1. Brain-warmer: Spectral representation at finite temperature.

In lecture we have derived a spectral representation for the two-point function of a scalar operator in the vacuum state

\[ iD(x) = \langle 0 | \mathcal{T} \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle \]

Derive a spectral representation for the two-point function of a scalar operator at a nonzero temperature:

\[ iD_\beta(x) \equiv \text{tr} \frac{e^{-\beta H}}{Z_\beta} \mathcal{T} \mathcal{O}(x) \mathcal{O}^\dagger(0) = \frac{1}{Z_\beta} \sum_n e^{-\beta E_n} \langle n | \mathcal{T} \mathcal{O}(x) \mathcal{O}^\dagger(0) | n \rangle \, . \]

Here \( Z_\beta \equiv \text{tr} e^{-\beta H} \) is the thermal partition function. Check that the zero temperature (\( \beta \to \infty \)) limit reproduces our previous result. Assume that \( \mathcal{O} = \mathcal{O}^\dagger \) if you wish.

The idea is again to insert a resolution of the identity in between the operators. All the steps as for the vacuum correlators go through, the only difference being that instead of arriving at a sum of squares of matrix elements of the operator between the vacuum and an arbitrary state, we get matrix elements between pairs of states:

\[ iD(x) = Z_\beta^{-1} \sum_n e^{-\beta E_n} \sum_m \| \mathcal{O}_{nm} \|^2 \left( e^{ix(p_n-p_m)} \theta(t) + e^{ix(p_m-p_n)} \theta(-t) \right) \, . \]

From here, the momentum space representation follows as before. When \( \beta \to \infty \), only the groundstate contributes (assume it is nondegenerate) and \( Z_\beta \to e^{-\beta E_0} \).

2. One more consequence of unitarity. [From Aneesh Manohar]

A general statement of the optical theorem is:

\[ -i \left( \mathcal{M}(a \to b) - \mathcal{M}(b \to a) \right) = \sum_f \int d\Phi_f \mathcal{M}^*(b \to f) \mathcal{M}(a \to f) \, . \]

Consider again QED with electrons and muons. The first part of the problem you basically did on the previous problem set, but I need to set up some notation.
(a) Consider scattering of an electron \((e^-)\) and a positron \((e^+)\) into \(e^-e^+\) (so \(a = b\) in the notation above). We wish to consider the contribution to the imaginary part of the amplitude for this process which is proportional to \(Q_e^2Q_\mu^2\) where \(Q_e\) and \(Q_\mu\) are the electric charges of the electron and muon (which are in fact numerically equal but never mind that). Draw the relevant Feynman diagram, and compute the imaginary part of this amplitude (just the \(Q_e^2Q_\mu^2\) bit) as a function of \(s \equiv (k_1 + k_2)^2\) where \(k_{1,2}\) are the momenta of the incoming \(e^+\) and \(e^-\).

(b) Use the optical theorem and the fact that the total cross section for \(e^+e^- \rightarrow \mu^+\mu^-\) must be positive

\[
\sigma(e^+e^- \rightarrow \mu^+\mu^-) \geq 0
\]

to show that a Feynman diagram with a fermion loop must come with a minus sign.

If we left out the minus sign, we would get a negative cross section.


(a) Show that the adjoint representation matrices

\[
(T^B)_{AC} \equiv -i f_{ABC}
\]

furnish a dim \(G\)-dimensional representation of the Lie algebra

\[
[T^A, T^B] = i f_{ABC} T^C
\]

Hint: commutators satisfy the Jacobi identity

\[
\]

The structure constants \(f_{C}^{AB}\) are part of the definition of the Lie algebra – in any representation, the generators satisfy \([T^A, T^B] = i f_{ABC} T^C\). This is a property of the algebra, not of any particular representation. The Jacobi identity follows from this fact, by taking the commutator of the BHS with \(T^D\). Reshuffling this identity gives the desired equation (up to a sign which may be flipped by redefining \(T \rightarrow -T\)).

More abstractly, the operation \(B \rightarrow ad_A(B) \equiv [A, B]\) is called the adjoint action of \(A\) on \(B\). The Jacobi identity is then the statement that \(ad_A ad_B(C) - ad_B ad_A(C) = ad_{[A,B]}(C)\), i.e.\([ad_A, ad_B] = ad_{[A,B]}\). This is the statement that the map \(A \rightarrow ad_A\) preserves the Lie algebra, and hence gives
a representation, which is inevitably called the adjoint representation. In terms of the generators of an arbitrary representation, \(ad_{T^A}T^B = [T^A, T^B] = if_{ABC}T^C\), we find an expression for the adjoint generators, which is indeed \((T_{\text{adj}}^A)_{BC} = if_{ABC}\) with the opposite sign from what I said.

(b) From the transformation law for \(A\), show that the non-abelian field strength transforms in the adjoint representation of the gauge group.

Mindlessly plugging in, we have

\[
F_{\mu\nu} A^A \leftrightarrow \partial_{\mu} \left( A^A_{\nu} + \partial_{\nu} \lambda^A - f_{ABC} \lambda^B A^C_{\nu} \right) - (\mu \leftrightarrow \nu)
\]

\[
+ f_{ABC} f \left( A^B_{\mu} + \partial_{\mu} \lambda^B - f_{BDE} \lambda^D A^E_{\mu} \right) \left( A^C_{\nu} + \partial_{\nu} \lambda^C - f_{CFG} \lambda^F A^G_{\nu} \right)
\]

\[
= F_{\mu\nu}^A - f_{ABC} \lambda^B \partial_{\mu} A^C_{\nu} + f_{ABC} \lambda^B \partial_{\nu} A^C_{\mu} - f_{ABC} f_{BDE} \lambda^D A^E_{\mu} A^C_{\nu} - f_{ABC} f^B f^C f^D
\]

(1)

\[
= F_{\mu\nu}^A - f_{ABC} \lambda^B \partial_{\mu} A^C_{\nu} + f_{ABC} \lambda^B \partial_{\nu} A^C_{\mu} - \lambda^D A^E_{\mu} A^C_{\nu} (f_{ABC} f_{BDE} + f_{ABC} f_{BDE})
\]

(2)

\[
= F_{\mu\nu}^A - f_{ABC} \lambda^B \partial_{\mu} A^C_{\nu} + f_{ABC} \lambda^B \partial_{\nu} A^C_{\mu} - \lambda^D A^E_{\mu} A^C_{\nu} f_{ADB} f_{BDC}
\]

(3)

\[
= F_{\mu\nu}^A - f_{ABC} \lambda^B \partial_{\mu} A^C_{\nu} + f_{ABC} \lambda^B \partial_{\nu} A^C_{\mu} - \lambda^B A^D_{\mu} A^E_{\nu} f_{ABC} f_{CDE}
\]

(4)

\[
= F_{\mu\nu}^A - \lambda^B f_{ABC} F_{\mu\nu}^C.
\]

(5)

Everywhere we ignored \(O(\lambda^2)\) terms. At step (2) we used the Jacobi identity.

At steps (1) and (3) we relabelled dummy indices.


Consider the Abelian Higgs model in \(D = 3 + 1\):

\[
L_h \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} |D_{\mu} \phi|^2 - V(|\phi|)
\]

where \(\phi\) is a scalar field of charge \(e\) whose covariant derivative is \(D_{\mu} \phi = (\partial_{\mu} - iq A_{\mu}) \phi\), and let’s take

\[
V(|\phi|) = \frac{\kappa}{2} (|\phi|^2 - v^2)^2
\]

for some couplings \(\kappa, v\). Here we are going to do some interesting classical field theory. Set \(q = 1\) for a bit.

(a) Consider a configuration which is independent of \(x^3\), one of the spatial co-ordinates, and static (independent of time). Show that its energy density (energy per unit length in \(x^3\)) is

\[
U = \int d^2x \left( \frac{1}{2} F_{12}^2 + \frac{1}{2} |D_t \phi|^2 + V(|\phi|) \right).
\]
(b) Consider the special case where $\kappa = 1$. Assuming that the integrand falls off sufficiently quickly at large $x^{1,2}$, show that

$$U_{\kappa=1} = \int d^2 x \left( \frac{1}{2} (F_{12} + |\phi|^2 - v^2)^2 + \frac{1}{4} |D_i \phi + i \epsilon_{ij} D_j \phi|^2 + v^2 F_{12} - \frac{1}{2} i \epsilon_{k\ell} \partial_k (\phi^* D_\ell \phi) \right).$$

(c) The first two terms in the energy density of the previous part are squares and hence manifestly positive, so setting to zero the things being squared will minimize the energy density. Show that the resulting first-order equations (they are called BPS equations after people with those initials, Bogolmonyi, Prasad, Sommerfeld)

$$0 = (D_i + i \epsilon_{ij} D_j) \phi, \quad F_{12} = -|\phi|^2 + v^2$$

are solved by ($x^1 + i x^2 \equiv re^{i \varphi}$)

$$\phi = e^{i n \varphi} f(r), \quad A_1 + i A_2 = -i e^{i \varphi} \frac{a(r) - n}{r}$$

if

$$f' = \frac{a}{r} f, \quad a' = r (f^2 - v^2)$$

with boundary conditions

$$a \to 0, f \to v + O(e^{-mr}), \quad \text{at } r \to \infty \quad (6)$$

$$a \to n + O(r^2), f \to r^n (1 + O(r^2)), \quad \text{at } r \to 0.$$

(For other values of $\kappa$, the story is not as simple, but there is a solution with the same qualitative properties. See for example §3.3 of E. Weinberg, Classical solutions in Quantum Field Theory.)

(d) The second BPS equation and (6) imply that all the action (in particular $F_{12}$) is localized near $r = 0$. Evaluate the magnetic flux through the $x^1 - x^2$ plane, $\Phi \equiv \int B \cdot da$ in the vortex configuration labelled by $n$. Show that the energy density is $U = \frac{v^2}{2} \cdot \Phi$.

The tricky bit is that the second term is

$$- \int d^2 x \frac{1}{2} i \epsilon_{k\ell} \partial_k (\phi^* D_\ell \phi) = -\frac{1}{2} i \int d (\phi^* D_\ell \phi dx^\ell) = -\frac{1}{2} i \oint_{C_\infty} (\phi^* D_\ell \phi dx^\ell)$$

$$= -\frac{1}{2} \oint_{C_\infty} A |\phi|^2 = -\frac{1}{2} \Phi v^2. \quad (7)$$
(e) Show that the previous result for the flux follows from demanding that the two terms in $D_i\phi$ cancel at large $r$ so that

$$D_i\phi \overset{r \to \infty}{\to} 0$$  \hspace{1cm} (9)$$

faster than $1/r$. Solve (9) for $A_i$ in terms of $\phi$ and integrate $\int d^2xF_{12}$.

Zee page 307 (with charge $q$):

$$A_i \overset{r \to \infty}{\to} -\frac{i}{qe} \frac{1}{\rho^2} \rho^i \partial_i \phi = \frac{1}{qe} \partial_i \varphi.$$ 

Therefore

$$\Phi = \int d^2xF_{12} = \oint dx A_i = \frac{2\pi}{qe}.$$ 

(f) Argue that a single vortex (string) in the ungauged theory (with global $U(1)$ symmetry)

$$\mathcal{L} = |\partial \phi|^2 + V(|\phi|)$$

does not have finite energy per unit length. By a vortex, I mean a configuration where $\phi \overset{r \to \infty}{\to} ve^{ik\varphi}$, where $x^1 + ix^2 = re^{i\varphi}$.

The kinetic energy density is

$$\partial_{\mu} \phi \partial^{\mu} \phi = |\partial_r \phi|^2 + r^{-2} |\partial_\varphi \phi|^2 = ... + r^{-2} v^2 n^2$$

so the energy per unit length is

$$U \geq \int_a^L dr r \frac{1}{r^2} v^2 n^2 = v^2 n^2 \ln \frac{L}{a}$$

where $L$ is the size of the box and $a$ is the short distance cutoff.

(g) Consider now the case where the scalar field has charge $q$. (Recall that in a superconductor made by BCS pairing of electrons, the charged field which condenses has electric charge two.) Show that the magnetic flux in the core of the minimal ($n = 1$) vortex is now (restoring units) $\frac{hc}{qe}$.

5. **BPS conditions from supersymmetry.** [bonus!] What’s special about $\kappa = 1$? For one thing, it is the boundary between type I and type II superconductors. More sharply, it means the mass of the scalar equals the mass of the vector, at least classically. Moreover, in the presence of some extra fermionic fields, the model with this coupling has an additional symmetry mixing bosons and fermions, namely supersymmetry. This symmetry underlies the special features we found above. Here is an outline (you can do some parts without doing others) of how this works. The logic in part (c) underlies a lot of the progress in string theory since the mid-1990s. Please do not trust my numerical factors.
(a) Add to $\mathcal{L}_h$ a charged fermion $\Psi$ (partner of $\phi$) and a neutral Majorana fermion $\lambda$ (partner of $A_\mu$):

$$\mathcal{L}_f = \frac{1}{2} i \bar{\Psi} \not{D} \Psi + i \bar{\lambda} \not{D} \lambda + \bar{\lambda} \Psi \phi + h.c..$$

Consider the transformation rules

$$\delta_\epsilon A_\mu = i \bar{\epsilon} \gamma_\mu \lambda, \quad \delta_\epsilon \Psi = D_\mu \phi \gamma^\mu \epsilon, \quad \delta_\epsilon \phi = -i \bar{\epsilon} \Psi, \quad \delta_\epsilon \lambda = -\frac{1}{2} i \sigma^{\mu\nu} F_{\mu\nu} \epsilon + i (|\phi|^2 - v) \epsilon$$

where the transformation parameter $\epsilon$ is a Majorana spinor (and a Grassmann variable). Show that (something like this) is a symmetry of $\mathcal{L} = \mathcal{L}_h + \mathcal{L}_f$. This is $\mathcal{N} = 1$ supersymmetry in $D = 4$.

(b) Show that the conserved charges associated with these transformations $Q_\alpha$ (they are Grassmann objects and spinors, since they generate the transformations, via $\delta_\epsilon$ fields $= [\epsilon_\alpha Q_\alpha + h.c., \text{fields}]$), satisfy the algebra

$$\{Q, \bar{Q}\} = 2 \gamma^\mu P_\mu + 2 \gamma^\mu \Sigma_\mu \quad (10)$$

where $P_\mu$ is the usual generator of spacetime translations and $\Sigma_\mu$ is the *vortex string charge*, which is nonzero in a state with a vortex string stretching in the $\mu$ direction. $\bar{Q} \equiv Q^\dagger \gamma^0$ as usual.

(c) If we multiply (10) on the right by $\gamma^0$, we get the positive operator $\{Q_\alpha, Q_\beta^\dagger\}$. This operator annihilates states which satisfy $Q \ket{BPS} = 0$ for some components of $Q$. Such a state is therefore invariant under some subgroup of the supersymmetry, and is called a BPS state. Now look at the right hand side of (10)$\times \gamma^0$ in a configuration where $\Sigma_3 = \pi n v^2$ and show that its energy density is $E \geq \pi |n| v^2$, with the inequality saturated only for BPS states.

(d) To find BPS configurations, we can simply set to zero the relevant supersymmetry variations of the fields. Since we are going to get rid of the fermion fields anyway, we can set them to zero and consider just the (bosonic) variations of the fermionic fields. Show that this reproduces the BPS equations.