambda = \int_{\Lambda/b} [D\phi^<] e^{-\int d^d x (\mathcal{L}(\phi^<) + \delta\mathcal{L}(\phi^<))} . \quad (6.8)$$

To get a feeling for the form of $\delta\mathcal{L}$ let's parametrize the LG integrand:

$$\mathcal{L} = \frac{1}{2}\gamma(\partial\phi)^2 + \sum_n g_n \phi^n + \dots \quad (6.9)$$

where we include *all* possible terms consistent with the symmetries ($\phi^{\lessgtr} \rightarrow -\phi^{\lessgtr}, h \rightarrow -h$, rotation invariance¹⁷). Then we can find an explicit expression for \mathcal{L}_1 :

$$\int d^d x \mathcal{L}_1(\phi^<, \phi^>) = \int d^d x \left(\frac{1}{2}\gamma(\partial\phi^>)^2 + \frac{1}{2}m^2(\phi^>)^2 + (\phi^>)^3 \phi^< + \dots \right)$$

(I write the integral so that I can ignore terms that integrate to zero, such as $\partial\phi^<\partial\phi^>.$) This is the action for a scalar field $\phi^>$ interacting with itself and with a (slowly-varying) background field $\phi^<$. But what can the result $\delta\mathcal{L}$ be but something of the form (6.9) again, with different coefficients?¹⁸ The result is to shift the couplings $g_n \rightarrow g_n + \delta g_n$. (This includes the coefficient of the kinetic term and also of the higher-derivative terms which are hidden in the ... in (6.9). You will see in a moment the logic behind which terms I hid.)

Finally, so that we can compare steps of the procedure to each other, we rescale our rulers. We'd like to change units so that $\int_{\Lambda/b}$ is a \int_Λ with different couplings; We accomplish this by changing variables: $k' = bk$ so now $|k'| < \Lambda$. So $x = x'b, \partial' \equiv \partial/\partial x' = b\partial_x$ and the Fourier kernel is preserved $e^{ikx} = e^{ik'x'}$. Plug this into the action

$$\int d^d x \mathcal{L}(\phi^<) = \int d^d x' b^d \left(\frac{1}{2}b^{-2}(\partial'\phi^<)^2 + \sum_n (g_n + \delta g_n)(\phi^<)^n + \dots \right)$$

We can make this look like \mathcal{L} again by rescaling the field variable: $b^{d-2}(\partial'\phi^<)^2 \equiv (\partial'\phi')^2$ (*i.e.* $\phi' \equiv b^{\frac{1}{2}(d-2)}\phi^<$):

$$\int d^d x' \mathcal{L}(\phi^<) = \int d^d x' \left(\frac{1}{2}(\partial'\phi')^2 + \sum_n (g_n + \delta g_n) b^{d-\frac{n(d-2)}{2}}(\phi')^n + \dots \right)$$

¹⁷Why impose rotation invariance here? For now, it's for simplicity. But (preview) we will see that the fixed points we find are stable to rotation-symmetry breaking perturbations. It's an *emergent symmetry*.

¹⁸Again we apply the Landau-Ginzburg-Wilson logic. The idea is the same as in our discussion of blocking for the Ising model. The result is local in space because the interactions between the slow modes mediated by the fast modes have a range of order b/Λ . The result is analytic in $\phi^<$ at small $\phi^<$ and there is no symmetry-breaking because we only integrate the short-wavelength modes.

So the end result is that integrating out a momentum shell of thickness $\delta\Lambda \equiv (1 - b^{-1})\Lambda$ results in a change of the couplings to

$$g'_n = b^{\Delta_n} (g_n + \delta g_n)$$

where

$$\Delta_n \equiv \frac{n(2-d)}{2} + d.$$

Ignore the interaction corrections, δg_n , for a moment. Then we can keep doing this and take $b \rightarrow \infty$ to reach macroscopic scales. Then, as b grows, the couplings with $\Delta_n < 0$ get smaller and smaller as we integrate out more shells. If we are interested in only the longest-wavelength modes, we can ignore these terms. They are *irrelevant*. Couplings (‘operators’) with $\Delta_n > 0$ get bigger and are *relevant*.

The ‘mass term’ has $n = 2$ and $r' = b^2 r$ is always relevant for any $d < \infty$.

This counting is the same as dimensional analysis: demand that βH is dimensionless, and demand that the kinetic term $(\partial\phi)^2$ stays fixed. Naive (length) dimensions:

$$[\beta S] = 0, \quad [x] \equiv 1, \quad [d^d x] = d, \quad [\partial] = -1$$

The kinetic term tells us the engineering dimensions of ϕ :

$$0 = [S_{\text{kinetic}}] = d - 2 + 2[\phi] \implies [\phi] = \frac{2-d}{2}.$$

Then an interaction term has dimensions

$$0 = [\beta g_n \phi^n] = d + [g_n] + n[\phi] \implies [g_n] = -(d + n[\phi]) = -\left(d + n\frac{2-d}{2}\right) = -\Delta_n$$

– couplings with negative length dimension are relevant. This result is the same as engineering dimensional analysis because we’ve left out the interaction terms. This is actually correct when $g_n = 0, n \geq 3$, which is the ...

An important conclusion from this discussion is that there is only a *finite* number of marginal and relevant couplings that we must include to parametrize the physics. Further, if the interactions produce small corrections, they will not change a very irrelevant operator to a relevant operator. This should mitigate some of the terror you felt when we introduced the horrible infinite-dimensional space of hamiltonians \mathcal{M} at the beginning of the course.

Another important conclusion is that the gaussian Ising critical point is *stable* to interactions in $d > 4$. It is of course unstable in the sense that $r\phi^2$ is relevant. And it is unstable if we allow terms with odd powers of ϕ which break the Ising symmetry. But what is the smallest-dimension operator which we haven’t added and which respects

the Ising symmetry? According to our Gaussian counting, each derivative counts for +1, and each power of ϕ counts for $\frac{2-d}{2}$. If we demand rotation invariance (or even just parity) so we can't have a single derivative, the next most important perturbation is $g_4\phi^4$. Its dimension is $\Delta_4 = 4 - d$ - it is irrelevant if $d > 4$ and relevant if $d < 4$. We should have expected this, since it coincides with the breakdown of mean field theory. In $d = 4$, the ϕ^4 term is marginal, and it is an opportunity for small interaction corrections to decide its fate.

Why does mean field theory get better in more dimensions?

Mean field theory is valid if in the energy depending on a spin s_i

$$H_i \equiv J s_i \left(\sum_{\langle i|j \rangle} s_j + h_i \right)$$

we can approximate the values of the neighboring spins by their average $s_j \stackrel{?}{=} \langle s_j \rangle$, and treat the coefficient of s_i as an effective ‘molecular’ field $h_i^{\text{eff}} = \sum_{\langle i|j \rangle} \langle s_j \rangle + h_i$.

More dimensions means more neighbors (for example, for the hypercubic lattice in d dimensions, each site has $2d$ neighbors); more neighbors means that there are more terms in the sum $\sum_{\langle i|j \rangle} s_j + h_i$. If the correlations between the terms in the sum are small enough, the central limit theorem tells us that the fractional error decays with the number of terms in the sum. And this assumption is self-consistent, since in MFT the spins s_j are statistically independent (the probability distribution factorizes).

The preceding argument says that at asymptotically large d , MFT becomes more and more correct. You saw on the homework that when the number of neighbors grows with N (namely with all-to-all interactions), then MFT is exact. When $d = 1$ MFT is completely wrong, since there is no ordering at all at finite T . So something must happen somewhere in between. We’ve just learned that that somewhere is $d = 4$.

$d = 4$ is maybe not so exciting for statistical mechanics applications. However, the same machinery can be used with one of the dimensions interpreted as time. For more on this, I refer you to references on QFT (such as my 215C notes).

6.5 Field theory without Feynman diagrams

[Brezin, Chapter 11] Before we do so systematically, let’s pursue the calculation we did in §6.1 a bit further, now that we’ve learned to organize the integrals over the fluctuations scale-by-scale.

Suppose we've already done the integrals over the shell: $\int_{\Lambda/b}^{\Lambda} dk$, so that the effects of the fluctuations with those wavenumbers are already incorporated into $\gamma^{(1)}[m, b]$. As we argued, $\gamma^{(1)}[m, b]$ will take the same form as the more microscopic effective potential γ , but with some new values of the couplings, depending on b . And then let's do the rest of the integrals using this action (still in the quadratic approximation) and ask how things depend on b . More specifically, we'll ask how the couplings in $\gamma(b)$ need to depend on b to *keep* the physics from depending on this fictional division we are making between UV and IR. That is, we make the demand (sometimes named after Callan and Symanzik) that physical quantities

$$\langle \text{PHYSICS} \rangle \equiv Z^{-1} \int_{\Lambda} D\phi e^{-S[\phi]} \text{PHYSICS}(\phi) = Z^{-1} \int_{\Lambda/b} D\phi^< e^{-S_b[\phi^<]} \text{PHYSICS}(\phi^<)$$

are independent of our arbitrary choice b of dividing line between small and large. (Note that the observable in question must be a long-wavelength observable, not depending on the fast modes $\phi^>$, or else we cannot compute it using just the slow modes $\phi^<$.)

If we're just integrating out the fluctuations with momenta in the shell above Λ/b , in the quadratic approximation, we can just replace (6.5) with

$$\gamma^{(1)}[m, b] = \frac{1}{2} r_0 m^2 + \frac{g_0}{4!} m^4 + \frac{1}{2} \int_{\Lambda/b}^{\Lambda} \mathrm{d}^d q \log \left(\frac{r_0 + \frac{g_0}{2} m^2 + q^2}{r + q^2} \right). \quad (6.10)$$

I also added some subscripts on the couplings to emphasize that r_0, g_0 are parameters in some particular zeroth-order accounting we are making of the physics, not some holy symbols whose values we can measure. Moreover, they will depend on b : $(r_0, g_0) = (r_0(b), g_0(b))$.

Something we *can* measure, and which should not depend on our accounting parameter b , is the susceptibility (for $T > T_c$):

$$r \equiv \chi^{-1} = \partial_m^2 \gamma_{m=0} = r_0 + \frac{g_0}{2} \int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d q}{q^2 + r_0}.$$

(Such an equation, relating a physical quantity like χ to something we can compute in terms of the running couplings $g_n(b)$, is sometimes called a *renormalization condition*.) We can invert this equation to figure out $r_0(b)$:

$$r_0 = r - \frac{g_0}{2} \int \frac{\mathrm{d}^d q}{q^2 + r} + \mathcal{O}(g_0^2).$$

The critical point occurs when $\chi^{-1} = r \rightarrow 0$, which happens when r_0 is

$$r_0^c = -\frac{g_0}{2} \int \frac{\mathrm{d}^d q}{q^2} + \mathcal{O}(g_0^2).$$

This gives the deviation in temperature from the critical point, by subtracting the previous two displayed equations:

$$t_0 \equiv r_0 - r_0^c = r \left(1 + \frac{g_0}{2} \underbrace{\int \frac{d^d q}{q^2(q^2 + r)}}_{\equiv I_d(r,b)} + \mathcal{O}(g_0^2) \right).$$

(Note that $t_0 = t_0(b)$ is merely a relabelling of the coordinate r ; the relation between them is analytic and t_0 depends on our zoom factor b . Hence the demeaning subscript.)

[End of Lecture 15]

Now we must study the integral I . We've observed that $I_d(r, b \rightarrow \infty)$ blows up (by taking $b \rightarrow \infty$ we include all the fluctuations) when $r \rightarrow 0$ for $d \leq 4$. Let's start at $d = 4$, where

$$I_4(r, b) = K_4 \int_{\Lambda/b}^{\Lambda} \frac{q^3 dq}{q^2(q^2 + r)} = -K_4 \log b. \quad (6.11)$$

Another quantity we can imagine measuring is the coupling

$$g \equiv \partial_m^4 \gamma|_{m=0} = g_0 - \frac{3g_0^2}{2} \int \frac{d^d q}{(q^2 + r_0)^2} + \mathcal{O}(g_0^3).$$

Again we can perturbatively invert this to find an expression for the running coupling $g_0(b)$:

$$g_0 = g + \frac{3g^2}{2} \int \frac{d^d q}{(q^2 + r)^2} + \mathcal{O}(g^3) \stackrel{d \rightarrow 4}{\simeq} g - \frac{3g^2}{2} K_4 \log b. \quad (6.12)$$

Combining (6.12) and (6.11), we conclude that for $d = 4$

$$t_0(b) = r \left(1 - \frac{g}{2} K_4 \log b \right).$$

$$g_0(b) = g - \frac{3g^2}{2} K_4 \log b.$$

(I may drop the subscripts on the t s sometimes.)

These expressions are useful because the b -dependence is explicit and we can derive from them an infinitesimal statement:

$$-b \partial_b t_0 = t_0(b) (\kappa(g_0(t_0))). \quad (6.13)$$

For the case above, $\kappa(g_0) = \frac{1}{2} g_0 + \mathcal{O}(g_0^2)$. Similarly,

$$-b \partial_b g_0 = \beta_{g_0} = K_4 \frac{3}{2} g_0^2 + \mathcal{O}(g_0^3). \quad (6.14)$$

The vector fields indicating the continuous flows of the couplings with the zoom factor are generally called *beta functions*.

The ordinary differential equation (6.14) is solved by

$$t(b) = t e^{-\int_1^b \frac{d\mu}{\mu} \kappa(g_0(\mu))}, \quad (6.15)$$

where $t \equiv t(b=1)$, naturally. If there exists a fixed point, $g_0 = g_*$ with $\kappa(g_*) \neq 0$, then its contribution to the exponent (the upper limit dominates) is

$$-\int_1^b \frac{d\mu}{\mu} \kappa(g_*) \xrightarrow{b \rightarrow \infty} \kappa(g_*) \underbrace{\int_1^b \frac{d\mu}{\mu}}_{=\log b}.$$

Hence, in this case

$$t(b) = t b^{-\kappa(g_*)} \quad (6.16)$$

– that is $\kappa(g_*)$ determines the critical exponent with which the IR value of $t(b)$ diverges. Why do we care about the IR value of $t(b)$? It determines the correlation length! We'll come back to this.

What is the solution of the beta function equation for the coupling in $d=4$? To save writing, let's redefine $\tilde{g}_0 \equiv K_4 g_0$ and drop the tilde. The equation is

$$-b \partial_b g_0 = \frac{3}{2} g_0^2 + \mathcal{O}(g_0^3)$$

which is solved by

$$g_0(b) = \frac{2}{3} \frac{1}{\log b} \xrightarrow{b \rightarrow \infty} 0. \quad (6.17)$$

There is an attractive IR fixed point at $g_0 = 0$. This is part of the way towards justifying my claim that perturbation theory would be useful to study the long-wavelength physics in this problem.

In the case of $d=4$, then, the interesting physics comes from the slow approach to the free theory in the IR. To get something interesting we must include the flow, for example in the solution for t , Eq. (6.15): since the flow of g_0 (6.17) never stops, we can parametrize the flow by g_0 and use the chain rule to write $\frac{d\mu}{\mu} = \frac{dg_0}{\beta(g_0)}$ so that

$$\int_1^b \frac{d\mu}{\mu} \kappa(g_0(\mu)) = \int_{g_0}^{g_0(b)} \underbrace{\frac{\kappa(g)}{\beta(g)}}_{=\frac{1}{3g}(1+\mathcal{O}(g))} dg \stackrel{b \gg 1}{\simeq} \frac{1}{3} \log \frac{g(b)}{g_0}$$

From which we conclude

$$t_0(b) \stackrel{b \gg 1}{\simeq} t (\log b)^{-1/3}. \quad (6.18)$$

This equation, a power law (remember I convinced that a log is a special case of a power law) relation between the effective temperature and the zoom factor, will be useful below.

Extracting physics from the running couplings. Let's use this information to study the susceptibility and understand when the quadratic approximation is under control.

First, physics is independent of where we start the flow:

$$\chi^{-1}(t, g_0, \Lambda) = \chi^{-1}(t(b), g_0(b), \Lambda/b) \quad (6.19)$$

– this is what I called the *Callan-Symanzik equation* during the random-walk discussion §1.3. Second, we use ordinary engineering dimensional analysis:

$$[\Lambda] = -1, [\chi^{-1}] = [t] = -2, [g_0] = 0$$

This implies that the RHS of (6.19) is

$$\chi^{-1}(t(b), g_0(b), \Lambda/b) = b^{-2} \chi^{-1}(t(b)b^2, g_0(b), \Lambda).$$

For $d \sim 4$, we found that

$$\chi^{-1} \sim t \left(1 - \frac{g_0}{2} \log \frac{\Lambda}{\sqrt{t}} \right). \quad (6.20)$$

This says perturbation theory (in g_0) breaks down (even for small g_0 !) when $g_0 \log \Lambda / \sqrt{t} \gtrsim 1$. But for fixed physics (as in (6.19)), we can *choose* the zoom factor $b = b^*$ so that the argument of the logarithm is

$$1 \stackrel{!}{=} \frac{\Lambda/b^*}{\sqrt{t(b^*)}}. \quad (6.21)$$

When does the zoom factor hit the sweet spot (6.21)? The answer is different in $d = 4$ and $d < 4$.

Using (6.18), this happens when

$$\Lambda/b^* = \sqrt{t(b^*)} = \sqrt{t} (\log(b^*))^{-1/6} \quad \leftrightarrow \quad (b^*)^{-2} (\log b^*)^{1/3} = \frac{t}{\Lambda^2}$$

which we can solve for b^* in the limit $t \ll \Lambda$ (closer to the critical point than the lattice scale):

$$(b^*)^{-2} \stackrel{t/\Lambda^2 \ll 1}{\simeq} \frac{t}{\Lambda^2 (\log(t/\Lambda^2))^{1/3}}. \quad (6.22)$$

Putting this information back into the Callan-Symanzik equation for the susceptibility, we have

$$\begin{aligned}
\chi^{-1}(t, g_0, \Lambda) &= (b^*)^{-2} \chi^{-1}(t(b^*) (b^*)^2 = \Lambda^2, g_0(b^*), \Lambda) \\
&\stackrel{(6.20)}{=} (b^*)^{-2} t \left(1 - \frac{g_0(b^*)}{2} \underbrace{\log \left(\frac{\Lambda}{\sqrt{t(b^*)}} \right)}_{=0, \text{ by design}} \right) \\
&\stackrel{(6.22)}{=} \frac{t}{(\log t/\Lambda^2)^{1/3}}.
\end{aligned} \tag{6.23}$$

This is a scaling law for how the susceptibility diverges as we tune the knob on our thermostat towards the critical value.

Two important generalizations. Now we make two easy but crucial generalizations of the $d = 4$ Ising calculation we've just done: namely $\mathbb{Z}_2 \rightarrow \mathbf{O}(n)$ and $d \rightarrow 4 - \epsilon$.

$\boxed{\mathbf{O}(n)}$: by the LG logic, a $\mathbf{O}(n)$ -invariant and translation-invariant free energy at fixed magnetization m^a must look like

$$S[\phi^a] = \int d^d x \left(\frac{1}{2} \vec{\nabla} \phi^a \cdot \vec{\nabla} \phi^a + \frac{1}{2} r_0 \phi^a \phi^a + \frac{g_0}{4!} (\phi^a \phi^a)^2 \right)$$

For $n > 1$, in expanding about the mean field configuration $\phi^a = m^a + \varphi^a$, we encounter a distinction between the one (longitudinal) fluctuation in the direction of $m^a \equiv m e_0^a$ and the $n - 1$ transverse fluctuations. The quadratic part of this action comes from the kernel

$$K_{xy}^{ab} = \frac{\delta S}{\delta \phi_x^a \delta \phi_y^b} \Big|_{\phi=m} = \left(\left(-\nabla^2 + r_0 + \frac{g_0}{6} m^2 \right) \delta_{ab} + \frac{g_0}{3} m_a m_b \right) \delta_{xy}.$$

Its eigenvectors can be constructed using an orthonormal basis $\{e^a\}$, in terms of which we can decompose $\varphi^a \equiv \varphi^L e_0^a + \varphi_\alpha^T e_\alpha^a$. Then K is block-diagonal in this basis:

$$K_{xy}^{ab} = \left((\delta_{ab} - e_a e_b) \left(-\nabla^2 + r_0 + \frac{g_0}{6} m^2 \right) + e_a e_b \left(-\nabla^2 + r_0 + \underbrace{1/2}_{=1/3+1/6} g_0 m^2 \right) \right) \delta_{xy}$$

This matrix is made of one copy of the $n = 1$ Ising case with coefficient of m^2 equal to $g_0/2$, and $n - 1$ degenerate copies of the same thing with $g_0/6$. So the sum of the logs of the eigenvalues is

$$\text{tr}_{x,a} \log K = V \int \mathbb{d}^d q \left(\log \left(r_0 + \frac{g_0}{2} m^2 + q^2 \right) + (n - 1) \log \left(r_0 + \frac{g_0}{6} m^2 + q^2 \right) + \text{const} \right).$$

Redoing the steps between (6.10) and (6.12), we find

$$g = g_0 - \left(\frac{3g_0^2}{2} + (n-1)\frac{g_0^2}{6} \right) \int \frac{d^d q}{(q^2 + r)^2} + \mathcal{O}(g_0^3) \stackrel{d \rightarrow 4}{\simeq} g_0 - g_0^2 \left(\frac{3}{2} + \frac{n-1}{6} \right) K_4 \log b$$

so that the beta function is

$$\beta(g_0) = K_4 \frac{n+8}{6} g_0^2 + \mathcal{O}(g_0^3)$$

The flow of the temperature is

$$t_0(b) = t_0 \left(1 - g_0 \frac{n+2}{6} K_4 \log b \right).$$

$d = 4 - \epsilon$: If $d \neq 4$ the self-coupling term $\int d^d x \varphi^4$ in the LG action is not dimensionless: $[\varphi] = \frac{2-d}{2}$, $\implies [g_0] = 4 - d \equiv \epsilon$. Let's extract powers of the UV cutoff to make a dimensionless coupling $g_0 \rightarrow \Lambda^\epsilon g_0$, so that the LG action is

$$\gamma[\phi] = \dots + \int d^d x \Lambda^\epsilon g_0 \phi^4.$$

Anticipating the result a bit, we are going to treat g_0 and ϵ as being of the same order in our expansion, so $\mathcal{O}(g_0) = \mathcal{O}(\epsilon)$ and $\mathcal{O}(g_0^2) = \mathcal{O}(\epsilon g_0)$ et cetera. Thinking of ϵ as small, then, the only change in (6.12) is

$$\Lambda^\epsilon g_0 = g + \mathfrak{b}_0 g^2 \int \frac{d^d q}{(q^2 + r)^2} \tag{6.24}$$

where $\mathfrak{b}_0 \equiv \frac{3}{2} + \frac{n-1}{6}$. Note that g is not dimensionless. Again we absorb the factors of K_d into g, g_0 .

Here comes the magic: the key fact is roughly that “ $\Lambda^\epsilon = 1 + \epsilon \log \Lambda + \mathcal{O}(\epsilon^2)$ ”; I put that in quotes because it is distasteful to take the log of a dimensional quantity. Here is the more accurate statement: suppose we start the RG at Λ/b instead of Λ . Then in order for g to be independent of b , we need $g_0 = g_0(b)$ according to:

$$(\Lambda/b)^\epsilon g_0(b) = g - \mathfrak{b}_0 K_d \Lambda^{-\epsilon} g^2 \log b + \mathcal{O}(g_0^2 \epsilon)$$

(The b -dependence on the LHS comes from starting the flow at Λ/b .) Systematically ignoring things that can be ignored (including the $\Lambda^{-\epsilon}$ which is needed in the previous equation for dimensions to work), this is:

$$g_0(b) = g_0 (1 + \epsilon \log b) - \mathfrak{b}_0 g_0^2 \log b + \mathcal{O}(g_0^2 \epsilon)$$

$$\beta_{g_0} \equiv -b \partial_b g_0 = -\epsilon g_0 + \mathfrak{b}_0 g_0^2 + \mathcal{O}(g_0^3).$$

The crucial extra term proportional to ϵg_0 comes from the engineering dimensions of g_0 . Where are the fixed points? There is still one at $g_0 = 0$, our old friend the Gaussian fixed point. But [there is another](#), at

$$g^* = \frac{\epsilon}{\mathbf{b}_0} + \mathcal{O}(\epsilon^2) = \frac{6}{n+8}\epsilon + \mathcal{O}(\epsilon^2) .$$

This is the Wilson-Fisher fixed point (really one for every n and $d \lesssim 4$). As was foretold, g_0 is of order ϵ .

The WF fixed point and the Gaussian critical point exchange roles as we decrease d through four. For $d > 4$, the Gaussian critical point is IR attractive and governs the critical behavior at long wavelengths: MFT is right. At $d = 4$, they collide and this produces the weird logarithms in the approach to $g = 0$ that we saw above. For $d < 4$, the Gaussian fixed point is unstable to the interaction term: the $g_0\phi^4$ term is a relevant perturbation, since g_0 grows as we zoom out.

Correlation length critical exponent. Now we can look at the behavior of the correlation length as we approach this critical point, by tuning the temperature. Again, physics is independent of where we start the flow:

$$\xi(t, g_0, \Lambda) = \chi^{-1}(t(b), g_0(b), \Lambda/b) \tag{6.25}$$

– this is what I called the *Callan-Symanzik equation* during the random-walk discussion §1.3. Second, we use ordinary engineering dimensional analysis:

$$[\Lambda] = -1, [\xi] = 1, [g_0] = 0, [t] = -2$$

– the correlation length is a length and so zooms like a length. From this, we deduce that (the RHS of (6.25) is)

$$\xi(t(b), g_0(b), \Lambda/b) = b\xi(t(b)b^2, g_0(b), \Lambda).$$

Now we can choose a convenient zoom factor, b . Again, we choose $b = b^*$ so that the argument of the logs are all 1 and they go away:

$$\frac{t(b^*)}{(\Lambda/b^*)^2} = 1. \tag{6.26}$$

If $b^* \rightarrow \infty$, then $g_0(b^*) \rightarrow g_*$, the IR fixed point value, where

$$t(b^*) \stackrel{(6.16)}{\simeq} (b^*)^{-\kappa(g_*)} t.$$

We can solve this equation for b^* , using (6.26):

$$b^* = \left(\frac{t}{\Lambda^2} \right)^{-\frac{1}{2-\kappa(g_*)}} \quad t \begin{array}{l} \ll \Lambda \\ \gg \end{array} \infty$$

which indeed blows up in the critical region $t \ll \Lambda$ – that is: this is an IR fixed point, a fixed point we reach by zooming out.

Therefore

$$\begin{aligned} \xi(t, g_0, \Lambda) &= b^* \xi(t(b^*), (b^*)^2 = \Lambda^2, g_*, \Lambda) \sim \left(\frac{t}{\Lambda^2}\right)^{-\frac{1}{2-\kappa(g_*)}} \\ &\equiv \left(\frac{t}{\Lambda^2}\right)^{-\nu} \end{aligned} \quad (6.27)$$

Explicitly, $\kappa(g_0) = \frac{n+2}{5}g_0 + \mathcal{O}(g_0^2)$ means $\kappa(g_*) = \frac{n+2}{n+8}\epsilon + \mathcal{O}(\epsilon^2)$ so that

$$\nu = \frac{1}{2} + \frac{n+2}{4(n+8)}\epsilon + \mathcal{O}(\epsilon) \quad (6.28)$$

Notice that all the information about the short-distance stuff has dropped out of (6.27) (except for the stuff hidden in the twiddle, i.e. the overall coefficient) – only the physics *at* the fixed point matters for the exponent.

We can do remarkably well by setting $\epsilon = 1$ in (6.28) and comparing to numerical simulations in $d = 3$.

[End of Lecture 16]

6.6 Perturbative momentum-shell RG

[Kardar, *Fields*, §5.5, 5.6] I will say a bit about how to develop this perturbative RG more systematically. We'll end up at the same place, but with more context. This calculation is important enough that it's worth doing many ways.

We'll do n -component fields, ϕ^a , $a = 1..n$ with $O(n)$ symmetry, in $d = 4 - \epsilon$ dimensions.

$$S[\phi] = S_0[\phi] + \mathcal{U}.$$

with S_0 the gaussian terms, as above. For n component fields, the gaussian term looks like

$$S_0[\phi] = \int_0^\Lambda \int d^d k \underbrace{\phi^a(k)\phi^a(-k)}_{\equiv |\phi|^2(k)} \frac{1}{2} (r_0 + r_2 k^2).$$

(If it is not diagonal, do a field redefinition to make it so.) We assume the model has a $O(n)$ symmetry which acts by $\phi^a \rightarrow R_b^a \phi^b$, with $R^t R = \mathbb{1}_{n \times n}$. The most relevant, symmetric interaction term (non-Gaussian perturbation) is the ϕ^4 term

$$\mathcal{U} = \int d^d x u_0 (\phi^a(x)\phi^a(x))^2 = u_0 \int \prod_{i=1}^4 \int d^d k_i \sum_{a_{1,2,3,4}=1}^n \phi^{a_1}(k_1)\phi^{a_2}(k_2)\phi^{a_3}(k_3)\phi^{a_4}(k_4) \delta\left(\sum_i k_i\right) \delta^{a_1 a_2} \delta^{a_3 a_4}.$$

(I've defined $\delta(q) \equiv (2\pi)^d \delta^d(q)$.)

We'll show that it's not actually necessary to ever do any momentum integrals to derive the RG equations.

Again we break up our fields into slow and fast, and we want to integrate out the fast modes first:

$$Z_\Lambda = \int [D\phi_{<}] e^{-\int_0^{\Lambda/b} \int d^d k |\phi_{<}(k)|^2 \left(\frac{r_0 + r_2 k^2}{2}\right)} Z_{0,>} \langle e^{-\mathcal{U}[\phi_{<}, \phi_{>}]} \rangle_{0,>}.$$

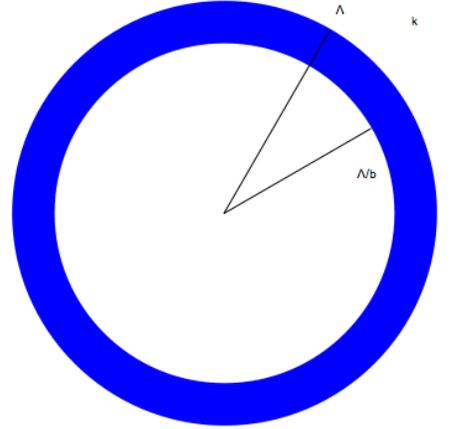
The $\langle \dots \rangle_{0,>}$ means averaging over the fast modes with their Gaussian measure, and $Z_{0,>}$ is an irrelevant normalization factor, independent of the objects of our fascination, the slow modes $\phi_{<}$.

The corrections to the effective action for $\phi_{<}$ can be obtained by a cumulant expansion:

$$\log \langle e^{-\mathcal{U}} \rangle_{0,>} = - \underbrace{\langle \mathcal{U} \rangle_{0,>}}_1 + \frac{1}{2} \underbrace{\left(\langle \mathcal{U}^2 \rangle_{0,>} - \langle \mathcal{U} \rangle_{0,>}^2 \right)}_2 + \mathcal{O}(\mathcal{U}^3)$$

Let's focus on the first-order term first:

$$1 = \langle \mathcal{U}[\phi_{<}, \phi_{>}] \rangle_{0,>} = u_0 \int \prod_{i=1}^4 \int d^d k_i \delta^d\left(\sum_i k_i\right) \left\langle \prod_i (\phi_{<} + \phi_{>})_i \right\rangle_{0,>}$$



Diagrammatically, these 16 terms decompose as in Fig. 1.

We can compute the averages over the fast modes by doing Wick contractions. This is a fact about Gaussian integrals, which can be summarized by noting that

$$\langle e^{h_A \phi_A} \rangle_0 = e^{\frac{1}{2} h_A \langle \phi_A \phi_B \rangle_0 h_B}$$

where A is a multi-index over space and flavor labels and whatever else (to prove it, complete the square). Then expand both sides to learn that

$$\langle \phi_{A_1} \cdots \phi_{A_m} \rangle_0 = \begin{cases} 0, & \text{if } m \text{ is odd} \\ \text{sum of all pairwise contractions,} & \text{if } m \text{ is even} \end{cases} .$$

Each pairwise contraction is given by the ‘propagator’, which in our case is

$$\langle \phi_{>}^a(q_1) \phi_{>}^b(q_2) \rangle_{0,>} = \frac{\delta^{ab} \phi(q_1 + q_2)}{r_0 + q_1^2 r_2} = \text{diagram: a wavy line with blue dots at each end and indices } a, b \text{ above the dots.}$$

In the figure, these are denoted by wiggly lines. The slow modes are denoted by straight lines. The 4-point interaction is denoted by a dotted line connecting two pairs of lines (straight or wiggly).

$$u_0 \delta^{a_1 a_2} \delta^{a_3 a_4} \phi \left(\sum_i q_i \right) = \text{diagram: a central dotted line connecting two pairs of lines (one straight, one wiggly) with indices } a_1, a_2, a_3, a_4 \text{ and } \phi \text{ above the central line.}$$

Although the four fields must be at the same point in space we separate the two pairs whose flavor indices are contracted, so that we can follow the conserved flavor index around the diagrams.

Let’s analyze the results of the first order correction: The interesting terms are

$$\mathbf{1}_3 = -u_0 \underbrace{2}_{\text{symmetry} = \delta^{aa}} \underbrace{n}_n \int_0^{\Lambda/s} d^d k |\phi_{<}(k)|^2 \int_{\Lambda/s}^{\Lambda} d^D q \frac{1}{r_0 + r_2 q^2}$$

$$\mathbf{1}_4 = \frac{4 \cdot 1}{2 \cdot n} \mathbf{1}_3$$

has a bigger symmetry factor but no closed flavor index loop. The result through $\mathcal{O}(u)$ is then just what we found previously:

$$\boxed{r_0 \rightarrow r_0 + \delta r_0 = r_0 + 4u_0(n+2) \int_{\Lambda/b}^{\Lambda} d^d q \frac{1}{r_0 + r_2 q^2} + \mathcal{O}(u_0^2)} .$$

r_2 and u are unchanged. The second part of the RG step is rescaling

$$\tilde{q} \equiv bq, \quad \tilde{\phi} \equiv \zeta^{-1} \phi_{<}$$

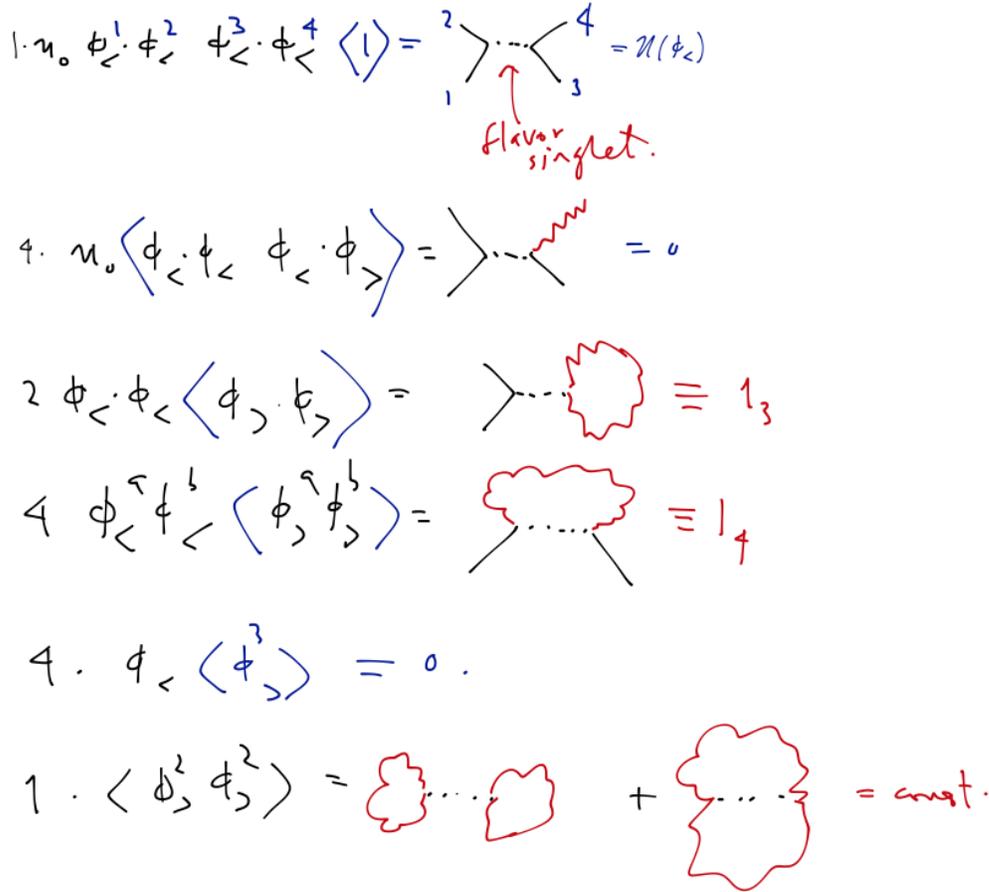


Figure 1: 1st order corrections from the quartic perturbation of the Gaussian fixed point of the $O(N)$ model. Wiggly lines denote propagation of fast modes $\phi_{>}$, straight lines denote (external) slow modes $\phi_{<}$. A further refinement of the notation is that we split apart the 4-point vertex to indicate how the flavor indices are contracted; the dotted line denotes a direction in which no flavor flows, i.e. it represents a coupling between the two flavor singlets, $\phi^a \phi^a$ and $\phi^b \phi^b$. The numbers at left are multiplicities with which these diagrams appear. (The relative factor of 2 between 1_3 and 1_4 can be understood as arising from the fact that 1_3 has a symmetry which exchanges the fast lines but not the slow lines, while 1_4 does not.) Notice that closed loops of the wiggly lines represent factors of N , since we must sum over which flavor is propagating in the loop – the flavor of a field running in a closed loop is not determined by the external lines, just like the momentum.

to restore the original action: we must choose $\zeta = b^{1+d/2}$ to keep $\tilde{r}_2 = r_2$.

The second-order-in- u_0 terms are displayed in Fig. 2. The interesting part of the second order bit

$$2 = \frac{1}{2} \langle \mathcal{U}[\phi_{<}, \phi_{>}]^2 \rangle_{0, >, \text{connected}}$$

is the correction to $\mathcal{U}[\phi_{<}]$. There are less interesting bits which are zero or constant or two-loop corrections to the quadratic term. The correction to the quartic term at

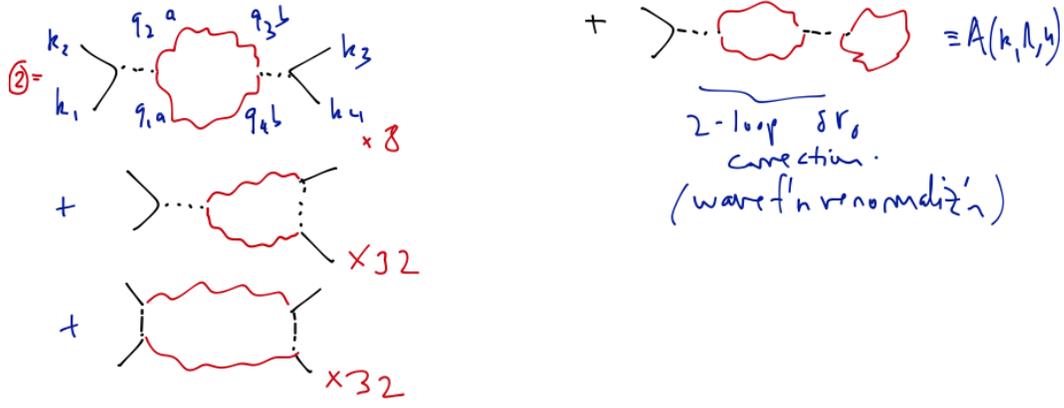


Figure 2: 2nd order corrections from the quartic perturbation of the Gaussian fixed point of the $O(N)$ model. Notice that the diagram at right has two closed flavor loops, and hence goes like n^2 , and it comes with two powers of u_0 . You can convince yourself by drawing some diagrams that this pattern continues at higher orders. If you wanted to define a model with large n you should therefore consider taking a limit where $n \rightarrow \infty$, $u_0 \rightarrow 0$, holding $u_0 n$ fixed. The quantity $u_0 n$ is often called the 't Hooft coupling.

2nd order is

$$\delta_2 S_4[\phi_<] = u_0^2 (4n + 32) \int_0^{\Lambda/b} \prod_i^4 (d^d k_i \phi_<(k_i)) \delta(\sum k_i) f(k_1 + k_2)$$

with

$$f(k_1 + k_2) = \int d^d q \frac{1}{(r_0 + r_2 q^2)(r_0 + r_2(k_1 + k_2 - q)^2)} \simeq \int d^d q \frac{1}{(r_0 + r_2 q^2)^2} (1 + \mathcal{O}(k_1 + k_2))$$

– the bits that depend on the external momenta give irrelevant derivative corrections, like $\phi_<^2 \partial^2 \phi_<^2$. We ignore them. This leaves behind just the correction to u we found before.

Altogether, the full result through $\mathcal{O}(u_0^2)$ is then the original action, with the parameter replacement

$$\begin{pmatrix} r_2 \\ r_0 \\ u_0 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{r}_2 \\ \tilde{r}_0 \\ \tilde{u}_0 \end{pmatrix} = \begin{pmatrix} b^{-d-2} \zeta^2 (r_2 + \delta r_2) \\ b^{-d} \zeta^2 (r_0 + \delta r_0) \\ b^{-3d} \zeta^4 (u_0 + \delta u_0) \end{pmatrix} + \mathcal{O}(u_0^3).$$

The shifts are:

$$\begin{cases} \delta r_2 = u_0^2 \frac{\partial_k^2 A(0)}{r_2} \\ \delta r_0 = 4u_0(n+2) \int_{\Lambda/b} d^d q \frac{1}{r_0 + r_2 q^2} - A(0)u_0^2 \\ \delta u_0 = -\frac{1}{2}u_0^2(8n+64) \int_{\Lambda/b} d^d q \frac{1}{(r_0 + r_2 q^2)^2} \end{cases} .$$

Here A is a new ingredient that we didn't notice earlier: it produces a correction to the kinetic term for $\phi_<$ (which is called 'wavefunction renormalization'): $A(k) = A(0) + \frac{1}{2}k^2\partial_k^2 A(0) + \dots$. We can choose to keep $\tilde{r}_2 = r_2$ by setting

$$\zeta^2 = \frac{b^{d+2}}{1 + u_0^2 \partial_k^2 A(0)/r_2} = b^{d+2} (1 + \mathcal{O}(u_0^2)).$$

Now let's make the RG step infinitesimal:

$$b = e^\ell \simeq 1 + \delta\ell$$

$$\begin{cases} \frac{dr_0}{d\ell} = 2r_0 + \frac{4(n+2)K_d\Lambda^d}{r_0+r_2\Lambda^2}u_0 - Au_0^2 + \mathcal{O}(u_0^3) \\ \frac{du_0}{d\ell} = (4-d)u_0 - \frac{4(n+8)K_d\Lambda^d}{(r_0+r_2\Lambda^2)^2}u_0^2 + \mathcal{O}(u_0^3) \end{cases} \quad (6.29)$$

To see how the previous thing arises, and how the integrals *all went away*, let's consider just the $\mathcal{O}(u_0)$ correction to the mass:

$$\begin{aligned} \tilde{r}_0 &= r_0 + \delta\ell \frac{dr_0}{d\ell} = s^2 \left(r_0 + 4u(n+2) \int_{\Lambda/b}^\Lambda \frac{d^d q}{r_0 + r_2 q^2} + \mathcal{O}(u_0^2) \right) \\ &= (1 + 2\delta\ell) \left(r_0 + 4u_0(n+2)K_d\Lambda^d \frac{1}{r_0 + r_2\Lambda^2} \delta\ell + \mathcal{O}(u_0^2) \right) \\ &= \left(2r_0 + \frac{4u_0(n+2)}{r_0 + r_2\Lambda^2} K_d\Lambda^d \right) \delta\ell + \mathcal{O}(u_0^2). \end{aligned} \quad (6.30)$$

Now we are home. (6.29) has two fixed points. One is the free fixed point at the origin where nothing happens. The other (Wilson-Fisher) fixed point is at

$$\begin{cases} r_0^* = -\frac{2u_0^*(n+2)K_d\Lambda^d}{r_0^*+r_2\Lambda^2} \stackrel{d=4-\epsilon}{=} -\frac{1}{2} \frac{n+2}{n+8} r_2\Lambda^2 \epsilon + \mathcal{O}(\epsilon^2) \\ u_0^* = \frac{(r_0^*+r_2\Lambda^2)^2}{4(n+8)K_d\Lambda^d} \epsilon \stackrel{d=4-\epsilon}{=} \frac{1}{4} \frac{r_2^2}{(d+8)K_4} \epsilon + \mathcal{O}(\epsilon^2) \end{cases}$$

which is at positive u_0^* if $\epsilon > 0$. In the second step we keep only leading order in $\epsilon = 4 - d$.

Now we follow protocol and *linearize* near the W-F fixed point:

$$\frac{d}{d\ell} \begin{pmatrix} \delta r_0 \\ \delta u_0 \end{pmatrix} = M \begin{pmatrix} \delta r_0 \\ \delta u_0 \end{pmatrix}$$

The matrix M is a 2x2 matrix whose eigenvalues describe the flows near the fixed point. It looks like

$$M = \begin{pmatrix} 2 - \frac{n+2}{n+8}\epsilon & \dots \\ \mathcal{O}(\epsilon^2) & -\epsilon \end{pmatrix}$$

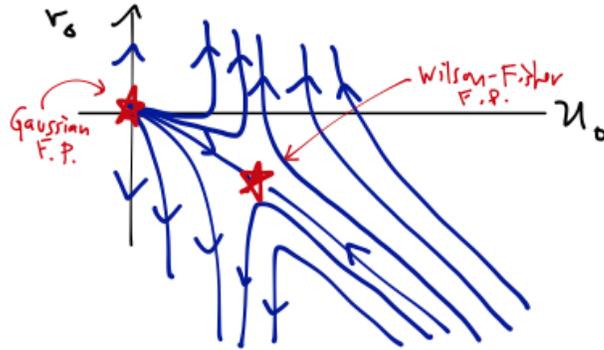


Figure 3: The ϕ^4 phase diagram, for $\epsilon > 0$.

Its eigenvalues (which don't care about the off-diagonal terms because the lower left entry is $\mathcal{O}(\epsilon^2)$) are

$$y_r = 2 - \frac{n+2}{n+8}\epsilon + \mathcal{O}(\epsilon^2) > 0$$

which determines the instability of the fixed point and

$$y_u = -\epsilon + \mathcal{O}(\epsilon^2) < 0 \text{ for } d < 4$$

which is a stable direction.

So y_r determines the correlation length exponent. Its eigenvector is δr_0 to $\mathcal{O}(\epsilon^2)$. This makes sense: r_0 is the relevant coupling which must be tuned to stay at the critical point. The correlation length can be found as follows (as we did around Eq. (??)). ξ is the value of $s = s_1$ at which the relevant operator has turned on by an order-1 amount, *i.e.* by setting $\xi \sim s_1$ when $1 \sim \delta r_0(s_1)$. According to the linearized RG equation, close to the fixed point, we have $\delta r_0(s) = s^{y_r} \delta r_0(0)$. Therefore

$$\xi \sim s_1^{-\frac{1}{y_r}} = (\delta r_0(0))^{-\nu} .$$

This last equality is the definition of the correlation length exponent (how does the correlation length scale with our deviation from the critical point $\delta r_0(0)$). Therefore

$$\nu = \frac{1}{y_r} = \left(2 \left(1 - \frac{1}{2} \frac{n+2}{n+8} \epsilon \right) \right)^{-1} + \mathcal{O}(\epsilon^2) \simeq \frac{1}{2} \left(1 + \frac{n+2}{2(n+8)} \epsilon \right) + \mathcal{O}(\epsilon^2).$$

[End of Lecture 17]

7 RG approach to walking

7.1 SAWs and $O(n \rightarrow 0)$ magnets

[Brézin, ch 8; Cardy ch 9; the original reference is (brief!) P. de Gennes, [Phys. Lett. A38 \(1972\) 339.](#)]

At each site i of your graph, place an n -component vector \vec{s}_i ; we'll normalize them so that for each site i $n = \vec{s}_i \cdot \vec{s}_i \equiv \sum_{a=1}^n (s_i^a)^2$, and we'll study the hamiltonian

$$H(s) = -K \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j$$

(I have named the coupling K to make contact with our previous discussion of SAWs). Denote by $d\Omega(s)$ the round (*i.e.* $O(n)$ -invariant) measure on an $(n-1)$ -sphere, normalized to $\int d\Omega(s) = 1$. The partition sum is

$$\begin{aligned} Z &= \int \prod_i d\Omega(s_i) e^{-H(s)} \\ &= \int \prod_i d\Omega(s_i) \sum_{k=0}^{\infty} \frac{K^k}{k!} \left(\sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j \right)^k \\ &= \sum_{\text{graphs, } G} \frac{K^{k \equiv N_l(G)}}{k!} \int \prod_i d\Omega(s_i) \prod_{\langle ij \rangle \in G} \vec{s}_i \cdot \vec{s}_j. \end{aligned} \quad (7.1)$$

Here we are doing the high-temperature expansion, and further expanding the product of factors of the Hamiltonian; we interpret each such term as a graph G covering a subset of links of the lattice. $N_l(G)$ is the number of links covered by the graph G . Now we can do the spin integrals. The integral table is

$$\begin{aligned} \int d\Omega(s) &= 1 \\ \int d\Omega(s) s_a s_b &= \delta_{ab} n \\ \int d\Omega(s) s_a s_b s_c s_d &= \frac{n}{n+2} (\delta_{ab} \delta_{cd} + 2 \text{ perms}) \end{aligned} \quad (7.2)$$

where the second follows by $O(n)$ invariance and taking partial traces. The generating function is useful:

$$f_n(x) \equiv \int d\Omega(s) e^{\vec{x} \cdot \vec{s}} = \frac{\int_0^\pi d\theta \sin^{n-2} \theta e^{x \cos \theta}}{\int_0^\pi d\theta \sin^{n-2} \theta} = \sum_{p=0}^{\infty} \frac{x^p}{p!} \underbrace{\frac{\int_0^\pi d\theta \sin^{n-2} \cos^p \theta}{\int_0^\pi d\theta \sin^{n-2} \theta}}_{=0, n \text{ odd}}$$

$$= 1 + \sum_{p=1}^{\infty} \frac{x^{2p}}{p!} \frac{n^p}{2^p n(n+2) \cdots (n+2p-2)} \xrightarrow{n \rightarrow 0} f_0(x) = 1 + \frac{x^2}{2}. \quad (7.3)$$

Let's interpret this last result: it says that in the limit $n \rightarrow 0$, each site is covered either zero times or two times. This means that the graphs which contribute at $n \rightarrow 0$ avoid themselves.¹⁹

Returning to $n > 0$, since $\langle s_i^a s_i^b \rangle = 0$ if $a \neq b$, the value of the spin is conserved along the closed loops. We get a factor of n from the spin sums $\sum_{a=1}^n$ from *closed* loops. Only closed loops contribute to Z . So $Z \xrightarrow{n \rightarrow 0} 1$, yay. Less trivially, however, consider

$$G^{ab=11}(r, K) \equiv \langle s_0^{a=1} s_r^{b=1} \rangle \equiv Z^{-1} \int \prod_i d\Omega(s_i) e^{-H(s)} s_0^1 s_r^1.$$

Doing the same high-temperature expansion to the numerator, we get contributions from loops which end only at sites 0 and r . In the $n \rightarrow 0$ limit all the closed loops go away from both numerator and denominator, leaving

$$G^{ab=11}(r, K) \xrightarrow{n \rightarrow 0} \sum_p K^p M_p(\vec{r}) \quad (7.4)$$

where $M_p(\vec{r})$ is (as in (2.4)) the number of SAWs going from 0 to \vec{r} in p steps. This is the generating function we considered earlier! The quantity G in (2.6) is actually the correlation function of the $O(n \rightarrow 0)$ magnet!

Summing the BHS of (7.4) over r , the LHS is $\sum_r G^{11}(r, K) = \chi^{11}(K) \sim (K_c - K)^{-\gamma}$ near the critical point of this magnet. The RHS is $\sum_p \sum_r K^p M_p(R) = \sum_p M_p K^p \rightarrow \infty$ which $K \rightarrow K_c$ (from below), from which concluded earlier that for large walks, $M_p \stackrel{p \gg 1}{\sim} p^{\gamma-1} a^p$ (with $a = 1/K_c$, a non-universal constant which is sometimes fetishized by mathematicians).

Furthermore, the quantity ξ in (2.7) is actually the correlation length, $G^{11}(r, K) \sim e^{-r/\xi}$. At the critical point, $\xi \sim (K_c - K)^{-\nu}$ means that $R_p \sim p^\nu$, which determines the fractal dimension of the SAW in d dimensions to be $D_{\text{SAW}} = \lim_{n \rightarrow 0} \frac{1}{\nu(n, d)}$, where $\nu(n, d)$ is the correlation-length critical exponent for the $O(n)$ Wilson-Fisher fixed point in d dimensions.

7.1.1 SAW:RW::WF:Gaussian

In the same way, the Gaussian fixed point determines the fractal dimension of the unrestricted walk. This can be seen by a high-temperature expansion of the Gaussian

¹⁹Cardy has a clever way of avoiding these spherical integrations by starting with a microscopic model with a nice high temperature expansion (namely $H(s) = \sum_{\langle ij \rangle} \log(1 + K \vec{s}_i \cdot \vec{s}_j)$) and appealing to universality.

model. (For more on this point of view, see Parisi §4.3 - 4.4.) Alternatively, consider unrestricted walks on a graph with adjacency matrix A_{ij} , starting from the origin 0. Denote the probability of being at site r after n steps by $P_n(r)$. Starting at 0 means $P_0(r) = \delta_{r,0}$. For an unrestricted walk, we have the one-step (Markov) recursion:

$$P_{n+1}(r) = \frac{1}{z} \sum_{r'} A_{r'r} P_n(r') \quad (7.5)$$

where the normalization factor $z \equiv \sum_{r'} A_{r'r}$ is the number of neighbors (more generally, the matrix A could be a weighted adjacency matrix and z could depend on r). Defining the generating function

$$G(r|q) \equiv \sum_{n=0}^{\infty} q^n P_n(r)$$

the recursion (7.5) implies

$$\left(\delta_{r'r} - \frac{q}{z} A_{r'r} \right) G(r|q) = \delta_{r',0}. \quad (7.6)$$

In words: G is the correlation function of the Gaussian model with

$$Z = \int \prod_r d\phi_r e^{-\sum_{r'r} \phi_{r'} (\delta_{r'r} - \frac{q}{z} A_{r'r}) \phi_r}.$$

For the hypercubic lattice with spacing a , this is

$$G(r|q) = \int_{\text{BZ}} d^d p \frac{e^{i\vec{r}\cdot\vec{p}}}{1 - \frac{q}{z} \sum_{\mu} \cos a p_{\mu}}.$$

The long-wavelength properties (for which purposes the denominator may be replaced by $p^2 + r$ as $r \sim q - 1$) of the Gaussian model near its critical point at $q \rightarrow 1$ determine the behavior of large unrestricted walks, and in particular the RMS size $\sim \sqrt{n}$ and fractal dimension is 2.

And the Gaussian answer is the right answer even for a SAW in $d > 4$. We could anticipate this based on our understanding of the fate of the WF fixed point as $d \rightarrow 4$ from below. How can we see the correctness of mean field theory for SAWs in $d > 4$ directly from the walk?

There is a simple answer, and also a more involved, quantitative answer. The simple answer is: the random walk has fractal dimension $D = 2$ (if it is embedded in two or more dimensions and is unrestricted). Two-dimensional subspaces of \mathbb{R}^d will generically intersect (each other or themselves) if $d \leq 4$ (generic intersection happens when the sum of the codimensions is ≤ 0 , so the condition for intersection is underdetermined). For $d \geq 4$, they generically miss each other, and the self-avoidance condition does not have a big effect.

7.1.2 Worksheet theory of the SAW

[Cardy §9.2] Consider the following statistical model of a chain of N monomers at positions \vec{r}_i in d dimensions:

$$Z = \int \prod_{i=1}^N d^d r_i \exp \left(- \sum_i \frac{(\vec{r}_{i+1} - \vec{r}_i)^2}{a^2} - ua^d \sum_i \sum_j \delta^d(\vec{r}_i - \vec{r}_j) \right).$$

The first term insists that neighboring monomers be spaced by a distance approximately a . The second term penalizes a configuration where any two monomers collide. We used factors of the chain-spacing a to render the coupling u dimensionless.

Now zoom out. Suppose that $a \ll \xi$ so that we may treat the polymer as a continuous chain, $\vec{r}(t_i \equiv ia^2) \equiv \vec{r}_i$. In taking the continuum limit we must take $t \sim a^2$ in order to keep the coefficient of the \dot{r}^2 term independent of a . The exponent becomes the Edwards hamiltonian:

$$H_E[r] = \int dt \left(\frac{dr}{dt} \right)^2 + ua^{d-4} \int \int dt_1 dt_2 \delta^d(\vec{r}_1 - \vec{r}_2).$$

This is a ‘worksheet’ point of view: it is a 1d system of size Na^2 , with a long (infinite) range interaction. a plays the role of a UV cutoff.

If $u = 0$, so the walk does not avoid itself, dimensional analysis $[r] = 1, [t] = 2$ (demanding the kinetic term be dimensionless) gives

$$r(t) \sim a \left(\frac{t}{a^2} \right)^{1/2} \tag{7.7}$$

and r does not scale when we rescale the cutoff $a \rightarrow ba, t \rightarrow t$.

RG: The interaction strength ua^{d-4} is independent of the cutoff when $d = 4$. It is irrelevant for $d > 4$, giving back the MFT result (7.7), as promised.

For $d < 4$ it seems to grow in the IR, and we might hope for an IR fixed point u^* , and a resulting anomalous dimension for the operator r :

$$a \rightarrow ba, \quad r \rightarrow b^{-x}r, \quad t \rightarrow t.$$

Here is a clever (though approximate) argument (due to Flory) that suggests a value for x . At a fixed point, the two terms in H must conspire, and so should scale the same way. For general x , the kinetic term and the potential scale respectively as

$$\text{KE} \rightarrow \text{KE}b^{-2x}, \quad V \rightarrow Vb^{d-4+dx}$$

suggesting that $x = \frac{4-d}{2+d}$. Dimensional analysis says

$$r(t) = af\left(\frac{t}{a^2}\right) \sim t^{\frac{1+x}{2}}$$

and therefore the RMS walk size is

$$R = r(t = N) \sim N^\nu, \quad \nu = \frac{1+x}{2} \Big|_{\text{Flory}} = \frac{3}{d+2}.$$

This isn't too bad; in fact it's exactly right in $d = 2$. (See Cardy Chapter 9 for more on this.)

[End of Lecture 18]

A comment on 'power counting'.

How did we know from the engineering dimensional analysis that u was irrelevant when $d > 4$?

Let me describe the analogous argument in the case of field theory with local interactions. Consider the gaussian critical point in d dimensions $S_0[\phi] = \int d^d x (\nabla\phi)^2$, so that the length dimensions of the field are $[\phi] = \frac{2-d}{2}$. Perturb by $S_{\text{interaction}} \equiv \int d^d x g\phi^p$.

$$0 = [S_{\text{interaction}}] = d + [g] + p[\phi] \implies [g] = -(d + p[\phi]) = -\left(d + p\frac{2-d}{2}\right).$$

The coupling is dimensionless when $[g] = 0$ which happens when

$$p = p_d \equiv \frac{2d}{d-2},$$

this case is naively scale invariant, at least until we study the fluctuations. For $d > 2$, the coupling g has length dimensions

$$[g] = d \cdot \frac{p - p_d}{p_d} \begin{cases} > 0 \text{ when } p > p_d, & \text{non-renormalizable or irrelevant} \\ = 0 \text{ when } p = p_d, & \text{renormalizable or marginal} \\ < 0 \text{ when } p < p_d, & \text{super-renormalizable or relevant.} \end{cases}$$

Consider the 'non-renormalizable' case. Suppose we calculate some physical quantity f with $[f]$ as its naive dimension, in perturbation theory in g , *e.g.* by Feynman diagrams. We'll get:

$$f = \sum_{n=0}^{\infty} g^n c_n$$

with c_n independent of g . So

$$[f] = n[g] + [c_n] \implies [c_n] = [f] - n[g]$$

So if $[g] > 0$, c_n must have more and more powers of some inverse length as n increases. What dimensionful quantity makes up the difference?? The dimensions are made up by dependence on the short-distance cutoff $\Lambda = \frac{2\pi}{a}$, which has $[\Lambda] = -1$. Generically: $c_n = \tilde{c}_n (\Lambda)^{n[g]}$, where \tilde{c}_n is dimensionless, and $n[g] > 0$ – it's higher and higher powers of the cutoff. But this means that if we integrate out shells down to Λ/b , in order for physics to be independent of the zoom parameter b , the microscopic coupling $g(b)$ will have to depend on b to cancel this factor. In particular, we'll have to have

$$g(b) = g_0 b^{-n[g]} \xrightarrow{b \rightarrow \infty} 0.$$

7.2 RG approach to unrestricted lattice walk

We showed above that the generating function $G(r|q)$ for unrestricted walks on a lattice (from 0 to r) satisfies (7.6), which says that it's a Green's function for the lattice laplacian. The data of the Green's function is encoded in the spectrum of the adjacency matrix

$$A_{ij}v_j^\epsilon = \epsilon v_i^\epsilon. \quad (7.8)$$

This determines G via

$$G(i|q) = \sum_{\epsilon} v_0^\epsilon v_i^\epsilon \frac{1}{1 - \epsilon q/z}.$$

The eigensystem of A encodes the solution to many physics problems. For example, we could consider a continuous-time random walk, where the probability $p_i(t)$ for a walker to be at site i at time t satisfies

$$\partial_t p_i = (\delta_{ij} z_j - A_{ij}) p_j \quad (7.9)$$

where $z_j \equiv \sum_i A_{ij}$ is coordination number at site j , which addition guarantees $0 = \sum_i \partial_t p_i$, the conservation of probability. The solution is then

$$p_i(t) = \sum_{\epsilon, j} e^{-(z-\epsilon)t} v_i^\epsilon v_j^\epsilon p_j(0).$$

Alternatively, we could think of these as the equations for the normal modes of the lattice vibrations of a collection of springs stretched along the bonds of the lattice. In that case, this spectrum determines the (phonon contribution to the) heat capacity of a solid with this microstructure.

Previously, we solved this problem using translation symmetry of the lattice, by going to momentum space. Here I would like to illustrate an RG solution to this eigenvalue problem which is sometimes available. It takes advantage of the scaling symmetry of the lattice. Sometimes both scaling symmetry and translation symmetry are both present, but they don't commute.

Sometimes, as for most fractals, only the self-similarity is present. So this method is useful for developing an analytic understanding of walks on fractal graphs, or more generally the spectrum of their adjacency matrix. I believe the original references are [this paper](#) and [this one](#). Roughly, we are going to learn how to compute the phonon contribution to the heat capacity of the broccoflower!



Let's solve (7.8) for the case of a chain, with $A_{ij} = t(\delta_{i,j+1} + \delta_{i,j-1})$. I've introduced a 'hopping amplitude' t which can be regarded as related to the length of the bonds.

The eigenvalue equation can be rewritten as

$$v_i = \frac{t}{\epsilon} (v_{i-1} + v_{i+1}). \quad (7.10)$$

Notice that if i is odd, then the entries on the RHS only involve even sites. So this equation eliminates v_i at the odd sites in terms of the values at the even sites. Plugging this back into the equation for an even site gives

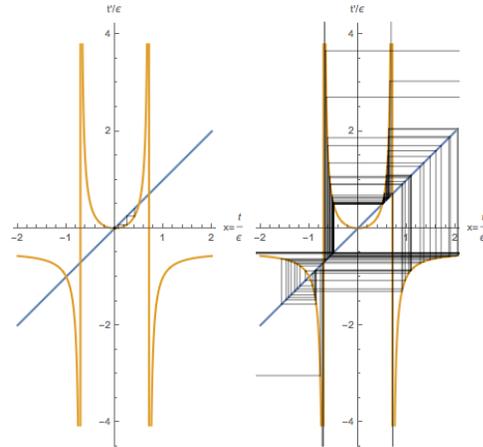
$$\begin{aligned} \epsilon v_{2l} &= t(v_{2l-1} + v_{2l+1}) = \frac{t^2}{\epsilon} (v_{2l-2} + v_{2l} + v_{2l} + v_{2l+2}) \\ \implies v_{2l} &= \underbrace{\frac{t^2}{\epsilon^2 - 2t^2}}_{\equiv t'/\epsilon} (v_{2l-2} + v_{2l+2}). \end{aligned}$$

This is the same equation as (7.10), but with half as many sites, *i.e.* the zoom factor is $b = 2$.

t' is a renormalized hopping amplitude:

$$\frac{t'}{\epsilon} = \frac{t^2}{\epsilon^2 - 2t^2} = \frac{(t/\epsilon)^2}{1 - 2(t/\epsilon)^2}.$$

This is a recursive map for the ratio $x = t/\epsilon$. Actually, it can be mapped to the logistic map $y \rightarrow ry(1 - y)$, with $r = 4$, by the change of variables $y = 4x^{-1} - 2$. A lot is known about this map.

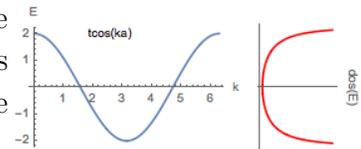


We can regard this recursion as a rule for growing the lattice (and all of its eigenvectors) starting from a small chunk of the stuff. How do we reconstruct the eigenvectors recursively? Suppose we start with a chain of 2^{n+1} sites and suppose we know an eigenstate v^{n, ϵ_n} for this case with $\epsilon_n \neq 0$. There is a solution on a lattice with twice as many sites with

$$v_{2j}^{n+1, \epsilon_{n+1}^\pm} = v_j^{n, \epsilon_n}, \quad v_{2j+1}^{n+1, \epsilon_{n+1}^\pm} = -\frac{v_j^{n, \epsilon_n} + v_{j-1}^{n, \epsilon_n}}{\epsilon_n^\pm}$$

where $\epsilon_{n+1}^\pm = \pm\sqrt{2 - \epsilon_n}$.

Let's cheat and remind ourselves of the known answer for the spectrum using translation invariance: $E(k) = 2t \cos ka$ ranges from $-2t$ to $2t$ as k varies over the BZ from 0 to $2\pi/a$. Let's use this to learn how to understand the iteration map.



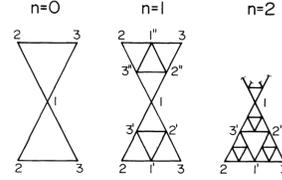
For the chain, the map has three fixed points, at $x = 0, \frac{1}{2}, -1$. Let's think of fixing E and varying the initial hopping rate. If $t_0 \in (-E/2, E/2)$ (that is, if $|E| > 2t$ is in the band gap) then $t_{n \rightarrow \infty} \rightarrow t^* = 0$ eventually reaches the fixed point at $x = 0$ (as in the left figure). More precisely, it goes like $t_n \stackrel{n \gg 1}{\sim} E e^{-2n\lambda}$ for some λ .

Such an orbit which asymptotes to $t \rightarrow 0$ can be described by decoupled clusters – the wavefunction is *localized*. I learned about this from [this paper](#).

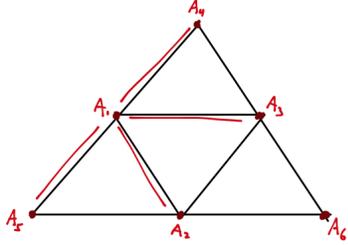
In contrast, one with finite or infinite asymptotic t is associated with an *extended* state. This happens if $|t_0| > E/2$ (so that $E \in (-2t, 2t)$ is in the band). Then $t_n > |E|/2$ for all n , and we have a nonzero effective hopping even between two sites that are arbitrarily far-separated.

The fixed point at $t_* = E/2$ is the state with $k = 0$, *i.e.* the uniform state.

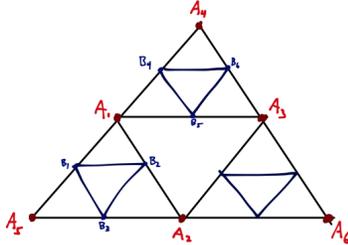
The procedure works for other examples, too, including some without translation invariance, where the spectrum can be quite different. Consider the Sierpinski triangle lattice.



[from Domany et al, linked above]



$$\epsilon A_1 = t(B_1 + B_2 + B_4 + B_5). \quad (7.11)$$



$$\begin{aligned} \epsilon B_1 &= t(A_1 + A_5 + B_2 + B_3), & \epsilon B_4 &= t(A_1 + A_4 + B_5 + B_6), \\ \epsilon B_2 &= t(A_1 + A_2 + B_1 + B_3), & \epsilon B_5 &= t(A_1 + A_3 + B_4 + B_6), \\ \epsilon B_3 &= t(A_2 + A_5 + B_1 + B_2), & \epsilon B_6 &= t(A_4 + A_3 + B_5 + B_4). \end{aligned}$$

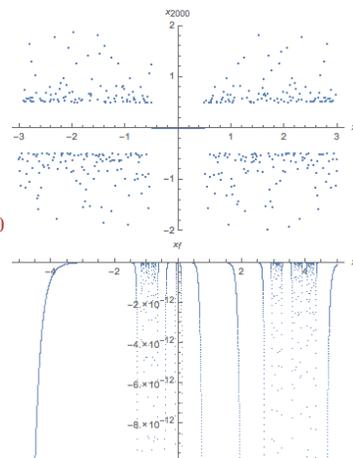
Eliminating the B sites by solving the previous six equations for them in terms of the A sites and plugging into (7.11) gives an equation of the same form on a coarser lattice

$$\epsilon A_1 = t'(A_2 + A_3 + A_4 + A_5), \quad t' = \frac{t^2}{\epsilon - 3t}.$$

Zoom factor is $b = \sqrt{2}$. In terms of the dimensionless ratio $x \equiv t/\epsilon$,

$$x \rightarrow \frac{x^2}{1 - 3x}$$

Here's a way to visualize the huge qualitative difference from this map relative to the result for the chain. Plot, as a function of some initial $x = t/\epsilon$, the value of the n th iterate, for some large value of n (here $10^5 \times 2$). For the chain (shown at the top), every x which starts in the band stays in the band ²⁰ ($x_n > 1/2$ if $x_0 > 1/2$), and vice versa. For the Sierpinski case, we get this Cantor-like set of localized states. Here the spacing on the x -axis is 10^{-2} ; if we scan more closely, we'll find more structure.



7.3 Spectral dimension

Here's one more notion of dimension, for a graph embedded in \mathbb{R}^d , following [Toulouse et al.](#) Think of the graph as a Debye solid, that is, put springs on the links of the graph, each with natural frequency $\omega_0^2 = K/m$. The normal modes of this collection of springs have frequencies ω with ω_n^2/ω_0^2 which are eigenvalues of the adjacency matrix.

The density of states of such modes for small ω is an ingredient in the heat capacity of the resulting model solid. Denote by $\rho(\omega)d\omega$ the number of modes with frequency in the interval $(\omega, \omega + d\omega)$.

For a translation-invariant system in d dimensions, the modes can be labelled by wavenumber and $\rho(\omega)d\omega = \mathfrak{d}^d k$ which at $\omega \rightarrow 0$ (in the thermodynamic limit) is governed by Goldstone's acoustic phonon with $\omega = v_s k$ and therefore $\rho(\omega) \propto \omega^{d-1}$. More generally, we define the *spectral dimension* d_s of the graph by the power law relation

$$\rho(\omega) \stackrel{N \rightarrow \infty}{\sim} \stackrel{\omega \rightarrow 0}{\sim} \omega^{d_s}.$$

Sometimes it's called the *diffusion dimension*. It is a useful idea! One cool application is to figuring out how many dimensions your average spacetime has when you do a simulation involving dynamical triangulations. (See §5.2 of [this paper](#).)

Now suppose that instead of translation-invariance, we have dilatation invariance, *i.e.* self-similarity. The number of sites for a graph Γ of linear size L scales as

$$N(L) \sim L^{D_\Gamma}$$

where D_Γ is the fractal dimension. This means that if we assemble a scaled up version whose linear size is scaled up by b , we have $N(bL) = b^{D_\Gamma} N(L)$ sites. And it means,

²⁰Thanks to Daniel Ben-Zion for help with these figures.

just by counting eigenvalues, that the density of states per unit cell must scale like

$$\rho_L(\omega) = b^{D_\Gamma} \rho_{L/b}(\omega). \quad (7.12)$$

Consider L finite so that the spectrum $\{\omega_n\}$ is discrete, and focus on the n th eigenvalue from the bottom, for some fixed n . If we knew that this eigenvalue scaled with system size like

$$\omega(L/b) = b^x \omega(L)$$

then

$$\begin{aligned} \rho_{L/b}(\omega) &= b^{-x} \rho_L(\omega b^{-x}) \\ \implies \rho_L(\omega) &\stackrel{(7.12), (7.13)}{=} b^{D_\Gamma - x} \rho_L(\omega b^{-x}) \stackrel{b=\omega^{1/x}}{\sim} \omega^{\frac{D_\Gamma - x}{x}}. \end{aligned} \quad (7.13)$$

Claim: The smooth part of the spectrum of the Sierpinski fractal solid does scale like b^x for some x which we can determine. A more earnest pursuit of the equations (7.11) implies that

$$\begin{aligned} \left(\frac{\omega^2}{\omega_0^2}\right) &\mapsto \left(\frac{(\omega')^2}{\omega_0^2}\right) = \left(\frac{\omega^2}{\omega_0^2}\right) \left(d + 3 - \left(\frac{\omega^2}{\omega_0^2}\right)\right) \\ \omega^2(L/2) &= \omega^2(L)(d + 3 + \mathcal{O}(\omega^2)) \equiv \omega^2(L)2^{2x} \\ \implies x &= \frac{\log(d + 3)}{2 \log 2}. \end{aligned}$$

(We used $b = 2$ since the number of sites per edge of the triangle is halved at each decimation step.) This means that the smooth part of the spectrum behaves as

$$\rho(\omega) \sim \omega^{\frac{D_\Gamma - x}{x}} = \omega^{d_s}.$$

7.4 Resistor networks

The resistor network on a Sierpinski d -gasket is studied [here](#). The scaling with size of the conductivity of stuff made from such a graph can be related to its spectral dimension.

Unlike the paragon of [nerd-sniping problems](#) (the resistor network on the square lattice), this problem cannot be solved by going to momentum space.

Consider sending a current I into one corner of a Sierpinski gasket. By symmetry, a current I/d must emerge from the other d corners.

Call $\rho(a)$ the resistance of one bond with lattice spacing a . Now we want to compute the effective, coarse-grained resistance $\rho(ba)$ for $b > 1$. The symmetry of the problem

forbids current from crossing the middle of the triangle, and this allows us to compute the voltage drop between the input corner and any of the others. Specifically, this voltage drop is preserved if

$$\rho(ba) = \frac{d+3}{d+1}\rho(a) \equiv b^\zeta \rho(a)|_{b=2}$$

$$\zeta = \frac{\log \frac{d+3}{d+1}}{\log 2}.$$

Now if we iterate this map ℓ times so that $b^\ell = \frac{L}{a}$ for some macroscopic L , then the resistance of the whole chunk of stuff is

$$\rho(L) \sim L^\zeta$$

and the conductivity of the stuff (in $\vec{J} = \sigma \vec{E}$, an intensive quantity) is

$$\sigma(L) = \frac{L^{2-d}}{\rho(L)} \sim L^{-t}$$

with scaling exponent $t = d - 2 + \zeta$.

Exercise: relate ζ to the spectral dimension d_s .

[End of Lecture 19]

8 RG sampler platter

In the last lecture, I want to convey some more the huge range of applications of the renormalization group perspective.

8.1 Desordre

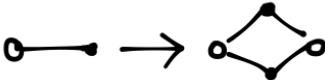
[McComb p.60; Creswick, chapter 3.]

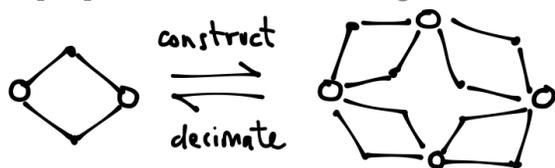
I want to emphasize that RG of couplings is a subset of RG of probability distributions.

So far in this course, we've been studying *clean* systems, ones whose couplings are the same at each location. It is often important and interesting to consider the case where the couplings are only uniform *on average*. Just encoding the values of the couplings in one realization of the system is then quite a job, never mind computing the resulting free energy. But for large systems, we can often appeal yet again to the central idea of statistical physics and choose the couplings from some probability distribution. (A physical quantity for which this assumption works is said to be 'self-averaging'.)

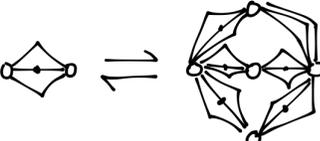
This probability distribution will then itself evolve under the RG.

Let's consider a case where we can study this in detail, namely the nearest-neighbor Ising ferromagnet on *hierarchical graphs*. Such a graph can be constructed by a sprouting rule: at each step of the construction, replace each link with some given motif.

For example, the sprouting rule  produces the *diamond hierarchical lattice*. I denote the new sites in black. The beauty of this construction for our purposes is that decimating the black sites precisely undoes the construction step:

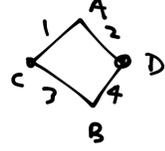


The generalization which replaces each link with q segments is called the Berker

lattice, I think. For $q = 3$, this looks like: 

Let $v_{\langle ij \rangle} \equiv \tanh \beta J_{\langle ij \rangle}$. Consider tracing over the black sites A and B in the figure at right. Using the high-temperature-expansion formula, this isn't hard:

$$\begin{aligned}
e^{-\Delta H_{\text{eff}}(s_C, s_D)} &= \sum_{s_A, s_B = \pm 1} e^{-H(s)} = \sum_{s_A, s_B \text{ links}, \langle ij \rangle} \prod (1 + v_{\langle ij \rangle} s_i s_j) \\
&= 2^2 (1 + v_1 v_2 s_C s_C) (1 + v_3 v_4 s_C s_D) \\
&= 2^2 ((1 + v_1 v_2 v_3 v_4) + (v_1 v_2 + v_3 v_4) s_C s_D) \\
&= 2^2 (1 + v_1 v_2 v_3 v_4) (1 + v' s_C s_D)
\end{aligned}$$



with

$$v'(v_1..v_4) = \frac{v_1 v_2 + v_3 v_4}{1 + v_1 v_2 v_3 v_4}. \quad (8.1)$$

In the clean limit where all couplings are the same, this is

$$v' = \frac{2v^2}{1 + v^4}.$$

This has fixed points at

$$v^* = 0, 1, 0.0437. \quad (8.2)$$

Just as we did for the Ising chain in §3, we can study the behavior near the nontrivial fixed point and find (here $b = \sqrt{2}$) that $\nu \simeq 1.338$. Redoing this analysis to include also a magnetic field, we would find $y_h = 1.758$ for the magnetization exponent.

But now suppose that the couplings are chosen from some initial product distribution, independently and identically distributed. Some examples for the bond distribution to consider are:

$$\begin{aligned}
\text{Random bond dilution:} & \quad P(J) = x\delta(J - J_0) + (1 - x)\delta(J) \\
\text{Source of frustration:} & \quad P(J) = x\delta(J - J_0) + (1 - x)\delta(J + J_0) \\
\text{Edwards-Anderson spin glass:} & \quad P(J) \propto \exp -\frac{J^2}{2J_0^2}
\end{aligned}$$

After the decimation step, the distribution for any link evolves according to the usual formula for changing variables in a probability distribution, using the RG relation (8.1):

$$P'(v') = \int dv_1 dv_2 dv_3 dv_4 \delta(v' - v'(v_1..v_4)) P(v_1) \cdots P(v_4).$$

The preceding relation is then an RG recursion equation for the *distribution* of couplings $P(v) \mapsto (\mathcal{R}(P))(v)$. As usual when confronted with such a recursion, we should ask about its fixed points, this case fixed distributions:

$$P_\star(v) \stackrel{!}{=} \int dv_1 dv_2 dv_3 dv_4 \delta(v - v'(v_1..v_4)) P_\star(v_1) \cdots P_\star(v_4). \quad (8.3)$$

We know some solutions of this equation. One is

$$P_\star(v) = 1\delta(v - v_\star)$$

with v_\star given by one of the solutions in (8.2).

Another set of fixed points is associated with bond percolation. Here's what I mean about percolation. Consider the $T \rightarrow 0$ (or $J \rightarrow \infty$) limit of a ferromagnetic NN Ising model on a graph, with one catch: For each pair of neighbors, we randomly decide whether or not to place a link. This is a model of *bond percolation*. This is realized in our system here by a distribution of the form

$$P_x(v) = x\delta(v - 1) + (1 - x)\delta(v).$$

Plugging this ansatz into the map gives

$$P'_x(v) = (x^4 + 4x^3(1 - x) + 2x^2(1 - x)^2) \delta(v-1) + (4x^2(1 - x^2) + 4x(1 - x)^3 + (1 - x)^4) \delta(v)$$

where the terms come from enumerating which of the four bonds is zero. So: the distribution is self-similar, but the bond-placing probability x evolves according to

$$x \mapsto x' = 2x^2 - x^4.$$

So each fixed point of this map gives a solution the fixed-distribution equation (8.3). They occur at

$$x^\star = \begin{cases} 0, & \text{nobody's home} \\ 1, & \text{everybody's home} \\ \frac{\sqrt{5}-1}{2}, & \text{percolation threshold on the DHL.} \end{cases}$$

We can study (a subset of) flows between these fixed points if we make the more general ansatz

$$p(v) = x\delta(v - v_0) + (1 - x)\delta(v)$$

with two parameters v_0, x . Then we get a 2d map, much more manageable, if the evolution preserves the form. It almost does. The evolution rule can be estimated by

$$\begin{aligned} x' &= -x^4 + 2x \\ x'v'_0 &= \langle v \rangle_{P'} \equiv \int dv v P'(v). \end{aligned} \tag{8.4}$$

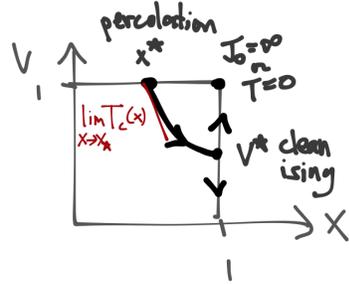
The result is

$$x'v'_0 = \int dv \int \prod_{i=1}^4 dv_i \delta(v - v'(v_1..v_4)) v \prod_i p(v_i)$$

$$\begin{aligned}
&= \int \prod_i dv_i \frac{v_1 v_2 + v_3 v_4}{1 + v_1 v_2 v_3 v_4} \prod_i p(v_i) \\
&= x^4 \frac{2v_0^2}{1 + v_0^4} + 4x^3(1-x)v_0^2 + 2x^2(1-x)^2v_0^2.
\end{aligned}$$

In the second line we used the delta function to do the v integral.

Indicated by the thick black link is the critical surface of the clean Ising fixed point at $x = 1, v = v^*$. The percolation fixed point at $x = x_*, v = 1$ is unstable to the clean Ising fixed point. Besides the structure of the phase diagram, we can infer the angle at which the Ising critical surface approaches $x = x_*$: $T_c(x) \xrightarrow{x \rightarrow x_*} -\frac{1}{\log(x-x_*)}$.



Strong-disorder RG. There is a simplifying limit where the distribution of the couplings is very broad. Such a distribution is sometimes an attractive fixed point of the RG, called a strong-disorder or even infinite-disorder fixed point depending on the extremity. This limit is simplifying because then we can order the RG analysis by looking at the largest coupling $t_i \equiv \Omega$ first, and we can use $\frac{t_{j \neq i}}{\Omega} \ll 1$ as a small parameter. A useful reference is [this paper by Altman and Refael](#). A more detailed but scarier discussion is [this review](#).

Let's analyze the example of an adjacency matrix of a graph with random values of the hopping parameter for each link. For simplicity let's think about the case where the graph is a chain.

So we want to solve

$$\begin{pmatrix} \ddots & & & & \\ & 0 & t_i & & \\ & t_i & 0 & t_{i+1} & \\ & & t_{i+1} & 0 & t_{i+2} \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ v_{i-1} \\ v_i \\ v_{i+1} \\ \vdots \end{pmatrix} = \epsilon \begin{pmatrix} \vdots \\ v_{i-1} \\ v_i \\ v_{i+1} \\ \vdots \end{pmatrix}$$

With some random t_i chosen from some broad distribution, so that individual t_i will be very different from each other. Consider the largest $t_i \equiv \mathcal{T}$, and assume that it is much bigger than all the others, including its neighbors. Then we can eliminate the two sites connected by the strong bond by solving the 2×2 problem

$$\begin{pmatrix} 0 & \mathcal{T} \\ \mathcal{T} & 0 \end{pmatrix} \begin{pmatrix} v_{i-1} \\ v_i \end{pmatrix} \simeq \epsilon \begin{pmatrix} v_{i-1} \\ v_i \end{pmatrix}.$$

More precisely, we can eliminate these two sites v_{i-1}, v_i in terms of their neighbors

using their two rows of the eigenvalue equation:

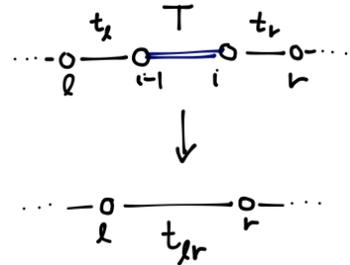
$$\begin{pmatrix} \ddots & & & & & & \\ & 0 & t_\ell & & & & \\ & t_\ell & 0 & \mathcal{T} & & & \\ & & \mathcal{T} & 0 & t_r & & \\ & & & t_r & 0 & & \\ & & & & & \ddots & \end{pmatrix} \begin{pmatrix} \vdots \\ v_\ell \\ v_{i-1} \\ v_i \\ v_r \\ \vdots \end{pmatrix} = \epsilon \begin{pmatrix} \vdots \\ v_\ell \\ v_{i-1} \\ v_i \\ v_r \\ \vdots \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} v_{i-1} \\ v_i \end{pmatrix} = \begin{pmatrix} -\epsilon & \mathcal{T} \\ \mathcal{T} & -\epsilon \end{pmatrix}^{-1} \begin{pmatrix} t_\ell \\ t_r \end{pmatrix}$$

The result of plugging this back into the neighboring rows of the equation is to make an effective hopping between ℓ and r of approximate strength

$$t_{\ell r} \sim \frac{t_\ell t_r}{\mathcal{T}}. \quad (8.5)$$

(I approximated $\epsilon \ll T$.)



This RG rule (8.5) (which we could name for Dasgupta and Ma in a slightly more fancy context) is very simple in terms of the logs of the couplings, $\zeta \equiv \log \mathcal{T}/t$:

$$\zeta' = \zeta_\ell + \zeta_r$$

– they just add. Here $\zeta \in (0, \infty)$ and $\zeta = 0$ is the strongest bond.

Let's make an RG for the probability distribution using this rule. The first step is to reduce the UV cutoff, by decimating the highest-energy or shortest distance degrees of freedom. Here the UV cutoff is just the largest hopping parameter, \mathcal{T} . Then the new effective bonds have the distribution (in the log)

$$P_{\text{new}}(\zeta) = \int_0^\infty d\zeta_\ell \int_0^\infty d\zeta_r P(\zeta_\ell) P(\zeta_r) \delta(\zeta - \zeta_\ell - \zeta_r).$$

Imagine we start the RG at some initial strongest bond \mathcal{T}_0 . Then $\Gamma = \log \mathcal{T}_0/\mathcal{T}$ says how much RGing we've done so far. The second rescaling step puts the distribution back in the original range, which requires shifting everyone

$$\zeta_i = \log \left(\frac{\mathcal{T}}{t_i} \right) \mapsto \log \left(\frac{\mathcal{T} - d\mathcal{T}}{t_i} \right) \simeq \zeta_i - \frac{d\mathcal{T}}{\mathcal{T}} = \zeta_i - d\Gamma$$

This moves the whole distribution to the left: $P(\zeta) \mapsto P(\zeta + d\Gamma) = P(\zeta) + d\Gamma P'(\zeta) + \mathcal{O}(d\Gamma^2)$, *i.e.*

$$d_{\text{rescale}} P(\zeta) = \frac{dP(\zeta)}{d\zeta} d\Gamma.$$

The change in the full distribution from adding in the new bonds is

$$d_{\text{new bonds}}P(\zeta) = \underbrace{d\Gamma P(0)}_{\text{strongest bond}} P_{\text{new}}(\zeta).$$

And the full evolution is

$$\frac{dP(\zeta)}{d\Gamma} = \frac{dP(\zeta)}{d\zeta} + P(0) \int_0^\infty d\zeta_\ell \int_0^\infty d\zeta_r P(\zeta_\ell) P(\zeta_r) \delta(\zeta - \zeta_\ell - \zeta_r).$$

This equation has a simple solution:

$$\begin{aligned} P_\Gamma(\zeta) = f(\Gamma)e^{-f(\Gamma)\zeta} &\implies \partial_\Gamma f = -f^2 \implies f(\Gamma) = \frac{1}{\Gamma}. \\ &\implies P_\Gamma(\zeta) = \frac{1}{\Gamma} e^{-\zeta/\Gamma}. \end{aligned}$$

In terms of the hoppings, this is

$$P_T(t) \sim \left(\frac{1}{t}\right)^{1-\frac{1}{\Gamma}} \xrightarrow{\Gamma \rightarrow \infty} \frac{1}{t}$$

– as we zoom out we approach a non-normalizable distribution. This is an infinite-randomness fixed point.

What is this analysis good for? For one thing, we can estimate the fraction of undecimated sites at each step. Each time we decimate a link, we remove two sites. Therefore, the number of undecimated sites evolves by

$$dN = -2P_\Gamma(0)N d\Gamma.$$

Approaching the fixed point, $P_\Gamma(0) = f(\Gamma) = \frac{1}{\Gamma}$, so the solution is

$$N(\Gamma) = \frac{N_0}{\Gamma^2} \sim \frac{N_0}{\log^2(\mathcal{T}_0/t)}.$$

The average distance between surviving sites is $L(\Gamma) \sim a \frac{N_0}{N(\Gamma)} \sim a\Gamma^2 \sim a \log^2(\mathcal{T}_0/t)$.

Here's an attempt at a physical application. Let's go back to using the spectrum of the (random-entry) adjacency matrix to determine the heat capacity of a Debye-Einstein solid. Let's add in a diagonal (Einstein) spring constant:

$$H = \sum_i \left(\frac{p_i^2}{2} + \omega_0^2 x_i^2 \right) + \omega_0^2 \sum_{ij} A_{ij} (x_i - x_j)^2.$$

So the spectrum of normal modes is

$$\omega_n^2 = \omega_0^2 (1 + \epsilon_n)$$

where ϵ_n are the eigenvalues of A . And we take $A_{ij} = (\delta_{i,i+1} + \delta_{i,i-1}) t_i$ and choose t_i from the strong-disorder distribution found above.

To find the heat capacity at temperature $1/\beta$, we should run the RG from some initial UV cutoff \mathcal{T}_0 down to the \mathcal{T} associated with temperature T , which is of order $\frac{T}{\omega_0}$. Because of the breadth of the distribution, the bonds with $t < T^2$ are likely to have $t \ll T^2$ and we can ignore them. Any site not participating in a bond produces a simple equipartition contribution $\Delta E = k_B T$ (i.e. it adds a constant to C_V) as long as $1/\beta > \Omega_0$. Sites participating in a bond have $\omega \ll T$ and are frozen out²¹ So the heat capacity is

$$C_V(\beta) = N(T)$$

where $N(T)$ is the number of undecimated sites when the temperature is T , which means here that the RG scale is $\mathcal{T} \sim T^2$. So this model produces a *crazy* dependence on the temperature,

$$C_V \sim \frac{1}{\log^2(T^2)}.$$

[End of Lecture 20]

8.2 RG viewpoint on matched asymptotic expansions

[Goldenfeld chapter 10]

It is possible to get anomalous dimensions even from systems without any stochastic element (*i.e.* thermal fluctuations or quantum fluctuations). It is even possible to get them from *linear* differential equations. The latter is demonstrated, for example, by [this analysis](#). (I apologize that I have not found the time to figure out how to explain this without all the surrounding complications.)

Goldenfeld gives an extended discussion of a diffusion equation, perturbed by a nonlinear, singular term, called the Barenblatt equation.

8.3 RG approach to the period doubling approach to chaos

[Creswick ch 2, Strogatz, Dan Arovav's 200B notes!]

²¹At the last possible moment, I have violated my promise not to use quantum mechanics. Sorry.