1. Heavy leptons get real.

Consider the contribution of a single loop of a heavy lepton of mass $M$ to the vacuum polarization. Find the imaginary part of $\text{Im}\Pi_L(q^2)$. Show that it is independent of the cutoff. Check that it agrees with the optical theorem result for the $e^+e^-\rightarrow L^+L^-$ cross section.

2. Another consequence of unitarity of the $S$ matrix.

(a) Show that unitarity of $S$, $S^\dagger S = 1 = SS^\dagger$, implies that the transition matrix is normal:

$$T T^\dagger = T^\dagger T.$$  \hfill (1)

(b) What does this mean for the amplitudes $M_{\alpha\beta}$ (defined as usual by $T_{\alpha\beta} = \delta(p_\alpha - p_\beta)|M_{\alpha\beta}|^2$)?

(c) The probability of a transition from $\alpha$ to $\beta$ is

$$P_{\alpha\rightarrow\beta} = |S_{\beta\alpha}|^2 = VT\delta(p_\alpha - p_\beta)|M_{\alpha\beta}|^2$$

which is IR divergent. More useful is the transition rate per unit time per unit volume:

$$\Gamma_{\alpha\rightarrow\beta} \equiv \frac{P_{\alpha\rightarrow\beta}}{VT}.$$  

Show that the the total decay rate of the state $\alpha$ is

$$\Gamma_{\alpha} \equiv \int d\beta \Gamma_{\alpha\rightarrow\beta} = 2\text{Im}M_{\alpha\alpha}.$$  

(d) Consider an ensemble of states $p_\alpha$ evolving according to the evolution rule

$$\partial_t p_\alpha = -p_\alpha \Gamma_{\alpha} + \int d\beta p_\beta \Gamma_{\beta\rightarrow\alpha}.$$  

$S[p] \equiv -\int d\alpha p_\alpha \ln p_\alpha$ is the Shannon entropy of the distribution. Show that

$$\frac{dS}{dt} \geq 0$$

as a consequence of (1). This is a version of the Boltzmann $H$-theorem.
(e) [Bonus] Notice that we are doing something weird in the previous part by using classical probabilities. This is a special case; more generally, we should describe such an ensemble by a density matrix $\rho_{\alpha\beta}$. Generalize the result of the previous part appropriately.