1. **Brain-warmer.** Show that
\[ \text{tr} F \wedge F = d\text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \]

Write out all the indices I’ve suppressed.

[Bonus] If you are feeling under-employed, find \( \omega_{2n-1} \) such that \( \text{tr} F^n = d\omega_{2n-1} \).

2. **The field of a magnetic monopole.**

A magnetic monopole of magnetic charge \( g \) is defined by the condition that \( \int_{S^2} F = g \), where \( S^2 \) any sphere surrounding the monopole. If the system is spherically symmetric, we can write
\[ F = \frac{g}{4\pi} d\cos \theta d\varphi. \]

(In this problem, we’ll work on a sphere at fixed distance from the monopole.)

(a) Show that the vector potential
\[ A_N = \frac{g}{4\pi} (\cos \theta - 1) d\varphi \]
gives the correct \( F = dA \). Show that it is a well-defined one-form on the sphere except at the south pole \( \theta = \pi \).

(b) Show that the one-form
\[ A_S = \frac{g}{4\pi} (\cos \theta + 1) d\varphi \]
also gives the correct \( F = dA \). Show that it is well-defined except at the north pole \( \theta = 0 \).

(c) Near the equator both \( A_N, S \) are well-defined. Show that as long as \( eg \in 2\pi\mathbb{Z} \), these two one-forms differ by a gauge transformation
\[ A_S - A_N = \frac{1}{ie} g^{-1}(\theta, \varphi) dg(\theta, \varphi) \]
for \( g(\theta, \varphi) \) a \( U(1) \)-valued function on the sphere, well-defined away from the poles.
3. **Wilson loops in abelian gauge theory at weak and strong coupling.**

(a) At weak coupling, the Wilson loop expectation value is a gaussian integral. In \( D = 4 \), study the continuum limit of a rectangular loop with time extent \( T \gg R \), the spatial extent. Show that this reproduces the Coulomb force. For help, see VI.B of the Kogut paper linked above.

(b) Compute the combinatorial factors in the first few terms of the strong-coupling expansion of the same quantity.

(c) Consider the case of two spacetime dimensions. In this case, the plaquette variables are actually independent variables (since the relations between them arise from the boundaries of 3-volumes).

(d) Consider the weak coupling calculation again for a Wilson loop coupled to a massive vector field. Show that this reproduces an exponentially-decaying force between external static charges.

4. **Chern-Simons theory, flux attachment, and anyons.**

(a) Consider the following action for a \( \text{U}(1) \) gauge field in \( D = 2 + 1 \):

\[
S[A] = \int \left( -\frac{1}{4g^2} F \wedge \star F + \frac{k}{4\pi} A \wedge F \right).
\]

What are the dimensions of \( g \) and \( k \)? Find the equations of motion for \( A \). Look for plane wave solutions. Show that the resulting particle excitations have a mass which grows with \( g \).

(b) For the rest of the problem, take \( g \to \infty \). Notice that the resulting theory does not require a metric, since the action is made only from exterior derivatives and wedge products of forms. Now add a matter current:

\[
S_j[A] = \int \left( \frac{k}{4\pi} A \wedge F + A \wedge \star j \right).
\]

Find the equations of motion. Show that the Chern-Simons term *attaches \( k \) units of flux* to the particles: \( F_{12} \propto \rho \).

(c) Show using the Bohm-Aharonov effect that the particles whose current density is \( j^\mu \) have anyonic statistics with exchange angle \( \frac{\pi}{k} \) (supposing they were bosons before we coupled them to \( A \)).

One way to do this is to consider a configuration of \( j \) which describes one particle adiabatically encircling another. Show that its wavefunction acquires a phase \( e^{i2\pi/k} \). This is twice the phase obtained by going halfway around, which (when followed by an innocuous translation) would exchange the particles.
5. **Gauge theory can emerge from a local Hilbert space.**

The Hilbert space of a gauge theory is a funny thing: states related by a gauge transformation are physically equivalent. In particular, it is not a tensor product over independent local Hilbert spaces associated with regions of space. Because of this, there is much hand-wringing about defining entanglement in gauge theory. The following is helpful for thinking about this. It is a realization of $\mathbb{Z}_2$ lattice gauge theory, beginning from a model with no redundancy in its Hilbert space. In this avatar it is due to Kitaev and is called the *toric code.*

To define the Hilbert space, put a qbit on every link $\ell$ of a lattice, say the 2d square lattice, so that $\mathcal{H} = \otimes_{\ell} \mathcal{H}_{\ell}$. Let $\sigma^x_{\ell}, \sigma^z_{\ell}$ be the associated Pauli operators, and recall that $\{\sigma^x_{\ell}, \sigma^z_{\ell}\} = 0$. $\mathcal{H}_{\ell} = \text{span}\{|\sigma^z_{\ell} = 1\rangle, |\sigma^z_{\ell} = -1\rangle\}$ is a useful basis for the Hilbert space of a single link.

One term in the Hamiltonian is associated with each site $j \rightarrow A_j \equiv \prod_{\ell \in j} \sigma^z_{\ell}$ and one with each plaquette $p \rightarrow B_p \equiv \prod_{\ell \in \partial p} \sigma^x_{\ell}$, as indicated in the figure at right.

$$
\mathbf{H} = -\Gamma_e \sum_j A_j - \Gamma_m \sum_p B_p.
$$

(a) Show that all these terms commute with each other.

(b) The previous result means we can diagonalize the Hamiltonian by minimizing one term at a time. Let’s imagine that $\Gamma_e \gg \Gamma_m$ so we’ll minimize the ‘star’ terms $A_j$ first. Which states satisfy the ‘star condition’ $A_j = 1$? In the $\sigma^x$ basis there is an extremely useful visualization: we say a link $l$ of $\hat{\Gamma}$ is covered with a segment of string (an electric flux line) if $\sigma^z_{\ell} = -1$ (so the electric field on the link is $e_{\ell} = 1$) and is not covered if $\sigma^z_{\ell} = +1$ (so the electric field on the link is $e_{\ell} = 0$): $\overline{l} \equiv (\sigma^z_{\ell} = -1)$. Draw all possible configurations incident on a single vertex $j$ and characterize which ones satisfy $A_j = 1$.

(c) [bonus] What is the effect of adding a term $\Delta \mathbf{H} = \sum_{\ell} g \sigma^x_{\ell}$? Convince yourself that in the limit $\Gamma_e \gg \Gamma_m$, for energies $E \ll \Gamma_e$, this is identical to $\mathbb{Z}_2$ lattice gauge theory, where $A_j = 1$ is a discrete version of the Gauss law

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A comment about notation: the notation $\sigma^x_{\ell}, \sigma^z_{\ell}$ is pretty terrible (at least for someone with deteriorating eyesight like me) because the crucial information ($x$ or $z$) is hidden in the superscript. Much better is to write

$$
\sigma^x_{\ell} \equiv X_{\ell}, \sigma^z \equiv Z_{\ell}.
$$
constraint. [This part is a bonus problem because I have not yet explained how to go from the euclidean lattice gauge theory to a Hamiltonian formulation. If you want to figure it out yourself, you can get help from section V.E of this Kogut review.]

(d) Set $g = 0$ again. In the subspace of solutions of the star condition, find the groundstate(s) of the plaquette term. First consider a simply-connected region of lattice, then consider periodic boundary conditions.