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0.1 Sources

The material in these notes is collected from many places, among which I should mention in particular the following:

- Peskin and Schroeder, *An introduction to quantum field theory* (Wiley)
- Banks, *Modern Quantum Field Theory: A Concise Introduction* (Cambridge)
- Schwartz, *Quantum field theory and the standard model* (Cambridge)

David Tong’s lecture notes

Many other bits of wisdom come from the Berkeley QFT courses of Prof. L. Hall and Prof. M. Halpern.
0.2 Conventions

Following most QFT books, I am going to use the $+---$ signature convention for the Minkowski metric. I am used to the other convention, where time is the weird one, so I’ll need your help checking my signs. More explicitly, denoting a small spacetime displacement as $dx^\mu \equiv (dt, d\vec{x})^\mu$, the Lorentz-invariant distance is:

$$ds^2 = +dt^2 - d\vec{x} \cdot d\vec{x} = \eta_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad \eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}.$$  

(spacelike is negative). We will also write $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\partial_t, \vec{\nabla}_x)^\mu$, and $\partial^\mu \equiv \eta^{\mu\nu} \partial_\nu$. I’ll use $\mu, \nu...$ for Lorentz indices, and $i, k,...$ for spatial indices.

The convention that repeated indices are summed is always in effect unless otherwise indicated.

A consequence of the fact that english and math are written from left to right is that time goes to the left.

A useful generalization of the shorthand $\hbar \equiv \frac{\hbar}{2\pi}$ is

$$dk \equiv \frac{dk}{2\pi}.$$  

I will also write $\delta^d(q) \equiv (2\pi)^d\delta^{(d)}(q)$. I will try to be consistent about writing Fourier transforms as

$$\int \frac{d^d k}{(2\pi)^d} e^{i k x} \tilde{f}(k) \equiv \int d^d k \ e^{i k x} \tilde{f}(k) \equiv f(x).$$

IFF $\equiv$ if and only if.

RHS $\equiv$ right-hand side. LHS $\equiv$ left-hand side. BHS $\equiv$ both-hand side.

IBP $\equiv$ integration by parts. WLOG $\equiv$ without loss of generality.

$+\mathcal{O}(x^n) \equiv$ plus terms which go like $x^n$ (and higher powers) when $x$ is small.

$h.c.$ $\equiv$ plus hermitian conjugate.

We work in units where $\hbar$ and the speed of light, $c$, are equal to one unless otherwise noted. When I say ‘Peskin’ I usually mean ‘Peskin & Schroeder’.

Please tell me if you find typos or errors or violations of the rules above.
7 To infinity and beyond

Last quarter we ended at a high point, computing the amplitudes and cross-sections for many processes using QED. More precisely, we studied the leading-order-in-\(\alpha\) amplitudes, using Feynman diagrams which were trees – no loops. The natural next step is to look at the next terms in the perturbation expansion in \(\alpha\), which come from diagrams with one loop. When we do that we’re going to encounter some confusing stuff, in fact some of the same confusing stuff we encountered in thinking about Casimir forces at the beginning of last quarter.

We didn’t encounter these short-distance issues in studying tree-level diagrams because in a tree-level diagram, the quantum numbers (and in particular the momenta) of the intermediate states are fixed by the external states. In contrast, once there is a loop, there are undetermined momenta which must be summed, and this sum includes, it seems, arbitrarily high momentum modes, about which surely we have no information yet.

In order to put ourselves in the right frame of mind to think about that stuff, let’s make a brief retreat to systems with finitely many degrees of freedom.

Then we’ll apply some of these lessons to a toy field theory example (scalar field theory). Then we’ll come back to perturbation theory in QED. Reading assignment for this chapter: Zee §III.

7.1 A parable from quantum mechanics on the breaking of scale invariance

Recall that the coupling constant in \(\phi^4\) theory in \(D = 3 + 1\) spacetime dimensions is dimensionless, and the same is true of the electromagnetic coupling \(e\) in QED in \(D = 3+1\) spacetime dimensions. In fact, the mass parameters are the only dimensionful quantities in those theories, at least in their classical avatars. This means that if we ignore the masses, for example because we are interested in physics at much higher energies, then these models seem to possess scale invariance: the physics is unchanged under zooming in.

Here we will study a simple quantum mechanical example (that is: an example with a finite number of degrees of freedom)\(^1\) with such (classical) scale invariance. It exhibits many interesting features that can happen in strongly interacting quantum field theory – asymptotic freedom, dimensional transmutation. Because the model is simple, we can understand these phenomena without resort to perturbation theory.

\(^1\)I learned this example from Marty Halpern.
They will nevertheless illuminate some ways of thinking which we’ll need in examples where perturbing is our only option.

Consider the following (‘bare’) action:

\[ S[\vec{q}] = \int dt \left( \frac{1}{2} \dot{\vec{q}}^2 + g_0 \delta^{(2)}(\vec{q}) \right) \equiv \int dt \left( \frac{1}{2} \dot{\vec{q}}^2 - V(\vec{q}) \right) \]

where \( \vec{q} = (x, y) \) are two coordinates of a quantum particle, and the potential involves \( \delta^{(2)}(\vec{q}) \equiv \delta(x)\delta(y) \), a Dirac delta function. I chose the sign so that \( g_0 > 0 \) is attractive. (Notice that I have absorbed the inertial mass \( m \) in \( \frac{1}{2}mv^2 \) into a redefinition of the variable \( q \), \( q \to \sqrt{mq} \).)

First, let’s do dimensional analysis (always a good idea). Since \( \hbar = c = 1 \), all dimensionful quantities are some power of a length. Let \(-[X]\) denote the number of powers of length in the units of the quantity \( X \); that is, if \( X \sim (\text{length})^\nu(X) \) then we have \([X] = -\nu(X)\), a number. We have:

\([t] = [\text{length}/c] = -1 \implies [dt] = -1\).

The action appears in the exponent in the path integrand, and is therefore dimensionless (it has units of \( \hbar \)), so we had better have:

\(0 = [S] = [\hbar]\)

and this applies to each term in the action. We begin with the kinetic term:

\(0 = [\int dt \dot{\vec{q}}^2] \implies [\dot{\vec{q}}^2] = +1 \implies [\dot{\vec{q}}] = +\frac{1}{2} \implies [\vec{q}] = -\frac{1}{2}\).

Since \( 1 = \int dq \delta(q) \), we have \(0 = [dq] + [\delta(q)]\) and

\([\delta^D(\vec{q})] = -[q]D = \frac{D}{2}, \quad \text{and in particular } [\delta^2(\vec{q})] = 1\).

This implies that the naive (“engineering”) dimensions of the coupling constant \( g_0 \) are \([g_0] = 0 – it is dimensionless. Classically, the theory does not have a special length scale; it is scale invariant.

The Hamiltonian associated with the Lagrangian above is

\[ H = \frac{1}{2} (p_x^2 + p_y^2) + V(\vec{q}). \]
Now we treat this as a quantum system. Acting in the position basis, the quantum Hamiltonian operator is

\[ H = -\frac{\hbar^2}{2} \left( \partial_x^2 + \partial_y^2 \right) - g_0 \delta^{(2)}(\vec{q}) \]

So in the Schrödinger equation \( H \psi = \left( -\frac{\hbar^2}{2} \nabla^2 + V(\vec{q}) \right) \psi = E \psi \), the second term on the LHS is

\[ V(\vec{q}) \psi(\vec{q}) = -g_0 \delta^{(2)}(\vec{q}) \psi(0). \]

To make it look more like we are doing QFT, let’s solve it in momentum space:

\[ \psi(\vec{q}) = \int \frac{d^2p}{(2\pi\hbar)^2} e^{i\vec{p} \cdot \vec{q}/\hbar} \varphi(\vec{p}) \]

The delta function is

\[ \delta^{(2)}(q) = \int \frac{d^2p}{(2\pi\hbar)^2} e^{i\vec{p} \cdot \vec{q}/\hbar}. \]

So the Schrödinger equation says

\[
\begin{align*}
\left( -\frac{1}{2} \nabla^2 - E \right) \psi(q) &= -V(q) \psi(q) \\
\int d^2pe^{ip\cdot q} \left( \frac{p^2}{2} - E \right) \varphi(p) &= +g_0 \delta^{(2)}(q) \psi(0)
\end{align*}
\]

\[ = +g_0 \left( \int d^2pe^{ip\cdot q} \right) \psi(0) \tag{7.1} \]

which (integrating the both-hand side of (7.1) over \( q \): \( \int d^2q e^{-ip\cdot q} ((7.1)) \) ) says

\[
\left( \frac{p^2}{2} - E \right) \varphi(p) = +g_0 \int \frac{d^2p'}{(2\pi\hbar)^2} \varphi(p') \\
-\psi(0)
\]

There are two cases to consider:

• \( \psi(\vec{q} = 0) = \int d^2p \varphi(\vec{p}) = 0 \). Then this case is the same as a free theory, with the constraint that \( \psi(0) = 0 \),

\[ \left( \frac{p^2}{2} - E \right) \varphi(\vec{p}) = 0 \]

i.e. plane waves which vanish at the origin, e.g. \( \psi \propto \sin \frac{px}{\hbar} e^{\pm ip_y y/\hbar} \). These scattering solutions don’t see the delta-function potential at all.
• $\psi(0) \equiv \alpha \neq 0$, some constant to be determined. This means $\vec{p}^2/2 - E \neq 0$, so we can divide by it:

$$\varphi(\vec{p}) = \frac{g_0}{\frac{\vec{p}^2}{2} - E} \left( \int \! \! d^2\vec{p}' \varphi(\vec{p}') \right) = \frac{g_0}{\frac{\vec{p}^2}{2} - E} \alpha.$$  

The integral of the RHS (for $\psi(0) = \alpha$) is a little problematic if $E > 0$, since then there is some value of $p$ where $p^2 = 2E$. Avoid this singularity by going to the boundstate region: consider $E = -\epsilon_B < 0$. So:

$$\varphi(\vec{p}) = \frac{g_0}{\frac{\vec{p}^2}{2} + \epsilon_B} \alpha.$$  

What happens if we integrate this $\int d^2p$ to check self-consistency – the LHS should give $\alpha$ again:

$$0 \overset{!}{=} \int d^2p \varphi(\vec{p}) \left( 1 - \int d^2p \frac{g_0}{\frac{\vec{p}^2}{2} + \epsilon_B} \right)$$  

$$\implies \int d^2p \frac{g_0}{\frac{\vec{p}^2}{2} + \epsilon_B} = 1$$

is a condition on the energy $\epsilon_B$ of possible boundstates.

But there’s a problem: the integral on the LHS behaves at large $p$ like

$$\int \frac{d^2p}{p^2} \sim \infty.$$  

At this point in an undergrad QM class, you would give up on this model. In QFT we don’t have that luxury, because this kind of thing happens all over the place. Here’s what we do instead.

We cut off the integral at some large $p = \Lambda$:

$$\int^{\Lambda} \frac{d^2p}{p^2} \sim \log \Lambda.$$  

This our first example of the general principle that a classically scale invariant system will exhibit logarithmic divergences (rather: logarithmic dependence on the cutoff). It’s the only kind allowed by dimensional analysis.

The introduction of the cutoff can be thought of in many ways: we could say there are no momentum states with $|p| > \Lambda$, or maybe we could say that the potential is not really a delta function if we look more closely. The choice of narrative here shouldn’t affect our answers to physics questions.
More precisely:

\[ \int_0^\Lambda \frac{d^2p}{\frac{p^2}{2} + \epsilon_B} = 2\pi \int_0^\Lambda \frac{pd\rho}{\frac{p^2}{2} + \epsilon_B} = 2\pi \log \left( 1 + \frac{\Lambda^2}{2\epsilon_B} \right). \]

So in our cutoff theory, the boundstate condition is:

\[ 1 = g_0 \int_0^\Lambda \frac{d^2p}{\frac{p^2}{2} + \epsilon_B} = \frac{g_0}{2\pi\hbar^2} \log \left( 1 + \frac{\Lambda^2}{2\epsilon_B} \right). \]

A solution only exists for \( g_0 > 0 \). This makes sense since only then is the potential attractive (recall that \( V = -g_0\delta \)).

Now here’s a trivial-seeming step that offers a dramatic new vista: solve for \( \epsilon_B \).

\[ \epsilon_B = \frac{\Lambda^2}{2} \frac{1}{\sqrt{\frac{2\pi\hbar^2}{g_0} - 1}}. \] (7.2)

As we remove the cutoff (\( \Lambda \to \infty \)), we see that \( E = -\epsilon_B \to -\infty \), the boundstate becomes more and more bound – the potential is too attractive.

Suppose we insist that the boundstate energy \( \epsilon_B \) is a fixed thing – imagine we’ve measured it to be 200 MeV\(^2\). We should express everything in terms of the measured quantity. Then, given some cutoff \( \Lambda \), we should solve for \( g_0(\Lambda) \) to get the boundstate energy we have measured:

\[ g_0(\Lambda) = \frac{2\pi\hbar^2}{\log \left( 1 + \frac{\Lambda^2}{2\epsilon_B} \right)}. \]

This is the crucial step: this silly symbol \( g_0 \) which appeared in our action doesn’t mean anything to anyone (see Zee’s dialogue with the S.E. in section III). We are allowing \( g_0 \equiv \text{the bare coupling to be cutoff-dependent} \).

Instead of a dimensionless coupling \( g_0 \), the useful theory contains an arbitrary \textit{dimensionful} coupling constant (here \( \epsilon_B \)). This phenomenon is called \textit{dimensional transmutation} (d.t.). The cutoff is supposed to go away in observables, which depend on \( \epsilon_B \) instead.

In QCD we expect that in an identical way, an arbitrary scale \( \Lambda_{QCD} \) will enter into physical quantities. (If QCD were the theory of the whole world, we would work in units where it was one.) This can be taken to be the rest mass of some mesons – boundstates of quarks. Unlike this example, in QCD there are many boundstates, but their energies are dimensionless multiplies of the one dimensionful scale, \( \Lambda_{QCD} \). Nature chooses \( \Lambda_{QCD} \simeq 200 \text{ MeV} \).

\(^2\)Spoiler alert: I picked this value of energy to stress the analogy with QCD.
[This d.t. phenomenon was maybe first seen in a perturbative field theory in S. Coleman, E. Weinberg, Phys Rev D7 (1973) 1898. We’ll come back to their example.]

There are more lessons in this example. Go back to (7.2):

\[ \epsilon_B = \frac{\Lambda^2}{2} \frac{1}{e^{2\pi \hbar^2 g_0} - 1} \neq \sum_{n=0}^{\infty} g_0^n \epsilon_B^n(\Lambda) \]

it is not analytic (i.e. a power series) in \( g_0(\Lambda) \) near small \( g_0 \); rather, there is an essential singularity in \( g_0 \). (All derivatives of \( \epsilon_B \) with respect to \( g_0 \) vanish at \( g_0 = 0 \).) You can’t expand the dimensionful parameter in powers of the coupling. This means that you’ll never see it in perturbation theory in \( g_0 \). Dimensional transmutation is an inherently non-perturbative phenomenon.

Look at how the bare coupling depends on the cutoff in this example:

\[ g_0(\Lambda) = \frac{2\pi \hbar^2}{\log(1 + \Lambda^2/2\epsilon_B)} \xrightarrow{\Lambda^2 \gg \epsilon_B} 2\pi \hbar^2 \log(\Lambda^2/2\epsilon_B) \xrightarrow{\Lambda^2 \gg \epsilon_B} 0 \]

– the bare coupling vanishes in this limit, since we are insisting that the parameter \( \epsilon_B \) is fixed. This is called asymptotic freedom (AF): the bare coupling goes to zero (i.e. the theory becomes free) as the cutoff is removed. This also happens in QCD.

**RG flow equations.** Define the beta-function as the logarithmic derivative of the bare coupling with respect to the cutoff:

**Def:** \( \beta(g_0) \equiv \Lambda \frac{\partial}{\partial \Lambda} g_0(\Lambda) \).

For this theory

\[ \beta(g_0) = \Lambda \frac{\partial}{\partial \Lambda} \left( \frac{2\pi \hbar^2}{\log(1 + \Lambda^2/2\epsilon_B)} \right) \xrightarrow{\text{calculate}} - \frac{g_0^2}{\pi \hbar^2} \left( \frac{1}{\text{perturbative}} - e^{-2\pi \hbar^2/g_0}/\text{not perturbative} \right). \]

Notice that it’s a function only of \( g_0 \), and not explicitly of \( \Lambda \). Also, in this simple toy theory, the perturbation series for the beta function happens to stop at order \( g_0^2 \).

\( \beta \) measures the failure of the cutoff to disappear from our discussion – it signals a quantum mechanical violation of scale invariance. What’s \( \beta \) for? Flow equations:

\[ \dot{g}_0 = \beta(g_0). \]
This is a tautology. The dot is
\[ \dot{A} = \partial_s A, \quad s \equiv \log \Lambda / \Lambda_0 \implies \partial_s = \Lambda \partial_\Lambda. \]
(\(\Lambda_0\) is some reference scale.) But forget for the moment that this is just a definition:
\[ \dot{g}_0 = -\frac{g_0^2}{\pi \hbar^2} \left(1 - e^{-2\pi\hbar^2/g_0}\right). \]
This equation tells you how \(g_0\) changes as you change the cutoff. Think of it as a nonlinear dynamical system (fixed points, limit cycles...)

**Def:** A fixed point \(g^*_0\) of a flow is a point where the flow stops:
\[ 0 = \dot{g}_0|_{g^*_0} = \beta(g^*_0), \]
a zero of the beta function. (Note: if we have many couplings \(g_i\), then we have such an equation for each \(g\): \(\dot{g}_i = \beta_i(g)\). So \(\beta_i\) is (locally) a vector field on the space of couplings.)

Where are the fixed points in our example?
\[ \beta(g_0) = -\frac{g_0^2}{\pi \hbar^2} \left(1 - e^{-2\pi\hbar^2/g_0}\right). \]
There’s only one: \(g^*_0 = 0\), near which \(\beta(g_0) \sim -\frac{g_0^2}{\pi \hbar}\), the non-perturbative terms are small. What does the flow look like near this point? For \(g_0 > 0\), \(\dot{g}_0 = \beta(g_0) < 0\). With this (high-energy) definition of the direction of flow, \(g_0 = 0\) is an attractive fixed point:

\[ *<---<---<---<---<---<---<---<---<---<---<---<---<---<---<---<---<---<---<---<---<---<---g_0 \]
\(g^*_0 = 0\).

We already knew this. It just says \(g_0(\Lambda) \sim \frac{1}{\log \Lambda} \to 0\) at large \(\Lambda\). A lesson is that in the vicinity of such an AF fixed point, the non-perturbative stuff \(e^{-2\pi\hbar^2/g_0}\) is small. So we can get good results near the fixed point from the perturbative part of \(\beta\). That is: we can compute the behavior of the flow of couplings near an AF fixed point *perturbatively*, and be sure that it is an AF fixed point. This is the situation in QCD.

---

\(^3\)Warning: The sign in this definition carries a great deal of cultural baggage. With the definition given here, the flow (increasing \(s\)) is toward the UV, toward high energy. This is the high-energy particle physics perspective, where we learn more physics by going to higher energies. As we will see, there is a strong argument to be made for the other perspective, that the flow should be regarded as going from UV to IR, since we lose information as we move in that direction – in fact, the IR behavior does not determine the UV behavior in general, but UV does determine IR.
On the other hand, the d.t. phenomenon that we’ve shown here is something that we can’t prove in QCD. However, the circumstantial evidence is very strong!

Another example where this happens is quantum mechanics in any number of variables with a central potential \( V = -\frac{g^2}{r^2} \). It is also classically scale invariant:

\[
[r] = -\frac{1}{2}, \quad \left[ \frac{1}{r^2} \right] = +1 \quad \implies \quad [g_0] = 0.
\]

This model was studied in K.M. Case, *Phys Rev* **80** (1950) 797 and you will study it on the first homework. The resulting boundstates and d.t. phenomenon are called Efimov states; this model preserves a *discrete* scale invariance.

Here’s a quote from Marty Halpern from his lecture on this subject:

*I want you to study this set of examples very carefully, because it’s the only time in your career when you will understand what is going on.*

In my experience it’s been basically true. For real QFTs, you get distracted by Feynman diagrams, gauge invariance, regularization and renormalization schemes, and the fact that you can only do perturbation theory.
7.2 A simple example of perturbative renormalization in QFT

[Zee §III.1, Schwartz §15.4] Now let’s consider an actual field theory but a simple one, namely the theory of a real scalar field in four dimensions, with

$$\mathcal{L} = -\frac{1}{2} \phi \square \phi - m^2 \phi^2 - \frac{g}{4!} \phi^4.$$  \hspace{1cm} (7.3)

Recall that $[\phi] = \frac{D-2}{2}$ so $[m] = 1$ and $[g] = \frac{4-D}{2}$, so $g$ is dimensionless in $D = 4$. As above, this will mean logarithms!

Let’s do $2 \leftrightarrow 2$ scattering of $\phi$ particles.

$$iM_{2\leftrightarrow 2} = \bigoplus \left[ \begin{array}{c c c c}
\phi^2 & \phi^2 & \phi^2 & \phi^2 \\
\phi^2 & \phi^2 & \phi^2 & \phi^2 \\
\phi^2 & \phi^2 & \phi^2 & \phi^2 \\
\phi^2 & \phi^2 & \phi^2 & \phi^2 \\
\end{array} \right] + O(g^3)$$

$$= -ig \left( iM_s + iM_t + iM_u \right) + O(g^3)$$

where, in terms of $q_s \equiv k_1 + k_2$, the s-channel 1-loop amplitude is

$$iM_s = \frac{1}{2}(-ig)^2 \int d^4k \frac{\epsilon}{k^2 - m^2 + i\epsilon} \frac{\epsilon}{(q_s - k)^2 - m^2 + i\epsilon} \sim \int d^4k \frac{\epsilon}{k^4}.$$  

**Parametrizing ignorance.** Recall our discovery of the scalar field at the beginning of last quarter by starting with a chain of springs, and looking at the long-wavelength (small-wavenumber) modes. In the sum, $\int d^4k$, the region of integration that’s causing the trouble is not the part where the system looks most like a field theory. That is: if we look closely enough (small enough $1/k$), we will see that the mattress is made of springs. In terms of the microscopic description with springs, there is a smallest wavelength, of order the inverse lattice spacing: the sum stops.

Field theories arise from many such models, which may differ dramatically in their short-distance physics. We’d like to not worry too much about which one, but rather say things which do not depend on this choice. Recall the discussion of the Casimir force from §1: in that calculation, many different choices of regulators for the mode sum corresponded to different material properties of the conducting plates. The leading Casimir force was independent of this choice; more generally, it is an important part of the physics problem to identify which quantities are UV sensitive and which are not.

Parametrizing ignorance is another way to say ‘doing science’. In the context of field theory, at least in the high-energy community it is called ‘regularization’.
Now we need to talk about the integral a little more. The part which is causing the trouble is the bit with large \( k \), which might as well be \( |k| \sim \Lambda \gg m \), so let’s set \( m = 0 \) for simplicity.

We’ll spend lots of time learning to do integrals below. Here’s the answer:

\[
iM = -ig + iCg^2 \left( \log \frac{\Lambda^2}{s} + \log \frac{\Lambda^2}{t} + \log \frac{\Lambda^2}{u} \right) + \mathcal{O}(g^3)
\]

If you must know, \( C = \frac{1}{16\pi^2} \).

**Observables can be predicted from other observables.** Again, the boldface statement might sound like some content-free tweet from some boring philosophy-of-science twitter feed, but actually it’s a very important thing to remember here.

What is \( g \)? As Zee’s Smart Experimentalist says, it is just a letter in some theorist’s lagrangian, and it doesn’t help anyone to write physical quantities in terms of it. Much more useful would be to say what is the scattering amplitude in terms of things that can be measured. So, suppose someone scatters \( \phi \) particles at some given \((s, t, u) = (s_0, t_0, u_0)\), and finds for the amplitude \( iM(s_0, t_0, u_0) = -ig_P \) where \( P \) is for ‘physical’.\(^4\)

This we can relate to our theory letters:

\[
-ig_P = iM(s_0, t_0, u_0) = -ig + iCg^2L_0 + \mathcal{O}(g^3)
\]

where \( L_0 \equiv \log \frac{\Lambda^2}{s_0} + \log \frac{\Lambda^2}{t_0} + \log \frac{\Lambda^2}{u_0} \). (Note that quantities like \( g_P \) are often called \( g_R \) where ‘R’ is for ‘renormalized,’ whatever that is.)

[End of Lecture 21]

**Renormalization.** Now here comes the big gestalt shift: Solve this equation (7.4) for the stupid letter \( g \)

\[
-ig = -ig_P - iCg^2L_0 + \mathcal{O}(g^3)
\]

\(
= -ig_P - iCg^2P_0 + \mathcal{O}(g^3_P)
\)

and eliminate \( g \) from the discussion:

\[
iM(s, t, u) = -ig + iCg^2L + \mathcal{O}(g^3) \quad \overset{(7.5)}{=} \quad -ig_P - iCg^2P_0 + iCg^2P_0 + \mathcal{O}(g^3_P)
\]

\(^4\)You might hesitate here about my referring to the amplitude \( M \) as an ‘observable’. The difficult and interesting question of what can actually be measured in experiments can be decoupled a bit from this discussion. I’ll say more later, but if you are impatient see the beginning of Schwartz, chapter 18.
\[ = -ig_P + ig_P^2 \left( \log \frac{s_0}{s} + \log \frac{t_0}{t} + \log \frac{u_0}{u} \right) + \mathcal{O}(g_P^3). \]  

(7.6)

This expresses the amplitude at any momenta (within the range of validity of the theory!) in terms of measured quantities, \( g_P, s_0, t_0, u_0 \). The cutoff \( \Lambda \) is gone! Just like in our parable in §7.1, it was eliminated by letting the coupling vary with it, \( g = g(\Lambda) \), according to (7.5). We’ll say a lot more about how to think about that dependence.

**Renormalized perturbation theory.** To slick up this machinery, consider the following Lagrangian density (in fact the same as (7.3), with \( m = 0 \) for simplicity):

\[ \mathcal{L} = -\frac{1}{2} \phi \square \phi - \frac{g_P}{4!} \phi^4 - \frac{\delta g}{4!} \phi^4 \]  

(7.7)

but written in terms of the measured coupling \( g_P \), and some as-yet-undetermined ‘counterterm’ \( \delta g \). Then

\[ \mathcal{M}(s, t, u) = -g_P - \delta g - Cg_P^2 \left( \log \frac{s}{\Lambda^2} + \log \frac{t}{\Lambda^2} + \log \frac{u}{\Lambda^2} \right) + \mathcal{O}(g_P^3). \]

If, in order to enforce the renormalization condition \( \mathcal{M}(s_0, t_0, u_0) = -g_P \), we choose \( \delta g = -g_P^2 C \left( \log \frac{s_0}{\Lambda^2} + \log \frac{t_0}{\Lambda^2} + \log \frac{u_0}{\Lambda^2} \right) \), then we find

\[ \mathcal{M}(s, t, u) = -g_P - Cg_P^2 \left( \log \frac{s}{s_0} + \log \frac{t}{t_0} + \log \frac{u}{u_0} \right) + \mathcal{O}(g_P^3) \]

– all the dependence on the unknown cutoff is gone, we satisfy the observational demand \( \mathcal{M}(s_0, t_0, u_0) = -g_P \), and we can predict the scattering amplitude (and others!) at any momenta.

The only price is that the ‘bare coupling’ \( g \) depends on the cutoff, and becomes infinite if we pretend that there is no cutoff. Happily, we didn’t care about \( g \) anyway. We can just let it go.

The step whereby we were able to absorb all the dependence on the cutoff into the bare coupling constant involved some apparent magic. It is not so clear that the same magic will happen if we study the next order \( \mathcal{O}(g_P^3) \) terms, or if we study other amplitudes. A QFT where all the cutoff dependence to all orders can be removed with a finite number of counterterms is called ‘renormalizable’. As we will see, such a field theory is less useful because it allows us to pretend that it is valid up to arbitrarily high energies. The alternative, where we must add more counterterms (such as something like \( \frac{\delta \phi}{\Lambda} \phi^6 \) at each order in perturbation theory, is called an effective field theory, which is a field theory that has the decency to predict its regime of validity.
7.3 Radiative corrections to the Mott formula

Recall from last quarter that by studying scattering of an electron from a heavy charged fermion (a muon is convenient) we reconstructed the cross section for scattering off a Coulomb potential (named after Mott). Our next goal is to figure out how this cross section is corrected by other QED processes.

Recall that

$$iM = \left( -ie\bar{\psi}(p')\gamma^\mu \psi(p) \right)_{\text{electrons}} \frac{-i}{q^2} \left( \eta_{\mu\nu} - \frac{(1-\xi)q'_\mu q'_\nu}{q^2} \right) \left( -ie\bar{\psi}(k)\gamma^\nu \psi(k') \right)_{\text{muons}}$$

with $q_t \equiv p - p' = k - k'$. After the spin sum,

$$\frac{1}{4} \sum_{s,s',r,r'} |M|^2 = 4\frac{e^4}{\ell^2} \left( -p_\mu p'_\nu - p'_\mu p_\nu - \eta_{\mu\nu}(-p \cdot p' + m_e^2) \right) \cdot \left( -k_\mu k'_\nu - k'_\mu k_\nu - \eta_{\mu\nu}(-k \cdot k' + m_\mu^2) \right)$$

Consider the limit where the target $\mu$ particle is much heavier than the electron. ‘Heavy’ here means that we can approximate the CoM frame by its rest frame, and its initial and final energy as $k'_0 = m_\mu, k_0 = \sqrt{m_\mu^2 + \vec{k}^2} = m_\mu + \frac{1}{2}\vec{k}^2/m_\mu + \cdots \approx m_\mu$. Also, this means the collision is approximately elastic. In the diagram of the kinematics at right, $c \equiv \cos \theta, s \equiv \sin \theta$.

The answer we found after some boiling was:

$$\frac{d\sigma}{d\Omega_{\text{Mott}}} = \alpha^2(1 - \beta^2 \sin^2 \theta/2) \frac{4\beta^2 p^2 \sin^4 \theta/2}{4\beta^2 p^2 \sin^4 \theta/2}.$$  

If we take $\beta \ll 1$ in this formula we get the Rutherford formula.

**Radiative corrections.** Now it’s time to think about perturbative corrections to this cross section. Given that the leading-order calculation reproduced the classical physics of the Coulomb potential, you can think of what we are doing as effectively discovering (high-energy or short-distance) quantum corrections to the Coulomb law. The diagrams we must include are these (I made the muon lines thicker and also red):

$$iM_{e\mu\rightarrow e\mu} = \quad + \quad \left( \quad + \right)$$
What do the one-loop diagrams in the second line have in common? They have an internal muon line. Why does this matter? When the energy going through the line is much smaller than the muon mass, then the propagator is \( \frac{k + m_\mu}{k^2 - m_\mu^2} \sim \frac{1}{m_\mu} \) and its relative contribution is down by \( k/m_\mu \ll 1 \). So let’s neglect these for now.

Why don’t we include diagrams like \( \text{?} \)? The LSZ formula tells us that their effects on the S-matrix are accounted for by the wavefunction renormalization factors \( Z \)

\[
S_{e\mu-e\mu} = \sqrt{Z_e^2} \sqrt{Z_\mu^2} \left( \begin{array}{c} \text{amputated, on-shell} \\
\end{array} \right)
\]
and in determining the locations of the poles whose residues are the S-matrix elements.

Notice that the one-loop amplitudes are suppressed relative to the tree level amplitude by two factors of \( e \), hence one factor of the fine structure constant \( \alpha = \frac{e^2}{4\pi} \). Their leading effects on the cross section come from

\[
\sigma \sim \left| \begin{array}{c} \text{amputated, on-shell} \\
\end{array} \right|^2 \sim \sigma_{\text{tree}} + O(\alpha^3)
\]
from the cross term between the tree and one-loop amplitudes.

In the above discussion, we encounter all three ‘primitive’ one-loop divergent amplitudes of QED, which we’ll study in turn:

- **electron self-energy:**

- **vertex correction:**

- **vacuum polarization (photon self-energy):**
7.4 Electron self-energy in QED

Let’s think about the electron two-point function in momentum space:

\[ \tilde{G}^{(2)}(p) = \]

As we did for the scalar field theory in §3 last quarter, we will denote the 1PI two-point function by

\[ -i\Sigma(p) \equiv \]

a blob with nubbins; for fermions with conserved particle number, the nubbins carry arrows indicating the particle number flow. Let me call the tree level propagator

\[ iS(p) \equiv \frac{i(p + m_0)}{p^2 - m_0^2 + i\epsilon} = \frac{i}{p - m_0} \]

– notice that I added a demeaning subscript to the notation for the mass appearing in the Lagrangian. Foreshadowing.

The full two point function is then:

\[ \tilde{G}^{(2)}(p) = iS + iS(-i\Sigma(p))iS + iS(-i\Sigma(p))iS(-i\Sigma(p))iS + \cdots \]

\[ = iS(1 + \Sigma S + \Sigma S\Sigma S + \cdots) = iS \frac{1}{1 - \Sigma S} \]

\[ = \frac{i}{p - m_0} \frac{1}{1 - \frac{1}{p - m_0}} = \frac{i}{p - m_0 - \Sigma(p)}. \]
we could do these manipulations in the eigenbasis of \( \hat{p} \). This fully corrected propagator has a pole at

\[
\hat{p} = m \equiv m_0 + \Sigma(m) .
\]

(7.12)

This means that the actual mass of the particle is this new quantity \( m \). But what is \( m \) (it is called the ‘renormalized mass’)\? To figure it out, we need to know about \( \Sigma \).

In QED we must study \( \Sigma \) in perturbation theory. As you can see from (7.10), the leading (one-loop) contribution is

\[
-i\Sigma_2(p) = (-ie)^2 \int d^4k \gamma^\mu \frac{ik + m_0}{k^2 - m_0^2 + ie} \gamma^\nu (p - k)^2 - \mu^2 + ie .
\]

Notice that I am relying on the Ward identity to enforce the fact that only the traverse bit of the photon propagator matters. Also, I added a mass \( \mu \) for the photon as an IR regulator. We must keep the external momentum \( p \) arbitrary, since we don’t even know where the mass-shell is!

Finally, I can’t put it off any longer: how are we going to do this loop-momentum integral?

Step 1: Feynman parameter trick. It is a good idea to consider the integral

\[
\int_0^1 dx \frac{1}{(xA + (1-x)B)^2} = \int_0^1 dx \frac{1}{x(A-B) + B}^2 = \left. \frac{1}{A-B x(A-B) + B} \right|_{x=0}^{x=1} = \frac{1}{A - B} \left( -\frac{1}{A} + \frac{1}{B} \right) = \frac{1}{AB}.
\]

This allows us to combine the denominators into one:

\[
I = \int_0^1 dx \frac{1}{k^2 - m_0^2 + ie} \frac{1}{(p-k)^2 - \mu^2 + ie} = \int_0^1 dx \frac{1}{x((p^2 - 2pk + k^2) - \mu^2 + ie) + (1-x)(k^2 - m_0^2 + ie))}^2.
\]

Step 2: Now we can complete the square

\[
I = \int_0^1 dx \frac{1}{((k - px)^2 - \Delta + ie)^2}
\]

with

\[
\ell^\mu \equiv k^\mu - p^\mu x, \quad \Delta \equiv +p^2 x^2 + x\mu^2 - xp^2 + (1-x)m_0^2 = x\mu^2 + (1-x)m_0^2 - x(1-x)p^2.
\]
Step 3: Wick rotate. Because of the $i\epsilon$ we’ve been dutifully carrying around, the poles of the $p^0$ integral don’t occur in the first and third octants of the complex $p^0$ plane. (And the integrand decays at large $|p^0|$.) This means that we can rotate the contour to euclidean time for free: $\ell^0 \equiv i\ell^4$. Equivalently: the integral over the contour at right vanishes, so the real time contour gives the same answer as the (upward-directed) Euclidean contour.

Notice that $\ell^2 = -\ell_E^2$. Altogether

$$-i\Sigma_2(p) = -e^2 \int d^4\ell \int_0^1 dx \frac{N}{(\ell^2 - \Delta + i\epsilon)^2} = -e^2 \int_0^1 dx i \int d^4\ell_E \frac{N}{(\ell_E^2 + \Delta)^2}$$

where the numerator is

$$N = \gamma^\mu (\ell + x\ell + m_0) \gamma_\mu = -2 (\ell + x\ell) + 4m_0.$$  

Here I used two Clifford algebra facts: $\gamma^\mu \gamma_\mu = 4$ and $\gamma^\mu \ell_\gamma^\mu = -2\ell$. Think about the contribution from the term with $\ell$ in the numerator: everything else is invariant under rotations of $\ell$

$$d^4\ell_E = \frac{1}{(2\pi)^4} d\Omega_3 \ell^3 d\ell \ell^2 d\ell^2 = \frac{d\Omega_3}{(2\pi)^4} \ell^2 d\ell^2,$$

so this averages to zero. The rest is of the form (using $\int d\Omega_3 = 2\pi^2$)

$$\Sigma_2(p) = e^2 \int_0^1 dx \int \frac{\ell^2 d\ell^2}{2} \frac{2(2m_0 - x\ell)}{(2\pi)^4} \frac{2(2m_0 - x\ell)}{(\ell^2 + \Delta)^2}$$

$$= \frac{e^2}{8\pi^2} \int_0^1 dx (2m_0 - x\ell) J$$  \hspace{1cm} (7.13)

with

$$J = \int_0^\infty d\ell^2 \ell^2 \frac{2(2m_0 - x\ell)}{(\ell^2 + \Delta)^2}.$$ 

In the large $\ell$ part of the integrand this is

$$\int_0^\Lambda d\ell^2 \frac{\ell^2}{(\ell^2 + \Delta)^2} \sim \log \Lambda.$$ 

You knew this UV divergence was coming. To be more precise, let’s add zero:

$$J = \int d\ell^2 \left( \ell^2 + \Delta \right) \frac{\Delta}{(\ell^2 + \Delta)^2} - \int_0^\infty d\ell^2 \left( \frac{\Delta}{(\ell^2 + \Delta)^2} \right) = \ln(\ell^2 + \Delta)|_{\ell^2=0}^\infty + \frac{\Delta}{\ell^2 + \Delta}|_{\ell^2=0}^\infty = \ln(\ell^2 + \Delta)|_{\ell^2=0}^\infty - 1.$$ 

Recall that

$$\Delta = x\mu^2 + (1-x)m_0^2 - x(1-x)p^2 \equiv \Delta(\mu^2).$$
Pauli-Villars regularization. Here is a convenient fiction: when you exchange a photon, you also exchange a very heavy particle, with mass \( m^2 = \Lambda^2 \), with an extra \((-1)\) in its propagator. This means that (in this Pauli-Villars regulation scheme) the Feynman rule for the wiggly line is instead

\[
\begin{align*}
\Gamma &= -i \eta_{\mu\nu} \left( \frac{1}{k^2 - \mu^2 + i\epsilon} - \frac{1}{k^2 - \Lambda^2 + i\epsilon} \right) \\
&= -i \eta_{\mu\nu} \left( \frac{\mu^2 - \Lambda^2}{(k^2 - \mu^2 + i\epsilon)(k^2 - \Lambda^2 + i\epsilon)} \right)
\end{align*}
\]

This goes like \( \frac{1}{k^4} \) at large \( k \), so the integrals are more convergent. Yay.

Notice that the contribution from the Pauli-Villars photon to tree-level amplitudes goes like \( \frac{1}{k^2 - \Lambda^2} \sim \frac{1}{\Lambda^2} \) (where \( k \) is the momentum going through the photon line, determined by the external momenta), which innocuously vanishes as \( \Lambda \to \infty \).

Remembering that the residue of the pole in the propagator is the probability for the field operator to create a particle from the vacuum, you might worry that this is a negative probability, and unitarity isn’t manifest. This particle is a ghost. However, we will choose \( \Lambda \) so large that the pole in the propagator at \( k^2 = \Lambda^2 \) will never be accessed and we’ll never have external Pauli-Villars particles. We are using this as a device to define the theory in a regime of energies much less than \( \Lambda \). You shouldn’t take the regulated theory too seriously: for example, the wrong-sign propagator means wrong-sign kinetic terms for the PV fields. This means that very wiggly configurations will be energetically favored rather than suppressed by the Hamiltonian. It will not make much sense non-perturbatively.

I emphasize that this regulator is one possibility of many. They each have their drawbacks. They all break scale invariance. A nice thing about PV is that it is Lorentz invariant. A class of regulators which make perfect sense non-perturbatively is the lattice (as in the model with masses on springs). The price is that it really messes up the spacetime symmetries.

Applying this to the self-energy integral amounts to the replacement

\[
\mathcal{J} \sim J_{\Delta(\mu^2)} - J_{\Delta(\Lambda^2)}
\]

\[
= \left[ \ln \left( \ell^2 + \Delta(\mu^2) \right) - 1 \right] - \left[ \ln \left( \ell^2 + \Delta(\Lambda^2) \right) - 1 \right] \bigg|_0^\infty
\]

\[
= \ln \frac{\ell^2 + \Delta(\mu^2)}{\ell^2 + \Delta(\Lambda^2)} \bigg|_0^\infty
\]

\[
= \ln \frac{\Delta(\mu^2)}{\Delta(\Lambda^2)} = \ln \frac{\Delta(\Lambda^2)}{\Delta(\mu^2)}. \quad [\text{End of Lecture 22}]
\]

Notice that we can take advantage of our ignorance of the microphysics to make the
cutoff as big as we like and thereby simplify our lives:

$$\Delta(\Lambda^2) = x\Lambda^2 + (1 - x)m_0^2 - x(1 - x)p^2 \Lambda \gg \text{everyone} \quad \approx x\Lambda^2.$$ 

Finally then

$$\Sigma_2(p)_{PV} = \frac{\alpha}{2\pi} \int_0^1 dx (2m_0 - x\not{p}) \ln \frac{x\Lambda^2}{x\mu^2 + (1 - x)m_0^2 + x(1 - x)m_0^2}.$$ 

(7.14)

Having arrived at this regulated expression for the self-energy we need to “impose a renormalization condition,” i.e. introduce some observable physics in terms of which to parametrize our answers. We return to (7.12): the shift in the mass as a result of this one-loop self-energy is

$$\delta m \equiv m - m_0 = \Sigma_2(p = m) + O(e^4) = \Sigma_2(p = m_0) + O(e^4)$$

$$= \frac{\alpha}{2\pi} \int_0^1 dx (2 - x)m_0 \ln \left( \frac{x\Lambda^2}{x\mu^2 + (1 - x)m_0^2 + x(1 - x)m_0^2} \right)$$

$$= \frac{\alpha}{2\pi} \int_0^1 dx (2 - x)m_0 \left( \ln \frac{\Lambda^2}{m_0^2} + \ln \frac{xm_0^2}{f(x, m_0, \mu)} \right)$$

$$\approx \frac{\alpha}{2\pi} \left( 2 - \frac{1}{2} \right) m_0 \ln \frac{\Lambda^2}{m_0^2} = \frac{3\alpha}{4\pi} m_0 \ln \frac{\Lambda^2}{m_0^2}.$$ 

(7.15)

In the penultimate step (with the ≈), we’ve neglected the finite bit (labelled ‘relatively small’) compared to the logarithmically divergent bit: we’ve already assumed $\Lambda \gg$ all other scales in the problem.

**Mass renormalization.** Now the physics input: The mass of the electron is 511 keV (you can ask how we measure it and whether the answer we get depends on the resolution of the measurement, and indeed there is more to this story; this is a low-energy answer, for example we could make the electron go in a magnetic field and measure the radius of curvature of its orbit and set $m_e v^2/r = evB/c$), so

$$511 \text{ keV} \approx m_e = m_0 \left( 1 + \frac{3\alpha}{4\pi} \ln \frac{\Lambda^2}{m_0^2} \right) + O(\alpha^2).$$

In this equation, the LHS is a measured quantity. In the correction on the RHS $\alpha \approx \frac{1}{137}$ is small, but it is multiplied by $\ln \frac{\Lambda^2}{m_0^2}$ which is arbitrarily large. This means that the bare mass $m_0$, which is going to absorb the cutoff dependence here, must actually be really small. (Notice that actually I’ve lied a little here: the $\alpha$ we’ve been using is
still the bare charge; we will need to renormalize that one, too, before we are done.) I emphasize: \( m_0 \) and the other fake, bare parameters in \( L \) depend on \( \Lambda \) and the order of perturbation theory to which we are working and other theorist bookkeeping garbage; \( m_e \) does not. At each order in perturbation theory, we eliminate \( m_0 \) and write our predictions in terms of \( m_e \). It is not too surprising that the mass of the electron includes such contributions: it must be difficult to travel through space if you are constantly emitting and re-absorbing photons.

**Wavefunction renormalization.** The actual propagator for the electron, near the electron pole is

\[
\tilde{G}^{(2)}(p) = \frac{i}{p - m_0 - \Sigma(p)} \overset{p \sim m}{\approx} \frac{iZ}{p - m} + \text{regular terms.} \tag{7.16}
\]

The residue of the pole at the electron mass is no longer equal to one, but rather \( Z \).

To see what \( Z \) actually is at this order in \( e^2 \), Taylor expand near the pole

\[
\Sigma(p) \overset{\text{Taylor}}{=} \Sigma(p = m) + \frac{\partial \Sigma}{\partial p} \bigg|_{p=m}(p - m) + \cdots
\]

\[
= \Sigma(p = m_0) + \frac{\partial \Sigma}{\partial p} \bigg|_{p=m_0}(p - m_0) + \cdots + \mathcal{O}(e^4)
\]

So then (7.16) becomes

\[
\tilde{G}^{(2)}(p) \overset{p \sim m}{\approx} \frac{i}{p - m - \frac{\partial \Sigma}{\partial p} \bigg|_{m_0}(p - m)} = \frac{i}{(p - m) \left(1 - \frac{\partial \Sigma}{\partial p} \bigg|_{m_0}\right)} \tag{7.17}
\]

So that

\[
Z = \frac{1}{1 - \frac{\partial \Sigma}{\partial p} \bigg|_{m_0}} \approx 1 + \frac{\partial \Sigma}{\partial p} \bigg|_{m_0} \equiv 1 + \delta Z
\]

and at leading nontrivial order

\[
\delta Z = \frac{\partial \Sigma_2}{\partial \psi} \bigg|_{m_0} \overset{(7.14)}{=} \frac{\alpha}{2\pi} \int_0^1 dx \left(-x \ln \frac{x \Lambda^2}{f(x, m_0, \mu)} + (2m_0 - x m_0) \frac{-2x(1 - x)}{f(x, m_0, \mu)\bigg)} \right)
\]

\[
= -\frac{\alpha}{4\pi} \left(\ln \frac{\Lambda^2}{m_0^2} + \text{finite}\right). \tag{7.18}
\]

Here \( f = f(x, m_0, \mu) \) is the same quantity defined in the second line of (7.15). We’ll see below that the cutoff-dependence in \( \delta Z \) plays a crucial role in making the \( S \) matrix (for example for the \( e\mu \rightarrow e\mu \) process we’ve been discussing) cutoff-independent and finite, when written in terms of physical variables.
7.5 Big picture interlude

OK, I am having a hard time just pounding away at one-loop QED. Let’s take a break and think about the self-energy corrections in scalar field theory. Then we will step back and think about the general structure of short-distance sensitivity in (relativistic) QFT, before returning to the QED vertex correction and vacuum polarization.

7.5.1 Self-energy in $\phi^4$ theory

[Zee §III.3] Let’s return to the $\phi^4$ theory in $D = 3 + 1$ for a moment. The $M_{\phi\phi\to\phi\phi}$ amplitude is not the only place where the cutoff appears.

Above we added a counterterm of the same form as the $\phi^4$ term in the Lagrangian. Now we will see that we need counterterms for everybody:

$$L = -\frac{1}{2} (\phi \square \phi + m^2 \phi^2) - \frac{g}{4!} \phi^4 - \frac{\delta g}{4!} \phi^4 + \frac{1}{2} \delta Z \phi \square \phi + \frac{1}{2} \delta m^2 \phi^2.$$

Here is a way in which $\phi^4$ theory is weird: At one loop there is no wavefunction renormalization. That is,

$$\delta \Sigma_1(k) = \frac{-ig}{q^2 - m^2 + i\epsilon} = \delta \Sigma_1(k = 0) \sim g\Lambda^2$$

which is certainly quadratically divergent, but totally independent of the external momentum. This means that when we Taylor expand in $k$ (as we just did in (7.17)), this diagram only contributes to the mass renormalization.

So let’s see what happens if we keep going:

$$\delta \Sigma_2(k) = \frac{-ig}{q^2 - m^2 + i\epsilon} = (\frac{-ig}{\sqrt{p^2 - m^2 + i\epsilon}} \int d^4 q D_0(p) D_0(q) D_0(k - p - q) \equiv I(k^2, m, \Lambda).$$

Here $iD_0(p) \equiv \frac{i}{\sqrt{p^2 - m^2 + i\epsilon}}$ is the free propagator (the factor of $i$ is for later convenience), and we’ve defined $I$ by this expression. The fact that $I$ depends only on $k^2$ is a consequence of Lorentz invariance. Counting powers of the loop momenta, the short-distance bit of this integral is of the schematic form $\int_0^\Lambda \frac{dp}{2p} \sim \Lambda^2$, also quadratically divergent, but this time $k^2$-dependent, so there will be a nonzero $\delta Z \propto g^2$. As we just did for the electron self-energy, we should Taylor expand in $k$. (We’ll learn more about
why and when the answer is analytic in \( k^2 \) at \( k = 0 \) later.) The series expansion in \( k^2 \) (let’s do it about \( k^2 = 0 \sim m^2 \) to look at the UV behavior) is

\[
\delta \Sigma_2(k^2) = A_0 + k^2 A_1 + k^4 A_2 + \cdots
\]

where \( A_0 = I(k^2 = 0) \sim \Lambda^2 \). In contrast, dimensional analysis says \( A_1 = \frac{\partial}{\partial k^2} I|_{k^2=0} \sim \Lambda^{0^+} \sim \ln \Lambda \) has two fewer powers of the cutoff. After that it’s clear sailing:

\[
A_2 = \left( \frac{\partial}{\partial k^2} \right)^2 I|_{k^2=0} \sim \int \frac{d^8 P}{P^2} \sim \Lambda^{-2} \text{ is finite as we remove the cutoff, and so are all the later coefficients.}
\]

Putting this together, the inverse propagator is

\[
D^{-1}(k) = D^{-1}_0(k) - \Sigma(k) = k^2 - m^2 - \left( \delta \Sigma_1(0) + A_0 \right) - k^2 A_1 - k^4 A_2 + \cdots \equiv a \sim \Lambda^2
\]

The \( \cdots \) here includes both higher orders in \( g (\mathcal{O}(g^3)) \) and higher powers of \( k^2 \), i.e. higher derivative terms. If instead the physical pole were at a nonzero value of the mass, we should Taylor expand about \( k^2 = m_P^2 \) instead:

\[
D^{-1}(k) = D^{-1}_0(k) - \Sigma(k) = k^2 - m_0^2 - \left( \delta \Sigma_1(0) + A_0 \right) - (k^2 - m_P^2) A_1 - (k^2 - m_P^2)^2 A_2 + \cdots \equiv a \sim \Lambda^2
\]

where now \( A_n \equiv \frac{1}{n!} \left( \frac{\partial}{\partial k^2} \right)^n \delta \Sigma_2(k^2)|_{k^2=m_P^2} \).

Therefore, the propagator is

\[
D(k) = \frac{1}{(1 - A_1)(k^2 - m_P^2)} + \cdots = \frac{Z}{k^2 - m_P^2} + \cdots
\]

with

\[
Z = \frac{1}{1 - A_1}, \quad m_P^2 = m^2 + a.
\]

Some points to notice:

- The contributions \( A_n \geq 2 (k^2)^n \) can be reproduced by counterterms of the form \( A_n \phi \Box^n \phi \). Had they been cutoff dependent we would have needed to add such (cutoff-dependent) counterterms.

- The mass-squared of the scalar field in \( D = 3 + 1 \) is quadratically divergent, while the mass of the spinor was only log divergent. This UV sensitivity of scalar fields is ubiquitous\(^5\) (see the homework) and is the source of many headaches.

- On the term ‘wavefunction renormalization’: who is \( \phi \)? Also just a theorist’s letter. Sometimes (in condensed matter) it is defined by some relation to observation (like

\(^5\)At least for most regulators. We’ll see that dim reg is special.
the height of a wave in the mattress), in high energy theory not so much. Classically, we fixed its (multiplicative) normalization by setting the coefficient of \( \phi \partial \phi \) to one. If we want to restore that convention after renormalization, we can make a redefinition of the field \( \phi_R \equiv Z^{-1/2} \phi \). This is the origin of the term ‘wavefunction renormalization’. A slightly better name would be ‘field renormalization’, but even better would be just ‘kinetic term renormalization’.

**Renormalized perturbation theory revisited.** The full story for the renormalized perturbation expansion in \( \phi^4 \) theory is

\[
\mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m_P^2 \phi^2 - \frac{g_P}{4!} \phi^4 + \mathcal{L}_{ct}
\]

with

\[
\mathcal{L}_{ct} = \frac{1}{2} \delta Z (\partial \phi)^2 + \frac{1}{2} \delta m^2 \phi^2 + \frac{\delta g}{4!} \phi^4.
\]

Here are the instructions for using it: The Feynman rules are as before: the coupling and propagator are

\[
\begin{align*}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\end{align*} = -i g_P, \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} = \frac{i}{k^2 - m_P^2 + i\epsilon}
\]

but the terms in \( \mathcal{L}_{ct} \) (the counterterms) are treated as new vertices, and treated perturbatively:

\[
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} = i \delta g, \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} = i(\delta Z k^2 + \delta m^2).
\]

All integrals are regulated, in the same way (whatever it is). The counterterm couplings \( \delta g, \delta Z, \delta m^2 \) are determined iteratively, as follows: given the \( \delta_{N-1} \)s up to \( \mathcal{O}(g_P^N) \), we fix each one \( \delta = \delta_{N-1} + g_P^N \Delta \delta_N + \mathcal{O}(g_P^{N+1}) \) by demanding that (7.19) are actually true up to \( \mathcal{O}(g_P^{N+1}) \). This pushes the cutoff dependence back into the muck a bit further.

I say this is the full story, but wait: we didn’t try to compute amplitudes with more than four \( \phi \)s (such as \( 3 \leftarrow 3 \) scattering of \( \phi \) quanta). How do we know those don’t require new counterterms (like a \( \phi^6 \) term, for example)?

### 7.5.2 Where is the UV sensitivity?

[still Zee §III.3, Peskin ch. 10. We’ll follow Zee’s discussion pretty closely for a bit.] Given some process in a relativistic, perturbative QFT, how do we know if it will depend on the cutoff? We’d like to be able answer this question for a theory with scalars, spinors, vectors. Here’s how: First, look at each diagram \( \mathcal{A} \) (order by order in the loop expansion). Define the ‘superficial’ degree of divergence of \( \mathcal{A} \) to be \( D_\mathcal{A} \) if
\(A \sim \Lambda^{D_A}\) (in the limit that \(\Lambda \ll all\ other\ scales\) – this is an asymptotic statement). A log divergent amplitude has \(D_A = 0\) (sometimes it’s called \(D_A = 0^+\)).

Let’s start simple, and study the \(\phi^4\) theory in \(D = 4\). Consider a connected diagram \(\mathcal{A}\) with \(B_E\) external scalar lines. I claim that \(D_A = 4 - B_E\). Why didn’t it depend on any other data of the diagram, such as

\[
\begin{align*}
B_I &\equiv \#\ of\ internal\ scalar\ lines\ (i.e.,\ propagators) \\
V &\equiv \#\ of\ \phi^4\ vertices \\
L &\equiv \#\ of\ loops
\end{align*}
\]

? We can understand this better using two facts of graph theory and some power counting. I recommend checking my claims below with an example, such as the one at right.

**Graph theory fact #1:** These quantities are not all independent. For a connected graph,

\[
L = B_I - (V - 1). \quad (7.20)
\]

Math proof\(^6\): Imagine placing the vertices on the page and adding the propagators one at a time. You need \(V - 1\) internal lines just to connect up all \(V\) vertices. After that, each *internal* line you add necessarily adds one more loop.

Another way to think about this fact makes clear that \(L = \#\ of\ loops = \#\ of\ momentum\ integrals\). Before imposing momentum conservation at the vertices, each internal line has a momentum which we must integrate: \(\prod_{\alpha=1}^{B_I} \int d^D q_{\alpha}\). We then stick a \(\delta^{(D)}(\sum q)\) for each vertex, but one of these gives the overall momentum conservation \(\delta^{(D)}(k_T)\), so we have \(V - 1\) fewer momentum integrals. For the example above, (7.20) says \(4 = 8 - (5 - 1)\).

**Graph theory fact #2:** Each external line comes out of one vertex. Each internal line connects two vertices. Altogether, the number of ends of lines sticking out of vertices is

\[
B_E + 2B_I = 4V
\]

where the RHS comes from noting that each vertex has four lines coming out of it (in \(\phi^4\) theory). In the example, this is \(4 + 2 \cdot 8 = 4 \cdot 5\). So we can eliminate

\[
B_I = 2V - B_E/2. \quad (7.21)
\]

\(^6\)I learned this one from my class-mate M.B. Schulz.
Now we count powers of momenta:

\[ A \sim \prod_{a=1}^{L} \int_{\Lambda}^{\Lambda} d^{D}k_{a} \prod_{a=1}^{B_{I}} \frac{1}{k_{a}^{2}}. \]

Since we are interested in the UV structure, I’ve set the mass to zero, as well as all the external momenta. The only scale left in the problem is the cutoff, so the dimensions of \( A \) must be made up by the cutoff:

\[ D_{A} = [A] = DL - 2B_{I} \]

\[ \overset{\text{(7.20)}}{=} B_{I}(D - 2) - D(V - 1) \]

\[ \overset{\text{(7.21)}}{=} D + \frac{2 - D}{2}B_{E} + V(D - 4). \]

If we set \( D = 3 + 1 = 4 \), we get \( D_{A} = 4 - B_{E} \) as claimed. Notice that with \( B_{E} = 2 \) we indeed reproduce \( D_{A} = 2 \), the quadratic divergence in the mass renormalization, and with \( B_{E} = 4 \) we get \( D_{A} = 0 \), the log divergence in the \( 2 \leftrightarrow 2 \) scattering. This pattern continues: with more than four external legs, \( D_{A} = 4 - B_{E} < 0 \), which means the cutoff dependence must go away when \( \Lambda \to 0 \). This is illustrated by the following diagram with \( B_{E} = 6 \):

\[ \sim \int_{\Lambda}^{\Lambda} \frac{d^{4}P}{P^{6}} \sim \Lambda^{-2}. \]

So indeed we don’t need more counterterms for higher-point interactions in this theory.

[End of Lecture 23]

Why is the answer independent of \( V \) in \( D = 4 \)? This has the dramatic consequence that once we fix up the cutoff dependence in the one-loop diagrams, the higher orders have to work out, \( i.e. \) it strongly suggests that the theory is renormalizable. \(^7\)

Before we answer this, let’s explore the pattern a bit more. Suppose we include also a fermion field \( \psi \) in our field theory, and suppose we couple it to our scalar by a

\(^7\)Why isn’t it a proof of renormalizability? Consider the following integral:

\[ \mathcal{I} = \int_{\Lambda}^{\Lambda} \frac{d^{4}p}{(p^{2} + m^{2})^{6}} \int_{\Lambda}^{\Lambda} d^{4}k. \]

According to our method of counting, we would say \( D_{\mathcal{I}} = 4 + 4 - 10 = -2 \) and declare this finite and cutoff-independent. On the other hand, it certainly does depend on the physics at the cutoff. (I bet it is possible to come up with more pathological examples.) The rest of the work involving ‘nested divergences’ and forests is in showing that the extra structure in the problem prevents things like \( \mathcal{I} \) from being Feynman amplitudes.
Yukawa interaction:

\[ S_{\text{bare}}[\phi, \psi] = -\int d^Dx \left( \frac{1}{2} \phi (\Box + m_\phi^2) \phi + \bar{\psi} (-\partial + m_\psi) \psi + y\phi\bar{\psi}\psi + \frac{g}{4!} \phi^4 \right). \]

To find the degree of divergence in an amplitude in this model, we have to independently keep track of the number fermion lines \( F_E, F_I \), since a fermion propagator has dimension \( \frac{1}{2} \) = -1, so that \( D_A = [A] = DL - 2B_I - F_I \). The number of ends-of-fermion-lines is \( 2V_y = 2F_E + F_I \) and the number of ends-of-boson-lines is \( V_y + 4V_g = B_E + 2B_I \). The number of loops is \( L = B_I + F_I - (V_y + V_g - 1) \). Putting these together (I used Mathematica) we get

\[ D_A = D + (D - 4) \left( V_g + \frac{1}{2}V_y \right) + B_E \left( \frac{2 - D}{2} \right) + F_E \left( \frac{1 - D}{2} \right). \] (7.22)

Again in \( D = 4 \) the answer is independent of the number of vertices! Is there something special about four spacetime dimensions?

To temper your enthusiasm, consider adding a four-fermion interaction: \( G(\bar{\psi}\psi)(\bar{\psi}\psi) \) (or maybe \( G_V(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi) \) or \( G_A(\bar{\psi}\gamma^\mu\gamma^5\psi)(\bar{\psi}\gamma^\mu\gamma^5\psi) \) or any other pile of gamma matrices in between, with the indices contracted). When you redo this calculation on the homework, you’ll find that in \( D = 4 \) a diagram (for simplicity, one with no \( \phi^4 \) or Yukawa interactions) has

\[ D_A = 4 - (1)B_E - \left( \frac{3}{2} \right) F_E + 2V_G. \]

This dependence on the number of four-fermi vertices means that there are worse and worse divergences as we look at higher-order corrections to a given process. Even worse, it means that for any number of external lines \( F_E \) no matter how big, there is a large enough order in perturbation theory in \( G \) where the cutoff will appear! This means we need \( \delta_n(\bar{\psi}\psi)^n \) counterterms for every \( n \), which we’ll need to fix with physical input. This is a bit unappetizing, and such an interaction is called “non-renormalizable”. However, when we remember that we only need to make predictions to a given precision (so that we only need to go to a finite order in this process) we will see that such theories are nevertheless quite useful.

So why were those other examples independent of \( V \)? It’s because the couplings were dimensionless. Those theories were classically scale invariant (except for the mass terms).
7.5.3 Naive scale invariance in field theory

[Halpern] Consider a field theory of a scalar field $\phi$ in $D$ spacetime dimensions, with an action of the form

$$S[\phi] = \int d^D x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - g \phi^p \right)$$

for some constants $p, g$. Which value of $p$ makes this scale invariant? (That is: when is $g$ dimensionless, and hence possibly the coupling for a renormalizable interaction.)

Naive dimensions:

$$[S] = [h] = 0, \quad [x] \equiv -1, \quad [d^D x] = -D, \quad [\partial] = 1$$

The kinetic term tells us the engineering dimensions of $\phi$:

$$0 = [S_{\text{kinetic}}] = -D + 2 + 2 [\phi] \implies [\phi] = \frac{D - 2}{2}.$$ 

Notice that the $D = 1$ case agrees with our quantum mechanics counting from §7.1. Quantum field theory in $D = 1$ spacetime dimensions is quantum mechanics.

Then the self-interaction term has dimensions

$$0 = [S_{\text{interaction}}] = -D + [g] + p[\phi] \implies [g] = D - p[\phi] = D + p \frac{2 - D}{2}.$$ 

We expect scale invariance when $[g] = 0$ which happens when

$$p = p_D \equiv \frac{2D}{D - 2}.$$ 

i.e. the scale invariant scalar-field self-interaction in $D$ spacetime dimensions is $\phi^{\frac{2D}{D - 2}}$.

<table>
<thead>
<tr>
<th>$D$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>$\ldots$</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\phi]$</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$1$</td>
<td>$\frac{3}{2}$</td>
<td>$2$</td>
<td>$\ldots$</td>
<td>$\frac{D}{2}$</td>
</tr>
<tr>
<td>scale-inv’t $p \equiv p_D$</td>
<td>$-2$</td>
<td>$\infty$</td>
<td>6</td>
<td>4</td>
<td>$\frac{10}{3}$</td>
<td>3</td>
<td>$\ldots$</td>
<td>2</td>
</tr>
</tbody>
</table>

* What is happening in $D = 2$? The field is dimensionless, and so any power of $\phi$ is naively scale invariant, as are more complicated interactions like $g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j$, where the coupling $g(\phi)$ is a function of $\phi$. This allows for scale-invariant non-linear sigma models, where the fields are coordinates on a curved manifold with metric $ds^2 = g_{ij} d\phi^i d\phi^j$.

In dimensions where we get fractional powers, this isn’t so nice.
Notice that the mass term $\Delta S = \int d^D x \frac{m^2}{2} \phi^2$ gives

$$0 = -D + 2[m] + 2[\phi] \implies [m] = 1 \ \forall D < \infty$$

- it’s a mass, yay.

What are the consequences of this engineering dimensions calculation in QFT? For $D > 2$, an interaction of the form $g\phi^p$ has

$$[g] = D \cdot \frac{p_D - p}{p_D} \begin{cases} < 0 \text{ when } p > p_D, & \text{non-renormalizable or irrelevant} \\ = 0 \text{ when } p = p_D, & \text{renormalizable or marginal} \\ > 0 \text{ when } p < p_D, & \text{super-renormalizable or relevant.} \end{cases}$$

(7.23)

Consider the ‘non-renormalizable’ case. Suppose we calculate in QFT some quantity $f$ with $[f]$ as its naive dimension, in perturbation theory in $g$, e.g. by Feynman diagrams. We’ll get:

$$f = \sum_{n=0}^{\infty} g^n c_n$$

with $c_n$ independent of $g$. So

$$[f] = n[g] + [c_n] \implies [c_n] = [f] - n[g]$$

So if $[g] < 0$, $c_n$ must have more and more powers of some mass (inverse length) as $n$ increases. What dimensionful quantity makes up the difference? Sometimes it is masses or external momenta. But generically, it gets made up by UV divergences (if everything is infinite, dimensional analysis can fail, nothing is real, I am the walrus). More usefully, in a meaningful theory with a UV cutoff, $\Lambda_{UV}$, the dimensions get made up by the UV cutoff, which has $[\Lambda_{UV}] = 1$. Generically: $c_n = \tilde{c}_n (\Lambda_{UV})^{-n[g]}$, where $\tilde{c}_n$ is dimensionless, and $n[g] < 0$ – it’s higher and higher powers of the cutoff.

Consider the renormalizable (classically scale invariant) case: $[c_n] = [f]$, since $[g] = 0$. But in fact, what you’ll get is something like

$$c_n = \tilde{c}_n \log^{\nu(n)} \left( \frac{\Lambda_{UV}}{\Lambda_{IR}} \right),$$

where $\Lambda_{IR}$ is an infrared cutoff or a mass, $[\Lambda_{IR}] = 1$.

Some classically scale invariant examples (so that $m = 0$ and the bare propagator is $1/k^2$) where you can see that we get logs from loop amplitudes:

$\phi^4$ in $D = 4$: \[ \sim \int d^4 k \left( \frac{1}{k^2} \right)^2 \]

$\phi^6$ in $D = 3$: \[ \sim \int d^3 k \left( \frac{1}{k^2} \right)^3 \]
φ³ in $D = 6$: In $D = 2$, even the propagator for a massless scalar field has logs:

$$\langle \phi(x)\phi(0) \rangle = \int \frac{d^2k}{k^2} e^{-ikx} \sim \log \frac{|x|}{\Lambda_{UV}}.$$  

The terms involving ‘renormalizable’ in (7.23) are somewhat old-fashioned and come from a high-energy physics point of view where the short-distance physics is unknown, and we want to get as far as we can in that direction with our limited knowledge (in which case the condition ‘renormalizability’ lets us get away with this indefinitely – it lets us imagine we know everything). The latter terms are natural in the opposite situation (like condensed matter physics) where we know some basically correct microscopic description but want to know what happens at low energies. Then an operator like $\frac{1}{M^2} \phi^2$ whose coefficient is suppressed by some large mass scale $M$ is irrelevant for physics at energies far below that scale. Inversely, an operator like $m^2 \phi^2$ gives a mass to the $\phi$ particles, and matters very much (is relevant) at energies $E < m$. In the marginal case, the quantum corrections have a chance to make a big difference.

7.6 Vertex correction in QED

[Peskin chapter 6, Schwartz chapter 17, Zee chapter III.6] Back to work on QED. The vertex correction has some great physics payoffs:

- We’ll cancel the cutoff dependence we found in the $S$ matrix from $\delta Z$.
- We’ll compute $g - 2$ (the anomalous magnetic moment) of the electron, the locus of some of the most precise agreement between theory and experiment. (Actually the agreement is so good that it’s used as the definition of the fine structure constant. But a similar calculation gives the leading anomalous magnetic moment of the muon.)
- We’ll see that the exclusive differential cross section $\left(\frac{d\sigma}{d\Omega}\right)_{e\mu\rightarrow e\mu}$ that we’ve been considering is not really an observable. Actually it is infinity!  

8 More accurately, the exclusive cross section is zero; the one-loop correction is minus infinity, which is perturbation theory’s clumsy attempt to correct the finite tree level answer to make it zero.
This is an example of an IR divergence. While UV divergences mean you’re overstepping your bounds (by taking too seriously your Lagrangian parameters or your knowledge of short distances), IR divergences mean you are asking the wrong question.

To get started, consider the following class of diagrams.

\[ \equiv iM = ie^2(\bar{u}(p')\Gamma^\mu(p,p')u(p)) \frac{1}{q^2}\bar{u}(K')\gamma_\mu u(K) \] (7.24)

The shaded blob is the vertex function \( \Gamma \). The role of the light blue factors is just to make and propagate the photon which hits our electron; let’s forget about them. Denote the photon momentum by \( q = p' - p \). We’ll assume that the electron momenta \( p, p' \) are on-shell, but \( q^\mu \) is not, as in the \( e\mu \) scattering process. Then \( q^2 = 2m^2 - 2p' \cdot p \).

Before calculating the leading correction to the vertex \( \Gamma^\mu = \gamma^\mu + \mathcal{O}(e^2) \), let’s think about what the answer can be. It is a vector made from \( p, p', \gamma^\mu \) and \( m, e \) and numbers. It can’t have any \( \gamma^5 \) or \( \epsilon_{\mu\nu\rho\sigma} \) by parity symmetry of QED. So on general grounds we can organize it as

\[ \Gamma^\mu(p,p') = A\gamma^\mu + B(p + p')^\mu + C(p - p')^\mu \] (7.25)

where \( A, B, C \) are Lorentz-invariant functions of \( p^2 = (p')^2 = m^2, p \cdot p', \gamma, \sigma \). But, for example, \( \gamma_\mu u(p) = (m\gamma^\mu - p^\mu)u(p) \) which just mixes up the terms; really \( A, B, C \) are just functions of the momentum transfer \( q^2 \). Gauge invariance, in the form of the Ward identity, says that contracting the photon line with the photon momentum should give zero:

\[ 0 \overset{\text{Ward}}{=} q^\mu\bar{u}(p')\Gamma^\mu u(p) \overset{(7.25)}{=} \bar{u}(p') \begin{pmatrix} A \\ q \\ =p'-p \end{pmatrix} \bar{u}(p) + B (p + p') \cdot (p - p') + C q^2 \overset{u(m^2-m^2=0)}{=} \begin{pmatrix} m \end{pmatrix} u(p) \]

Therefore \( 0 = Cq^2\bar{u}(p')u(p) \) for general \( q^2 \) and general spinors, so \( C = 0 \). This is the moment for the Gordon identity to shine:

\[ \bar{u}(p')\gamma^\mu u(p) = \bar{u}(p') \left( \frac{p^\mu + p'^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m} \right) u(p) \]
(where \( \sigma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu] \)) can be used to eliminate the \( p + p' \) term\(^9\). The Gordon identity shows that the QED interaction vertex \( \bar{u}(p')\gamma^\mu u(p)A_\mu \) contains a magnetic moment bit in addition to the \( p + p' \) term (which is there for a charged scalar field).

It is then convenient (and conventional) to parametrize the vertex in terms of the two form factors \( F_{1,2} \):

\[
\Gamma^\mu(p,p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2). \tag{7.26}
\]

This little monstrosity has the complete information about the coupling of the electron to the electromagnetic field, such as for example a background electromagnetic field. It is a parametrization of the matrix elements of the current between two one-electron states, incorporating the fact of gauge invariance.

The first term at zero momentum \( e F_1(q^2 = 0) \) is the electric charge of the electron (if you don’t believe it, use the vertex \((7.26)\) to calculate the Coulomb field of the electron; there are some details on page 186 of Peskin). Since the tree-level bit of \( F_1 \) is 1, if by the letter \( e \) here we mean the actual charge, then we’d better include counterterms \((\mathcal{L}_{ct} \ni \bar{\psi} \delta_\epsilon \gamma^\mu A_\mu \psi)\) to make sure it isn’t corrected: \( F_1(0) = 1 \).

On the homework last quarter you showed (or see Peskin p. 187) that the magnetic moment of the electron is

\[
\vec{\mu} = g \frac{e}{2m} \vec{S},
\]

where \( \vec{S} \equiv \xi^\dagger \frac{\sigma}{2} \xi \) is the electron spin. Comparing with the vertex function, this says that the \( g \) factor is

\[
g = 2(F_1(0) + F_2(0)) = 2 + 2F_2(0) = 2 + \mathcal{O}(\alpha).
\]

We see that the anomalous magnetic moment of the electron is \( 2F_2(q^2 = 0) \).

\(^9\)Actually this is why we didn’t include a \( \sigma^{\mu\nu} \) term. You could ask: what about a term like \( \sigma^{\mu\nu}(p + p')^\nu \)? Well, there’s another Gordon identity that relates that to things we’ve already included:

\[
\bar{u}_2\sigma^{\mu\nu}(p_1 + p_2)^\nu u_1 = i\bar{u}_2 (q_\mu - (m_1 - m_2)\gamma_\mu) u_1.
\]

It is proved the same way: just use the Dirac equation \( p_1 u_1 = m_1 u_1 \), \( \bar{u}_2 p_2 = \bar{u}_2 m_2 \) and the Clifford algebra. We are interested here in the case where \( m_1 = m_2 \).
about what it’s for, we sketch the evaluation of the one-loop QED vertex correction:

\[ \text{with } k' \equiv k + q. \]

(1) Feynman parameters again. The one we showed before can be rewritten more symmetrically as:

\[ \frac{1}{AB} = \int_0^1 dx \int_0^1 dy \delta(x + y - 1) \frac{1}{(xA + yB)^2} \]

Now how can you resist the generalization\(^\text{10}\):

\[ \frac{1}{ABC} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(x + y + z - 1) \frac{2}{(xA + yB + zC)^3} \]

\(^\text{10}\)Peskin outlines a proof by induction of the whole family of such identities on page 190. But here’s a simpler proof using \textit{Schwinger parameters}. You’ll agree that

\[ \frac{1}{A} = \int_0^\infty ds \ e^{-sA}. \quad (7.27) \]

Applying this identity to each factor gives

\[ \frac{1}{A_1 A_2 \cdots A_n} = \int_0^\infty ds_1 \cdots \int_0^\infty ds_n \ e^{-\sum_{i=1}^n s_i A_i}. \]

Now use scaling to set \( \tau \equiv \sum_{i=1}^n s_i \), and \( x_i \equiv s_i/\tau \). Then

\[ \frac{1}{A_1 A_2 \cdots A_n} = \int_0^\infty d\tau \tau^{n-1} \frac{n!}{\prod_{i=1}^n x_i} \int_0^1 dx_i \delta \left( \sum_{i=1}^n x_i - 1 \right) e^{-\tau \sum_i x_i A_i}. \]

Now do the integral over \( \tau \), using \( \int_0^\infty d\tau \tau^{n-1} e^{-\tau x} = \frac{(n-1)!}{x^n} \) (differentiate (7.27) wrt \( A \)), to arrive at

\[ \frac{1}{A_1 A_2 \cdots A_n} = \frac{n!}{\prod_{i=1}^n x_i} \int_0^1 dx_i \delta \left( \sum_{i=1}^n x_i - 1 \right) \frac{(n-1)!}{(\sum_i x_i A_i)^n}. \]

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So, set \( A = (k')^2 - m_e^2, B = k^2 - m_e^2, C = (p - k)^2 - m^2 \) (with the appropriate i.e.s), so that the integral we have to do is

\[
\int \frac{d^4kN^\mu}{(k^2 + k \cdot (\cdots) + \cdots)^3}.
\]

(2) Complete the square, \( \ell = k - zp + xq \) to get \( \int \frac{d^4\ell N^\mu}{(\ell^2 - \Delta)^m} \) where

\[
\Delta = -xyq^2 + (1 - z)^2m^2 + zm^2.
\]

The \( \ell \)-dependence in the numerator is either 1 or \( \ell^\mu \) or \( \ell^\mu \ell^\nu \). In the integral over \( \ell \), the second averages to zero, and the third averages to \( \eta^\mu\nu\ell^2 \frac{1}{4} \). As a result, the momentum integrals we need are just

\[
\int \frac{d^D\ell}{(\ell^2 - \Delta)^m} \text{ and } \int \frac{d^D\ell \ell^2}{(\ell^2 - \Delta)^m}.
\]

Right now we only need \( D = 4 \) and \( m = 3 \), but it turns out to be quite useful to think about them all at once. Like in our discussion of the electron self-energy diagram, we can evaluate them by Wick rotating (which changes the denominator to \( \ell^2_E + \Delta \)) and going to polar coordinates. This gives:

\[
\int \frac{d^D\ell}{(\ell^2 - \Delta)^m} = (-1)^m \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(m - \frac{D}{2})}{\Gamma(m)} \left( \frac{1}{\Delta} \right)^{m - \frac{D}{2}}.
\]

(7.28)

\[
\int \frac{d^D\ell \ell^2}{(\ell^2 - \Delta)^m} = (-1)^m \frac{D}{2} \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(m - \frac{D}{2} - 1)}{\Gamma(m)} \left( \frac{1}{\Delta} \right)^{m - \frac{D}{2} - 1}.
\]

(7.29)

Notice that these integrals are not equal to infinity when the parameter \( D \) is not an integer. This is the idea behind dimensional regularization.

(0) But for now let’s persist in using the Pauli Villars regulator. (I call this step (0) instead of (3) because it should have been there all along.) Here this means we subtract from the amplitude the same quantity with \( m_\gamma \) replaced by \( \Lambda^2 \). The dangerous bit comes from the \( \ell^2 \) term we just mentioned, since \( m - D/2 - 1 = 3 - 4/2 - 1 = 0 \) means logs.

The numerator is

\[
N^\mu = \bar{u}(p')\gamma^\nu (k + q + m_e) \gamma^\mu (k + m_e) \gamma_\nu u(p)
= -2 (\mathcal{A}\bar{u}(p')\gamma^\nu u(p) + \mathcal{B}\bar{u}(p')\sigma^\mu\nu q_\nu u(p) + \mathcal{C}\bar{u}(p')q^\mu u(p))
\]

(7.30)

where

\[
\mathcal{A} = -\frac{1}{2} \ell^2 + (1 - x)(1 - y)q^2 + (1 - 4z + z^2)m^2
\]

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\[ B = i m z (1 - z) \]
\[ C = m (z - 2) (y - x). \]  
(7.31)

The blood of many men was spilled to arrive at these simple expressions (actually most of the algebra is done explicitly on page 319 of Schwartz). Now you say: but you promised there would be no term like C because of the Ward identity. Indeed I did and indeed there isn’t because C is odd in \( x \leftrightarrow y \) while everything else is even, so this term integrates to zero.

The first term (with \( A \)) is a correction to the charge of the electron and will be UV divergent. More explicitly, we get, using Pauli-Villars,
\[
\int d^4 \ell \left( \frac{\ell^2}{(\ell^2 - \Delta_m)^3} - \frac{\ell^2}{(\ell^2 - \Delta)^3} \right) = \frac{i}{(4\pi)^2} \ln \frac{\Delta}{\Delta_m}.
\]
The other bits are finite, and we ignore the terms that go like negative powers of \( \Lambda \). More on this cutoff dependence soon. But first something wonderful:

### 7.6.1 Anomalous magnetic moment

The second term \( B \) contains the anomalous magnetic moment:

\[
F_2(q^2) = \frac{2m_e}{e} \cdot \text{(the term with } B \text{)}
= \frac{2m}{e} 4e^3 m \int dxdydz \delta(x + y + z - 1) z(1-z) \int \frac{d^4 \ell}{(\ell^2 - \Delta)^3}
= \frac{\alpha}{\pi} m^2 \int dxdydz \delta(x + y + z - 1) \frac{z(1-z)}{(1-z)^2 m^2 - xyq^2}. \tag{7.32}
\]
The magnetic moment is the long-wavelength bit of this:

\[
F_2(q^2 = 0) = \frac{\alpha}{\pi} m^2 \int_0^1 dz \int_0^{1-z} dy \frac{z}{(1-z)m^2} = \frac{\alpha}{2\pi}.
\]

\[
g = 2 + \frac{\alpha}{\pi} + \mathcal{O}(\alpha^2).
\]

A rare opportunity for me to plug in numbers: \( g = 2.00232 \).

### 7.6.2 IR divergences mean wrong questions.

There is a term in the numerator from the \( A_{\gamma^\mu} \) bit
\[
\int \frac{d^4 \ell}{(\ell^2 - \Delta)^3} = c \frac{1}{\Delta}
\]

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(with \( c = -\frac{i}{32\pi^2} \) again), but without the factor of \( z(1-z) \) we had in the magnetic moment calculation. It looks like we’ve gotten away without having to introduce a UV regulator here, too (so far). But now look at what happens when we try to do the Feynman parameter integrals. For example, at \( q^2 = 0 \), we get (if we had set \( m_\gamma = 0 \))

\[
\int dx dy dz \delta(x + y + z - 1) \frac{m^2(1-4z+z^2)}{\Delta} = m^2 \int_0^1 dz \int_0^{1-z} dy \frac{-2 + 2(1-z) + (1-z)^2}{(1-z)^2m^2}
\]

\[
= \int_0^1 dz \frac{-2}{(1-z)} + \text{finite}, \tag{7.33}
\]

which diverges at the upper limit of integration. In fact it’s divergent even when \( q^2 \neq 0 \). This is a place where we actually need to include the photon mass, \( m_\gamma \), for our own safety.

The (IR singular bit of the) vertex (to \( \mathcal{O}(\alpha) \)) is of the form

\[
\Gamma^\mu = \gamma^\mu \left( 1 - \frac{\alpha}{2\pi} f_{IR}(q^2) \ln \left( \frac{-q^2}{m_\gamma^2} \right) \right) + \text{stuff which is finite as } m_\gamma \to 0. \tag{7.34}
\]

Notice that the IR divergent stuff depends on the electron momenta \( p, p' \) only through \( q \), the momentum of the photon. So it looks like we are led to conclude

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\mu e \to \mu e} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} \left( 1 - \frac{\alpha}{\pi} f_{IR}(q^2) \ln \left( \frac{-q^2}{m_\gamma^2} \right) \right) + \mathcal{O}(\alpha^2)
\]

which blows up when we remove the fake photon mass \( m_\gamma \to 0 \).

[Schwartz §20.1] I wanted to just quote the above result for (7.34) but I lost my nerve, so here is a bit more detail leading to it. The IR dangerous bit comes from the second term in \( A \) above. That is,

\[
F_1(q^2) = 1 + f(q^2) + \delta_1 + \mathcal{O}(\alpha^2)
\]

with

\[
f(q^2) = \frac{e^2}{8\pi^2} \int_0^1 dx dy dz \delta(x+y+z-1) \left( \ln \frac{z\Lambda^2}{\Delta} + \frac{q^2(1-x)(1-y) + m_e^2(1-4z+z^2)}{\Delta} \right).
\]

\( \delta_1 \) here is a counterterm for the \( \Psi \gamma^\mu A_\mu \Psi \) vertex.

We can be more explicit if we consider \( -q^2 \gg m_e^2 \) so that we can ignore the electron mass everywhere. Then we would choose the counterterm \( \delta_1 \) so that

\[
1 = F_1(0) \implies \delta_1 = -f(0) \frac{m_e}{q^2} \to -\frac{e^2}{8\pi^2} \frac{1}{2} \ln \frac{\Lambda^2}{m_\gamma^2},
\]
And the form of $f(q^2)$ is

$$f(q^2) = \frac{e^2}{8\pi^2} \int dx dy dz \delta(x + y + z - 1) \left( \ln \frac{(1 - x - y)\Lambda^2}{\Delta_{\text{IR finite}}} + \frac{q^2(1 - x)(1 - y)}{-xyq^2 + (1 - x - y)m^2_\gamma} \right)$$

$$F_1(q^2) = 1 - \frac{e^2}{16\pi^2} \left( \ln^2 \frac{-q^2}{m^2_\gamma} + 3 \ln \frac{-q^2}{m^2_\gamma} \right) + \text{finite.}$$

In doing the integrals, we had to remember the $i\epsilon$ in the propagators, which can be reproduced by the replacement $q^2 \rightarrow q^2 + i\epsilon$. This $\ln^2(q^2/m^2_\gamma)$ is called a Sudakov double logarithm. Notice that taking differences of these at different $q^2$ will not make it finite.

**Diversity and inclusion to the rescue.** Before you throw up your hands in despair, I would like to bring to your attention another consequence of the masslessness of the photon: It means real (as opposed to virtual) photons can be made with arbitrarily low energy. But a detector has a minimum triggering energy: the detector works by particles doing some physical something to stuff in the detector, and it has a finite energy resolution – it takes a finite amount of energy for those particles to do stuff. This means that a process with exactly one $e$ and one $\mu$ in the final state cannot be distinguished from a process ending in $e\mu$ plus a photon of arbitrarily small energy, such as would result from (final-state radiation) or (initial-state radiation). This ambiguity is present for any process with external charged particles.

Being more inclusive, then, we cannot distinguish amplitudes of the form

$$\bar{u}(p')\mathcal{M}_0(p',p)u(p) \equiv -i\left( \begin{array}{c} \end{array} \right),$$

from more inclusive amplitudes like

$$-i\left( \begin{array}{c} \end{array} \right) = \bar{u}(p')\gamma^\mu p' + \bar{u}(p')(p+\bar{p})\gamma^\mu u(p)\epsilon^*_\mu(k) + \bar{u}(p')\gamma^\mu p' + \bar{u}(p')\mathcal{M}_0(p',p)\gamma^\mu u(p)\epsilon^*_\mu(k).$$
Now, by assumption the photon is real \((k^2=0)\) and it is soft, in the sense that \(k^0 < E_c\), the detector cutoff. So we can approximate the numerator of the second term as
\[
(p - k + m_e) \gamma^\mu u(p) \simeq (p + m_e) \gamma^\mu u(p) = (2p^\mu + \gamma^\mu (p + m_e))u(p) = 2p^\mu u(p).
\]
In the denominator we have e.g. \((p - k)^2 - m^2 = p^2 - m^2 - 2p \cdot k + k^2 \sim -2p \cdot k\) since the electron is on shell and \(k \ll p\). Therefore
\[
\mathcal{M}(e \mu \leftrightarrow \text{one soft } \gamma) = e\bar{\nu}(p') \mathcal{M}_0(p', p)u(p) \left( \frac{p' \cdot \epsilon}{p' \cdot k + i\epsilon} - \frac{p \cdot \epsilon}{p \cdot k - i\epsilon} \right).
\]
This is bremsstrahlung. Before we continue this calculation to find the inclusive amplitude which a real detector actually measures, let’s pause to relate the previous expression to some physics we know. Where have we seen this kind of expression before? Notice that the \(i\epsilon\) are different because one comes from final state and one from initial. Well, this object is the Fourier transform \(\tilde{j}_\mu(k) = \int d^4x e^{ikx} j_\mu(x)\) of the current
\[
j_\mu(x) = e \int d\tau dy \frac{d^\mu}{d\tau} \delta^4(x - y(\tau))
\]
associated with a particle which executes a piecewise linear motion \(^{11}\)
\[
y(\tau) = \begin{cases} 
\frac{p^\mu}{m} \tau, & \tau < 0 \\
\frac{p'^\mu}{m} \tau, & \tau > 0 
\end{cases}.
\]
This is a good approximation to the motion a free particle which experiences a sudden acceleration; sudden means that the duration of the pulse is short compared to \(\omega^{-1}\) for any frequency we’re going to measure. The electromagnetic radiation that such an accelerating charge produces is given classically by Maxwell’s equation: \(\tilde{A}_\mu(k) = -\frac{1}{k^2} \tilde{j}_\mu(k)\).

I claim further that the factor \(f_{IR}(q^2) = \frac{\alpha}{\pi} \ln \left( \frac{-q^2}{m^2} \right)\) (which entered our lives in \((7.34)\)) arises classically as the number of soft photons produced by such a process in

\(^{11}\)Check it:
\[
\int d^4x j_\mu(x)e^{ikx} = e \int d\tau \frac{d^\mu}{d\tau} (\tau) e^{ikx(\tau)} = e \int_{-\infty}^{\infty} d\tau \frac{p'^\mu}{m} e^{i \left( \frac{p'^\mu}{m} + i\nu \right) \tau} + e \int_0^{\infty} d\tau \frac{p^\mu}{m} e^{i \left( \frac{p^\mu}{m} - i\nu \right) \tau} = \tilde{j}_\mu(k).
\]
Notice that the \(i\epsilon\) are convergence factors in the Fourier transforms.
each decade of wavenumber. You can figure this out by plugging \( \hat{A}^\mu(k) = -\frac{1}{k^2} \hat{j}^\mu(k) \) into the electromagnetic energy \( \frac{1}{2} \int d^3x (E^2 + B^2) \). See Peskin §6.1 for help.

\[
\left( \frac{d\sigma}{d\Omega} \right)_{E_\gamma < E_c}^{\mu \epsilon_{\text{soft}} \leftrightarrow \mu e} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}}^{\gamma} e^2 \int_0^{E_c} \frac{d^3k}{2E_k} \left[ \frac{2p \cdot \epsilon' - 2p' \cdot \epsilon'}{2p \cdot k} - \frac{2p \cdot \epsilon'}{2p' \cdot k} \right]^2 \sim \int_0 \frac{d^3k}{k^3} = \infty.
\]

This is another IR divergence. (One divergence is bad news, but two is an opportunity for hope.) Just like we must stick to our UV regulators like religious zealots, we must cleave tightly to the consistency of our IR regulators: we need to put back the photon mass:

\[
E_k = \sqrt{k^2 + m_\gamma^2}
\]

which means that the lower limit of the \( k \) integral gets cut off at \( m_\gamma \):

\[
\int_0^{E_c} \frac{dk}{E_k} = \left( \int_0^{m_\gamma} + \int_{m_\gamma}^{E_c} \right) \frac{dk}{\sqrt{k^2 + m_\gamma^2}} \sim \int_0^{m_\gamma} \frac{dk}{m_\gamma} + \int_{m_\gamma}^{E_c} \frac{dk}{k}.
\]

Being careful about the factors, the actual cross section measured by a detector with energy resolution \( E_c \) is\(^{12}\)

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{observed}} = \left( \frac{d\sigma}{d\Omega} \right)_{\epsilon_{\text{mu}} \leftrightarrow \mu e} + \left( \frac{d\sigma}{d\Omega} \right)_{\mu \epsilon_{\text{soft}} \leftrightarrow \mu e}^{E_\gamma < E_c} + \mathcal{O}(\alpha^3)
\]

\[
= \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}}^{\text{vertex correction}} 1 - \frac{\alpha}{\pi} f_{IR}(q^2) \ln \left( \frac{-q^2}{m_\gamma^2} \right) + \frac{\alpha}{\pi} f_{IR}(q^2) \ln \left( \frac{E_c^2}{m_\gamma^2} \right)
\]

\[
= \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}}^{\text{soft photons}} 1 - \frac{\alpha}{\pi} f_{IR}(q^2) \ln \left( \frac{-q^2}{E_c^2} \right)
\]

The thing we can actually measure is independent of the IR regulator photon mass \( m_\gamma \), and finite when we remove it. On the other hand, it depends on the detector resolution. Like in the plot of some kind of Disney movie, an apparently minor character whom you may have been tempted to regard as an ugly detail has saved the day.

\(^{12}\)Notice that we add the cross-sections, not the amplitudes, for these processes with different final states. Here’s why: even though we don’t measure the existence of the photon, something does: it gets absorbed by some part of the apparatus or the rest of the world and therefore becomes entangled some of its degrees of freedom; when we fail to distinguish between those states, we trace over them, and this erases the interference terms we would get if we summed the amplitudes.
I didn’t show explicitly that the coefficient of the log is the same function \( f_{IR}(q^2) \). In fact this function is \( f_{IR}(q^2) = \frac{1}{2} \log(-q^2/m^2) \), so the product \( f_{IR} \ln q^2 \sim \ln^2 q^2 \) is the Sudakov double logarithm. A benefit of the calculation which shows that the same \( f_{IR} \) appears in both places (Peskin chapter 6.5) is that it also shows that this pattern persists at higher order in \( \alpha \): there is a \( \ln^2(q^2/m^2) \) dependence in the two-loop vertex correction, and a matching \(-\ln^2(E_c^2/m^2)\) term in the amplitude to emit two soft photons. There is a \( \frac{1}{2!} \) from Bose statistics of these photons. The result exponentiates, and we get

\[
e^{-\frac{\alpha}{\pi} f \ln(-q^2/m^2)} e^{-\frac{\alpha}{\pi} f (E_c^2/m^2)} = e^{-\frac{\alpha}{\pi} f \ln(-q^2/E_c^2)}.
\]

You may be bothered that I’ve made all this discussion about the corrections from the electron line, but said nothing about the muon line. But the theory should make sense even if the electron and muon charges \( Q_e, Q_m \) were different, so the calculation should make sense term-by-term in an expansion in \( Q_m \).

Some relevant names for future reference: The name for the guarantee that this always works in QED is the Bloch-Nordsieck theorem. Closely-related but more serious issues arise in QCD, the theory of quarks and gluons; this is the beginning of the story of jets (a jet is some IR-cutoff dependent notion of a QCD-charged particle plus the cloud of stuff it carries with it) and parton distribution functions.

[End of Lecture 25]

**Sketch of exponentiation of soft photons.** [Peskin §6.5] Consider a diagram with \( n \) soft external photons, summed over ways of distributing them on an initial and final electron line:

\[
\sum_{n_f=1}^{n} = \bar{u}(p') i \mathcal{M}_0 u(p) e_n \prod_{\alpha=1}^{n_f} \left( \frac{p'^\mu_\alpha}{p' \cdot k_\alpha} - \frac{p^{\mu_\alpha}}{p \cdot k_\alpha} \right) \equiv \mathcal{A}_n.
\]

Here the difference in each factor is just as in (7.35), one term from initial and one from final-state emission; expanding the product gives the sum over \( n_f = 1 - n_i \), the number coming from the final-state line. From this expression, we can make a diagram with a soft-photon loop by picking an initial line \( \alpha \) and a final line \( \beta \) setting \( k_\alpha = -k_\beta \equiv k \) and tying them together with a propagator and summing over \( k \):

\[
\mathcal{A}_{n-2} = \frac{e^2}{2} \int d^4k \frac{-i \eta_{\mu\alpha}}{k^2} \left( \frac{p'}{p' \cdot k} - \frac{p}{p \cdot k} \right)^\nu \left( \frac{p'}{-p' \cdot k} - \frac{p}{-p \cdot k} \right)^\sigma.
\]
The factor of $\frac{1}{2}$ accounts for the symmetry under exchange of $\alpha \leftrightarrow \beta$. For the case of $n = 2$, this is the whole story, and this is

$$\bar{u}iM_0u \cdot X = \begin{pmatrix} \text{soft part} \end{pmatrix}$$

(where here ‘soft part’ means the part which is singular in $m_\gamma$) from which we conclude that

$$X = -\frac{\alpha}{2\pi} f_{IR}(q^2) \ln \left( \frac{-q^2}{m_\gamma^2} \right) + \text{finite.}$$

Taking the most IR-divergent bit with $m$ virtual soft photons (order $\alpha^m$) for each $m$ gives

$$\mathcal{M}_{\text{virtual soft}} = \sum_{m=0}^{\infty} \left( \begin{pmatrix} \text{soft part} \end{pmatrix} \right) \sum_{m} \frac{1}{m!} X^m$$

where the $1/m!$ is a symmetry factor from interchanging the virtual soft photons.

Now consider the case of one real external soft ($E \in [m_\gamma, E_c]$) photon in the final state. The cross section is

$$d\sigma_{1\gamma} = \int d\Pi \sum_{\text{pols}} \epsilon^\mu \epsilon^{*\nu} \mathcal{M}_\mu \mathcal{M}^{*}_\nu \quad \equiv d\sigma_0 Y,$$

$$Y = \alpha \pi f_{IR}(q^2) \ln \left( \frac{E_c^2}{m_\gamma^2} \right).$$

(The integral is done in Peskin, page 201.) Therefore, the exclusive cross section, including contributions of soft real photons gives

$$\sum_{n=0}^{\infty} d\sigma_{n\gamma} = d\sigma_0 \sum_{n} \frac{1}{n!} Y^n = d\sigma_0 e^Y.$$ 

Here the $n!$ is because the final state contains $n$ identical bosons.

Putting the two effects together gives the promised cancellation of $m_\gamma$ dependence to all orders in $\alpha$:

$$d\sigma = d\sigma_0 e^{2X} e^Y$$
\[ d\sigma_0 \exp \left( -\frac{\alpha}{\pi} f_{IR}(q^2) \ln \frac{-q^2}{m_\gamma^2} + \frac{\alpha}{\pi} f_{IR}(q^2) \ln \frac{E_c^2}{m_\gamma^2} \right) \]
\[ = d\sigma_0 \exp \left( -\frac{\alpha}{\pi} f_{IR}(q^2) \ln \frac{-q^2}{E_c^2} \right) \]

This might seem pretty fancy, but unpacking the sum we did, the basic statement is that the probability of finding \( n \) photons with energy in a given (low-energy) range \([E_-, E_+]\) is

\[ P_{[E_-, E_+]} = \frac{1}{n!} \lambda^n e^{-\lambda}, \quad \lambda = \frac{\alpha}{\pi} f_{IR}(q^2) \ln \frac{E_+}{E_-} = \langle n \rangle = \langle n^2 \rangle - \langle n \rangle^2 \]

a Poisson distribution. This is just what one finds in a coherent state of the radiation field.

### 7.6.3 Some magic from gauge invariance of QED

We found that the self-energy of the electron gave a wavefunction renormalization factor

\[ Z_2 = 1 + \frac{\partial \Sigma}{\partial \not{p}} \big|_{\not{p} = m_0} + \mathcal{O}(e^4) = 1 - \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \text{finite} + \mathcal{O}(\alpha^2). \]

We care about this because there is a factor of \( Z_2 \) in the LSZ formula for an S-matrix element with two external electrons. On the other hand, we found a cutoff-dependent correction to the vertex \( e\gamma^\mu F_1(q^2) \) of the form

\[ F_1(q^2) = 1 + \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \text{finite} + \mathcal{O}(\alpha^2). \]

Combining these together

\[ S_{e\mu_e\gamma \gamma} = \left( \sqrt{Z_2(e)} \right)^2 \left( \bar{\gamma} \cdot \not{p}_e + \bar{\gamma} \cdot \not{p}_\gamma \right) + \cdots \]
\[ = \left( 1 - \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \cdots \right) e^2 \bar{u}(p') \left( \gamma^\mu \left( 1 + \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \cdots \right) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right) u(p) \]

the UV divergence from the vertex cancels the one in the self-energy. Why did this have to happen? During our discussion of the IR divergences, I mentioned a counterterm \( \delta_1 \) for the vertex. But how many counterterms do we get here? Is there a point of view which makes this cancellation obvious? Notice that the \( \cdots \) multiplying the \( \gamma^\mu \) term still contain the vacuum polarization diagram, which is our next subject, and which may be (is) cutoff dependent. Read on.
7.7 Vacuum polarization

[Zee, III.7] We’ve been writing the QED lagrangian as
\[ \mathcal{L} = \bar{\psi} \left( \partial - i e / \tilde{\mathbf{A}} - m \right) \psi - \frac{1}{4} \tilde{F}_{\mu \nu} \tilde{F}^{\mu \nu}. \]

I’ve put tildes on the photon field because of what’s about to happen: Suppose we rescale the definition of the photon field \( e \tilde{A}_\mu \equiv A_\mu, e \tilde{F}_{\mu \nu} \equiv F_{\mu \nu} \). Then the coupling \( e \) moves to the photon kinetic term:
\[ \mathcal{L} = \bar{\psi} \left( \partial - i A - m \right) \psi - \frac{1}{4 e^2} F_{\mu \nu} F^{\mu \nu}. \]

With this normalization, instead of measuring the coupling between electrons and photons, the coupling constant \( e \) measures the difficulty a photon has propagating through space:
\[ \langle A_\mu A_\nu \rangle \sim -i \eta_{\mu \nu} e^2 q^2. \]

None of the physics is different, since each internal photon line still has two ends on a \( \bar{\psi} A \psi \) vertex.

But from this point of view it is clear that the magic of the previous subsection is a consequence of gauge invariance, here’s why: the demand of gauge invariance relates the coefficients of the \( \bar{\psi} \partial \psi \) and \( \bar{\psi} A \psi \) terms\(^{13}\). Therefore, any counterterm we need for the \( \bar{\psi} \partial \psi \) term (which comes from the electron self-energy correction and is traditionally called \( \delta Z_2 \)) must be the same as the counterterm for the \( \bar{\psi} A \psi \) term (which comes from the vertex correction and is called \( \delta Z_1 \)). No magic, just gauge invariance.

A further virtue of this reshuffling of the factors of \( e \) (emphasized by Zee on page 205) arises when we couple more than one species of charged particle to the electromagnetic field, e.g. electrons and muons or, more numerously, protons: once we recognize that charge renormalization is a property of the photon itself, it makes clear that quantum corrections cannot mess with the ratio of the charges. A deviation from \(-1\) of the ratio of the charges of electron and proton as a result of interactions might seem plausible given what a mess the proton is, and would be a big deal for atoms. Gauge invariance forbids it.

Just as we defined the electron self-energy (amputated 2-point function) as
\[ \Sigma^\uparrow = -i \Sigma(p) \]
(with two spinor indices implied), we define the photon self-energy as
\[ +i \Pi_{\mu \nu}(q^2) \equiv \begin{array}{c} \text{Diagram} \end{array} = \begin{array}{c} \text{Vertex} \end{array} + \mathcal{O}(e^4) \]

\(^{13}\)Notice that the gauge transformation of the rescaled \( A_\mu \) is \( A_\mu \rightarrow A_\mu + \partial_\mu \lambda(x), \psi(x) \rightarrow e^{i q \lambda(x)} \psi(x) \) so that \( D_\mu \psi \equiv (\partial + q A_\mu) \psi \rightarrow e^{i q \lambda} D_\mu \psi \) where \( q \) is the charge of the field (\( q = -1 \) for the electron).

This is to be contrasted with the transformation of \( \tilde{A}_\mu \rightarrow \tilde{A}_\mu - \partial_\mu \lambda(x)/e. \)
(the diagrams on the RHS are amputated). It is a function of \( q^2 \) by Lorentz symmetry. (The reason for the difference in sign is that the electron propagator is \( \frac{-i\gamma_{\mu\nu}}{q^2} \) while the photon propagator is \( \frac{-i\eta_{\mu\nu}}{q^2} \).) We can parametrize the answer as

\[
\Pi^{\mu\nu}(q^2) = A(q^2)\eta^{\mu\nu} + B(q^2)q^\mu q^\nu.
\]

The Ward identity says

\[
0 = q_\mu \Pi^{\mu\nu}(q^2) \implies 0 = Aq^\nu + Bq^2 q^\nu \implies B = -A/q^2.
\]

Let \( A \equiv \Pi^2 \) so that

\[
\Pi^{\mu\nu}(q^2) = \Pi(q^2) q^2 \left( \eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right).
\]

This object \( \Delta_T^{\mu\nu} \) is a projector

\[
\Delta_T^{\mu\rho}\Delta_T^{\nu\sigma} = \Delta_T^{\mu\rho}
\]

onto modes transverse to \( q^\mu \). Recall that we can take the bare propagator to be

\[
\frac{-i\Delta_T}{q^2}
\]

without changing any gauge-invariant physics. This is useful because then

\[
\tilde{G}^{(2)}(q) = \sum + \sum + \sum + \cdots \tag{7.36}
\]

\[
\Delta_T^2 = \frac{-i\Delta_T}{q^2} \left( 1 + \Pi\Delta_T + \Pi^2 \Delta_T + \cdots \right) = \frac{-i\Delta_T}{q^2} \frac{1}{1 - \Pi(q^2)}.
\]

Does the photon get a mass? If the thing I called \( A \) above \( q^2 \Pi(q^2) \rightarrow 0 \) \( A_0 \neq 0 \) (that is, if \( \Pi(q^2) \sim \frac{A_0}{q^2} \) or worse), then \( \tilde{G} \rightarrow \frac{1}{q^2 - A_0} \) does not have a pole at \( q^2 = 0 \). If \( \Pi(q^2) \) is regular at \( q^2 = 0 \), then the photon remains massless. In order to get such a singularity in the photon self energy \( \Pi(q^2) \sim \frac{A_0}{q^2} \) we need a process like \( \delta\Pi \sim \), where the intermediate state is a massless boson with propagator \( \sim \frac{A_0}{q^2} \). As I will explain below, this is the Higgs mechanism (not the easiest way to understand it).

The Ward identity played an important role here. Why does it work for the vacuum polarization?

\[
q_\mu \Pi^{\mu\nu}(q^2) = q_\mu \sum \propto e^2 \int d^4p \frac{1}{p + q - m} \frac{1}{p - m} \gamma^\nu.
\]

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But here is an identity:

\[
\frac{1}{p+q-m} - \frac{m}{p-m} = \frac{1}{p-m} - \frac{1}{p+q-m}.
\]

(7.38)

Now, if we shift the integration variable \( p \to p+q \) in the second term, the two terms cancel.

Why do I say ‘if’? If the integral depends on the UV limit, this shift is not innocuous. So we have to address the cutoff dependence.

In addition to the (lack of) mass renormalization, we’ve figured out that the electromagnetic field strength renormalization is

\[
Z_\gamma \equiv Z_3 = \frac{1}{1-\Pi(0)} \sim 1 + \Pi(0) + O(e^4).
\]

We need \( Z_\gamma \) for example for the \( S \)-matrix for processes with external photons, like Compton scattering.

Claim: If we do it right\(^\text{14}\), the cutoff dependence looks like\(^\text{15}\):

\[
\Pi_2(q^2) = \frac{\alpha_0}{4\pi} \left( -\frac{2}{3} \ln \Lambda^2 + 2D(q^2) \right)_{\text{finite}}
\]

where \( \Lambda \) is the UV scale of ignorance. The photon propagator gets corrected to

\[
\frac{e_0^2 \Delta_T}{q^2} \sim \frac{Z_3 e_0^2 \Delta_T}{q^2},
\]

and \( Z_3 = \frac{1}{1-\Pi(0)} \) blows up logarithmically if we try to remove the cutoff. You see that the fine structure constant \( \alpha_0 = \frac{e^2}{4\pi} \) has acquired the subscript of deprecation: we can make the photon propagator sensible while removing the cutoff if we are willing to recognize that the letter \( e_0 \) we’ve been carrying around is a fiction, and write everything

\(^{14}\)What I mean here is: if we do it in a way which respects the gauge invariance and hence the Ward identity. The simple PV regulator we’ve been using does not quite do that. However, an only slightly more involved implementation, explained in Zee page 202-204, does. Alternatively, we could use dimensional regularization everywhere.

\(^{15}\)The factor in front of the \( \ln \Lambda \) can be made to look like it does in other textbooks using \( \alpha = \frac{e^2}{12\pi^2} \), so that

\[
\frac{\alpha_0}{4\pi} \left( \frac{2}{3} \ln \Lambda^2 \right) = \frac{e_0^2}{12\pi^2} \ln \Lambda.
\]
in terms of \( e \equiv \sqrt{Z_3}e_0 \) where \( \frac{e^2}{4\pi} = \frac{1}{137} \) is the measured fine structure constant. To this order, then, we write

\[
e_0^2 = e^2 \left( 1 + \frac{\alpha_0}{4\pi} \frac{2}{3} \ln \Lambda^2 \right) + \mathcal{O}(\alpha^2). \tag{7.39}
\]

\[
m_0 = m + \mathcal{O}(\alpha_0) = m + \mathcal{O}(\alpha). \tag{7.40}
\]

Since the difference between \( \alpha_0 \) and \( \alpha \) is higher order (in either), our book-keeping is unchanged. Inverting the relationship perturbatively, the renormalized charge is

\[
e^2 = e_0^2 \left( 1 - \frac{\alpha_0}{4\pi} \frac{2}{3} \ln \Lambda^2 + \mathcal{O}(\alpha^2) \right)
\]

– in QED, the quantum fluctuations reduce the charge, as you might expect from the interpretation of this phenomenon as dielectric screening.

[End of Lecture 26]

In the example case of \( e\mu \leftrightarrow e\mu \) scattering, the UV cutoff dependence looks like

\[
S_{e\mu\leftrightarrow e\mu} = \sqrt{Z_e} \left( 1 - \frac{\alpha_0}{4\pi} \ln \Lambda^2 + \frac{\alpha_0}{2\pi} A(m_0) \right) e_0^2 L_\mu \bar{u}(p') \left[ \gamma^\mu \left( 1 + \frac{\alpha_0}{4\pi} \ln \Lambda^2 + \frac{\alpha_0}{2\pi} (B + D) + \frac{\alpha_0}{4\pi} \left( -\frac{2}{3} \ln \Lambda^2 \right) \right) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \frac{\alpha_0}{2\pi} C(q^2, m_0) \right] u(p)
\]

\[
= e^2 L_\mu \bar{u}(p') \left[ \gamma^\mu \left( 1 + \frac{\alpha}{2\pi} (A + B + D) \right) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \frac{\alpha}{2\pi} C \right] u(p) + \mathcal{O}(\alpha^2) \tag{7.41}
\]

where \( L_\mu \) is the stuff from the muon line, and \( A, B, C, D \) are finite functions of \( m, q^2 \).

In the second step, two things happened: (1) we cancelled the UV divergences from the \( Z \)-factor and from the vertex correction: this had to happen because there was no possible counterterm. (2) we used (7.39) and (7.40) to write everything in terms of the measured \( e, m \).

Claim: this works for all processes to order \( \alpha^2 \). For example, Bhabha scattering gets a contribution of the form

\[
\propto e_0 \frac{1}{1 - \Pi(0)} e_0 = e^2.
\]

In order to say what are \( A, B, D \) we need to specify more carefully a renormalization scheme. To do that, I need to give a bit more detail about the integral.

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7.7.1 Under the hood

The vacuum-polarization contribution of a fermion of mass $m$ and charge $e$ at one loop is

$$q,\mu \rightsquigarrow q,\nu = -\int d^Dk \text{tr} \left( \frac{i e\gamma^\mu}{k^2 - m^2} \frac{i e\gamma^\nu}{(q + k)^2 - m^2} \right) \left( q + \frac{k}{q} + \frac{k}{q} + m \right)$$

The minus sign out front is from the fermion loop. Some boiling, which you can find in Peskin (page 247) or Zee (§III.7), reduces this to something manageable. The steps involved are: (1) a trick to combine the denominators, like the Feynman trick

$$\int_0^1 dx \left( \frac{1}{(1-x)} + \frac{x}{1-x} \right) \frac{1}{AB} = \int_0^1 dx \left( \frac{1}{(1-x)A + xB} \right)^2.$$  

(2) some Dirac algebra, to turn the numerator into a polynomial in $k, q$. As Zee says, our job in this course is not to train to be professional integrators. The result of this boiling can be written

$$i\Pi^{\mu\nu}(q) = -e^2 \int d^D \ell \int_0^1 dx \frac{N^{\mu\nu}}{\ell^2 - \Delta^2}$$

with $\ell = k + xq$ is a new integration variable, $\Delta \equiv m^2 - x(1-x)q^2$, and the numerator is

$$N^{\mu\nu} = 2\ell^\mu \ell^\nu - \eta^{\mu\nu} \ell^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu} \left( m^2 + x(1-x)q^2 \right) + \text{terms linear in } \ell^\mu.$$ 

At this point I have to point out a problem with applying the regulator we’ve been using (I emphasize that this is a distinct issue from the choice of RG scheme). With a euclidean momentum cutoff, the diagram $$\rightsquigarrow$$ gives something of the form

$$i\Pi^{\mu\nu}(q) \propto e^2 \int d^D \ell E \frac{\ell^2 E \eta^{\mu\nu}}{(\ell^2 E - \Delta)^2} + \ldots \propto e^2 \Lambda^2 \eta^{\mu\nu}$$

This is NOT of the form $\Pi^{\mu\nu} = \Delta^{\mu\nu} T \Pi(p^2)$; rather it produces a correction to the photon mass proportional to the cutoff. What happened? Our cutoff was not gauge invariant. Oops.\footnote{Two points: How could we have predicted that the cutoff on euclidean momentum $\ell_E^2 < \Lambda^2$ would break gauge invariance? Its violation of the Ward identity here is a proof, but involved some work. The idea is that the momentum of a charged field shifts under a gauge transformation. Second: it is possible to construct a gauge invariant regulator with an explicit UV cutoff, using a lattice. The price, however, is that the gauge field enters only via the link variables $U(x, \hat{e}) = e^{i \int_{x}^{x+\hat{e}} A}$ where $x$ is a site in the lattice and $\hat{e}$ is the direction to a neighboring site in the lattice. For more, look up ‘lattice gauge theory’ in Zee’s index. More on this later.}

**Fancier PV regularization.**  [Zee page 202] We can fix the problem by adding also heavy Pauli-Villars electron ghosts. Suppose we add a bunch of them with masses...
$m_a$ and couplings $\sqrt{e_a}e$ to the photon. Then the vacuum polarization is that of the electron itself plus

$$-\sum_a c_a \int d^D k \text{tr} \left( (ie_{\gamma^\mu}) \frac{i}{q + \not{k} - m_a} (ie_{\gamma^\nu}) \frac{i}{q - m_a} \right) \sim \int \Lambda^\Lambda k \left( \sum_a c_a + \sum_a c_a m_a^2 \frac{1}{p^4} + \ldots \right).$$

So, if we take $\sum_a c_a = -1$ we cancel the $\Lambda^2$ term, and if we take $\sum_a c_a m_a^2 = -m^2$, we also cancel the $\ln \Lambda$ term. This requires at least two PV electron fields, but so what? Once we do this, the momentum integral converges, and the Ward identity applies, so the answer will be of the promised form $\Pi^{\mu\nu} = q^2 \Pi^{\mu\nu}_{\Delta T}$. After some more boiling, the answer is

$$\Pi_2(q^2) = \frac{1}{2\pi^2} \int dx x(1 - x) \ln \frac{M^2}{m^2 - x(1 - x)q^2}$$

where $\ln M^2 \equiv -\sum_a c_a \ln m_a^2$. This $M$ plays the role of the UV scale of ignorance thenceforth.

Notice that this is perfectly consistent with our other two one-loop PV calculations: in those, the extra PV electrons never get a chance to run. At higher loops, we would have to make sure to be consistent.

**Dimensional regularization.** A regulator which is more automatically gauge invariant is dimensional regularization (dim reg). I have already been writing many of the integrals in $D$ dimensions. One small difference when we are considering this as a regulator for an integral of fixed dimension is that we don’t want to violate dimensional analysis, so we should really replace

$$\int d^4 \ell \longrightarrow \int \frac{d^{4-\epsilon} \ell}{\bar{\mu}^{-\epsilon}}$$

where $D = 4 - \epsilon$ and $\bar{\mu}$ is an arbitrary mass scale which will appear in the regulated answers, which we put here to preserve dim’l analysis — i.e. the couplings in dim reg will have the same engineering dimensions they had in the unregulated theory (dimensionless couplings remain dimensionless). $\bar{\mu}$ will parametrize our RG, i.e. play the role of the RG scale. (It is often called $\mu$ at this step and then suddenly replaced by something also called $\mu$; I will instead call this $\bar{\mu}$ and relate it to the thing that ends up being called $\mu$.)

[Zinn-Justin 4th ed page 233] Dimensionally regularized integrals can be defined systematically with a few axioms indicating how the $D$-dimensional integrals behave under

1. translations $\int d^D p f(p + q) = \int d^D p f(p)$\(^{17}\)

\(^{17}\)Note that this rule fails for the euclidean momentum cutoff. Also note that this is the property we needed to demonstrate the Ward identity for the vertex correction using (7.38).
2. scaling $\int d^D p f(sp) = |s|^{-D} \int d^D p f(p)$

3. factorization $\int d^D p \int d^D q f(p)g(q) = \int d^D p f(p) \int d^D q g(q)$

The (obvious?) third axiom implies the following formula for the sphere volume as a continuous function of $D$:

$$\left(\frac{\pi}{a}\right)^{D/2} = \int d^D x e^{-ax^2} = \Omega_{D-1} \int_0^\infty x^{D-1} dx e^{-ax^2} = \frac{1}{2} a^{-\frac{D}{2}} \Gamma\left(\frac{D}{2}\right) \Omega_{D-1}. \tag{7.42}$$

This defines $\Omega_{D-1}$ for general $D$.

In dim reg, the one-loop vacuum polarization correction does satisfy the gauge-invariance Ward identity $\Pi_{\mu\nu} = \Delta_{\mu\nu} T \delta \Pi$. A peek at the tables of dim reg integrals shows that $\Pi_2$ is:

$$\Pi_2(p^2) \overset{\text{Peskin}}{=} -\frac{8e^2}{(4\pi)^{D/2}} \int_0^1 dx x(1-x) \frac{\Gamma(2-D/2)}{\Delta^{2-D/2}} \mu^\epsilon \tag{7.43}$$

where $\mu^2 \equiv 4\pi \mu^2 e^{-\gamma_E}$ where $\gamma_E$ is the Euler-Mascheroni constant; we define $\mu$ in this way so that, like Rosen- crantz and Guildenstern, $\gamma_E$ both appears and disappears from the discussion at this point.

In the second line of (7.43), we expanded the $\Gamma$-function about $D = 4$. Notice that what was a log divergence, becomes a $\frac{1}{2}$ pole in dim reg. There are other singularities of this function at other integer dimensions. It is an interesting question to ponder why the integrals have such nice behavior as a function of $D$. That is: they only have simple poles. A partial answer is that in order to have worse (e.g. essential) singularities at some $D$, the perturbative field theory would have to somehow fail to make sense at larger $D$.

Now we are in a position to choose a renormalization condition (also known as a renormalization scheme), which will specify how much of the finite bit of $\Pi$ gets subtracted by the counterterm. One possibility is to demand that the photon propagator is not corrected at $q = 0$, i.e. demand $Z_\gamma = 1$. Then the resulting one-loop shift is

$$\delta\Pi_2(q^2) \equiv \Pi_2(q^2) - \Pi_2(0) = \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \log \left( \frac{m^2 - x(1-x)q^2}{m^2} \right).$$

We’ll use this choice below.
Another popular choice, about which more later, is called the \(\overline{\text{MS}}\) scheme, in which \(\Pi\) is defined by the rule that we subtract the \(1/\epsilon\) pole. This means that the counterterm is

\[
\delta [\Pi \overline{\text{MS}}]^{(f^2)} = -\frac{e^2}{2\pi^2} \frac{2}{\epsilon} \int_0^1 dx x (1 - x) \quad = \frac{1}{6}.
\]

(Confession: I don’t know how to state this in terms of a simple renormalization condition on \(\Pi_2\). Also: the bar in \(\overline{\text{MS}}\) refers to the (not so important) distinction between \(\bar{\mu}\) and \(\mu\).) The resulting vacuum polarization function is

\[
\delta \Pi_2^{(\overline{\text{MS}})} (p^2) = \frac{e^2}{2\pi^2} \int_0^1 dx x (1 - x) \log \left( \frac{m^2 - x (1 - x) p^2}{\mu^2} \right).
\]

7.7.2 Physics from vacuum polarization

One class of physical effects of vacuum polarization arise from attaching the corrected photon propagator to a static delta-function charge source. The resulting effective Coulomb potential is the fourier transform of

\[
\tilde{V}(q) = \frac{1}{q^2} \frac{e^2}{1 - \Pi(q^2)} \equiv \frac{e^2}{q^2} \frac{\epsilon_{\text{eff}}(q)}{q^2}.
\]

This has consequences in both IR and UV.

**IR:** In the IR \((q^2 \ll m^2)\), it affects the spectra of atoms. The leading correction is

\[
\delta \Pi_2(q) = \int dx x (1 - x) \ln \left( 1 - \frac{q^2}{m^2} x (1 - x) \right) \approx \int dx x (1 - x) \left( \frac{-q^2}{m^2} x (1 - x) \right) = \frac{q^2}{30 m^2}
\]

which means

\[
\tilde{V}(q) \approx \frac{e^2}{q^2} + \frac{e^2}{q^2} \left( -\frac{q^2}{30 m^2} \right) + \cdots
\]

and hence

\[
V(r) = -\frac{e^2}{4 \pi r^2} - \frac{e^4}{60 \pi^2 m^2} \delta(r) + \cdots \equiv V + \Delta V.
\]

This shifts the energy levels of hydrogen \(s\)-orbitals (the ones with support at the origin) by \(\Delta E_s = \langle s | \Delta V | s \rangle\) which contributes to lowering the \(2S\) state relative to the \(2P\) state (the Lamb shift).

This delta function is actually a long-wavelength approximation to what is called the Uehling potential; its actual range is \(1/m_e\), which is the scale on which \(\Pi_2\) varies. The
delta function approximation is a good idea for atomic physics, since $\frac{1}{m_e} \ll a_0 = \frac{1}{\alpha m_e}$, the Bohr radius. See Schwartz p. 311 for a bit more on this.

**UV:** In the UV limit ($q^2 \gg m^2$), we can approximate $\ln \left( 1 - \frac{q^2}{m^2} x(1-x) \right) \simeq \ln \left( -\frac{q^2}{m^2} \right)$ to get\(^\text{18}\)

$$\Pi_2(q^2) = \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \left( 1 - \frac{q^2}{m^2} x(1-x) \right) \simeq \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \left( -\frac{q^2}{m^2} \right) = \frac{e^2}{12\pi^2} \ln \left( -\frac{q^2}{m^2} \right).$$

Therefore, the effective charge in (7.44) at high momentum exchange is

$$e^2_{\text{eff}}(q^2) \stackrel{q^2 \gg m^2}{\simeq} \frac{e^2}{1 - \frac{e^2}{12\pi^2} \ln \left( -\frac{q^2}{m^2} \right)}.$$ 

(Remember that $q^2 < 0$ for t-channel exchange, as in the static potential, so the argument of the log is positive and this is real.)

Two things: if we make $q^2$ big enough, we can make the loop correction as big as the 1. This requires $|q| \sim 10^{286}$ eV. Good luck with that. This is called a Landau pole. The second thing is: this perspective of a scale-dependent coupling is very valuable, and is a crucial ingredient in the renormalization group. I’ll say more about it after we discuss the Wilsonian perspective in §10.

\(^{18}\)The last step is safe since the $x(1-x)$ suppresses the contributions of the endpoints of the $x$ integral, so we can treat $x(1-x)$ as finite.
8 Consequences of unitarity

Next I would like to fulfill my promise to show that conservation of probability guarantees that some things are positive (for example, $Z$ and $1 - Z$, where $Z$ is the wavefunction renormalization factor). We will show that amplitudes develop an imaginary part when the virtual particles become real. (Someone should have put an extra factor of $i$ in the definition to resolve this infelicity.) We will discuss the notion of density of states in QFT (this should be a positive number!), and in particular the notion of the density of states contributing to a correlation function $G = \langle \mathcal{O} \mathcal{O} \rangle$, also known as the spectral density of $G$ (or of the operator $\mathcal{O}$). In high-energy physics this idea is associated with the names Källen-Lehmann and is part of a program of trying to use complex analysis to make progress in QFT. These quantities are also ubiquitous in the theory of condensed matter physics and participate in various sum rules. This discussion will be a break from perturbation theory; we will say things that are true with a capital ‘t’.

8.1 Spectral density

[Zee III.8, Appendix 2, Xi Yin’s notes for Harvard Physics 253b] In the following we will consider a (time-ordered) two-point function of an operator $\mathcal{O}$. We will make hardly any assumptions about this operator. We will assume it is a scalar under rotations, and will assume translation invariance in time and space. But we need not assume that $\mathcal{O}$ is ‘elementary’. This is an extremely loaded term, a useful definition for which is: a field governed by a nearly-quadratic action. Also: try to keep an eye out for where (if anywhere) we assume Lorentz invariance.

So, let

$$-iD(x) \equiv \langle 0| T\mathcal{O}(x)\mathcal{O}^\dagger(0) |0 \rangle.$$ 

Notice that we do not assume that $\mathcal{O}$ is hermitian. Use translation invariance to move the left operator to the origin: $\mathcal{O}(x) = e^{iP_x} \mathcal{O}(0) e^{-iP_x}$. This follows from the statement that $P$ generates translations

$$\partial_\mu \mathcal{O}(x) = i[P_\mu, \mathcal{O}(x)].$$

Note that $P$ here is a $D$-component vector of operators

$$P_\mu = (H, \vec{P})_\mu,$$

which includes the Hamiltonian – we are using relativistic notation – but we haven’t actually required any assumption about the action of boosts.
And let’s unpack the time-ordering symbol:

\[- i \mathcal{D}(x) = \theta(t) \langle 0 | e^{iP_x} \mathcal{O}(0) e^{-iP_x} \mathcal{O}^\dagger(0) | 0 \rangle + \theta(-t) \langle 0 | \mathcal{O}^\dagger(0) e^{iP_x} \mathcal{O}(0) e^{-iP_x} | 0 \rangle. \]  

(8.1)

Now we need a resolution of the identity operator on the entire QFT \( \mathcal{H} \):

\[ \mathbb{1} = \sum_n |n\rangle \langle n|. \]

This innocent-looking \( n \) summation variable is hiding an enormous sum! Let’s also assume that the groundstate \( |0\rangle \) is translation invariant:

\[ \mathbf{P} |0\rangle = 0. \]

We can label each state \( |n\rangle \) by its total momentum:

\[ \mathbf{P} |n\rangle = p_n |n\rangle. \]

Let’s examine the first term in (8.1); sticking the 1 in a suitable place:

\[ \langle 0 | e^{iP_x} \mathcal{O}(0) \mathbb{1} e^{-iP_x} \mathcal{O}^\dagger(0) | 0 \rangle = \sum_n \langle 0 | \mathcal{O}(0) |n\rangle \langle n| e^{-iP_x} \mathcal{O}^\dagger(0) | 0 \rangle = \sum_n e^{-i\mathbf{p}_n \cdot \mathbf{x}} \| \mathcal{O}_{0n} \|^2, \]

with \( \mathcal{O}_{0n} \equiv \langle 0 | \mathcal{O}(0) |n\rangle \) the matrix element of our operator between the vacuum and the state \( |n\rangle \). Notice the absolute value: unitarity of our QFT requires this to be positive and this will have valuable consequences.

Next we work on the time-ordering symbol. I claim that:

\[ \theta(x^0) = \theta(t) = -i \int \frac{d\omega}{\omega - i\epsilon} e^{+i\omega t}; \quad \theta(-t) = +i \int \frac{d\omega}{\omega + i\epsilon} e^{+i\omega t}. \]

Just like in our discussion of the Feynman contour, the point of the \( i\epsilon \) is to push the pole inside or outside the integration contour. The half-plane in which we must close the contour depends on the sign of \( t \). There is an important sign related to the orientation with which we circumnavigate the pole. Here is a check that we got the signs and factors right:

\[ \frac{d\theta(t)}{dt} = -i \partial_t \int \frac{d\omega}{\omega - i\epsilon} e^{i\omega t} = \int d\omega e^{i\omega t} = \delta(t). \]

Consider now the fourier transform of \( \mathcal{D}(x) \):

\[ -i \mathcal{D}(q) = \int d^{D}xe^{iqx}i \mathcal{D}(x) = i(2\pi)^{D-1} \sum_n \| \mathcal{O}_{0n} \|^2 \left( \frac{\delta^{(D-1)}(q - \mathbf{p}_n)}{q^0 - p_n^0 + i\epsilon} - \frac{\delta^{(D-1)}(q + \mathbf{p}_n)}{q^0 + p_n^0 - i\epsilon} \right). \]

(8.2)
With this expression in hand, you could imagine measuring the $O_{\text{ons}}$s and using that to determine $D$.

Now suppose that our operator $O$ is capable of creating a single particle (for example, suppose, if you must, that $O = \phi$, a perturbative quantum field). Such a state is labelled only by its spatial momentum: $|\vec{k}\rangle$. The statement that $O$ can create this state from the vacuum means

$$\langle \vec{k} | O(0) | 0 \rangle = \frac{Z^{\frac{1}{2}}}{\sqrt{(2\pi)^{D-1} 2\omega^*_k}}$$

where $Z \neq 0$ and $\omega^*_k$ is the energy of the particle as a function of $\vec{k}$. For a Lorentz invariant theory, we can parametrize this as

$$\omega^*_k \equiv \sqrt{\vec{k}^2 + m^2}$$

in terms of $m$, the mass of the particle. 

What is $Z$? From (8.3) and the axioms of QM, you can see that it’s the probability that $O$ creates this 1-particle state from the vacuum. In the free field theory it’s 1, and it’s positive because it’s a probability. $1 - Z$ measures the extent to which $O$ does anything besides create this 1-particle state.

The identity of the one-particle Hilbert space (relatively tiny!) $\mathcal{H}_1$ is

$$\mathbb{1}_1 = \int d^{D-1}k |\vec{k}\rangle \langle \vec{k}|, \quad \langle \vec{k} | \vec{k}' \rangle = \delta^{(D-1)}(\vec{k} - \vec{k}')$$

This is a summand in the whole horrible resolution:

$$\mathbb{1} = \mathbb{1}_1 + \cdots$$

---

$^{20}$It’s been a month or two since we spoke explicitly about free fields, so let’s remind ourselves about the appearance of $\omega^{-\frac{1}{2}}$ in (8.3), recall the expansion of a free scalar field in creation and annihilation operators:

$$\phi(x) = \int \frac{d^{D-1}p}{\sqrt{2\omega^p}} \left( a_p e^{-ipx} + a_p^* e^{ipx} \right)$$

For a free field $|\vec{k}\rangle = a_{\vec{k}}^1 |0\rangle$, and $\langle \vec{k} | \phi(0) | 0 \rangle = \frac{1}{\sqrt{(2\pi)^{D-1} 2\omega^*_k}}$. The factor of $\omega^{-\frac{1}{2}}$ is required by the ETCRs:

$$[\phi(\vec{x}), \pi(\vec{x}')] = i\delta^{D-1}(\vec{x} - \vec{x}'), \quad [a_{\vec{k}}, a_{\vec{k}'}^1] = \delta^{D-1}(\vec{k} - \vec{k}')$$

where $\pi = \partial_t \phi$ is the canonical field momentum. It is just like in the simple harmonic oscillator, where

$$\mathbf{q} = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger), \quad \mathbf{p} = i\sqrt{\frac{\hbar\omega}{2}} (a - a^\dagger).$$
I mention this because it lets us define the part of the horrible $\sum_n$ in (8.2) which comes from 1-particle states:

$$
\implies -iD(q) = \ldots + i(2\pi)^{D-1} \int d^{D-1}k \frac{Z}{2\omega_k} \left( \frac{\delta^{D-1}(\vec{q} - \vec{k})}{q^0 - \omega_k^2 + i\epsilon} - (\omega_k \to -\omega_k) \right) \\
= \ldots + i \frac{Z}{2\omega_q} \left( \frac{1}{q^0 - \omega_q^2 + i\epsilon} - \frac{1}{q^0 + \omega_q^2 + i\epsilon} \right) \\
= \ldots + i \frac{Z}{q^2 - m^2 + i\epsilon}
$$

(Here again ... is contributions from states involving something else, e.g. more than one particle.) The big conclusion here is that even in the interacting theory, even if $O$ is composite and complicated, if $O$ can create a 1-particle state with mass $m$ with probability $Z$, then its 2-point function has a pole at the right mass, and the residue of that pole is $Z$. (This result was promised last quarter when we discussed LSZ.)

---

The imaginary part of $D$ is called the spectral density $\rho$ (beware that different physicists have different conventions for the factor of $i$ in front of the Green’s function; the spectral density is not always the imaginary part, but it’s always positive (in unitary theories))!

Using

$$
\text{Im} \frac{1}{Q + i\epsilon} = \pm \pi \delta(Q), \quad \text{(for } Q \text{ real).} \quad (8.4)
$$

we have

$$
\text{Im} D(q) = \pi (2\pi)^{D-1} \sum_n \| O_{0n} \|^2 \left( \delta^D(q - p_n) + \delta^D(q + p_n) \right).
$$

More explicitly:

$$
\text{Im} \int d^Dx \ e^{iqx} \langle 0 | \mathcal{T} O(x) O^\dagger(0) | 0 \rangle = \pi (2\pi)^{D-1} \sum_n \| O_{0n} \|^2 \left( \begin{array}{c}
\delta^D(q - p_n) \\
\delta^D(q + p_n)
\end{array} \right) = 0 \text{ for } q^0 > 0 \text{ since } p^0_n > 0.
$$

The second term on the RHS vanishes when $q^0 > 0$, since states in $\mathcal{H}$ have energy bigger than the energy of the groundstate. Therefore, the contribution of a 1-particle state to the spectral density is:

$$
\text{Im} D(q) = \ldots + \pi Z \delta(q^2 - m^2).
$$

21 If we hadn’t assumed Lorentz invariance, this would be replaced by the statement: if the operator $O$ can create a state with energy $\omega$ from the vacuum with probability $Z$, then its Green’s function has a pole at that frequency, with residue $Z$. 

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This quantity \( \text{Im} \mathcal{D}(q) \) (the spectral density of \( \mathcal{O} \)) is positive because it is the number of states (with \( D \)-momentum in an infinitesimal neighborhood of \( q \)), weighted by the modulus of their overlap with the state engendered by the operator on the groundstate.

Now what about multiparticle states? The associated sum over such states involves multiple (spatial) momentum integrals, not fixed by the total momentum \( e.g. \) in \( \phi^4 \) theory: the three particles must share the momentum \( q \). In this case the sum over all 3-particle states is

\[
\sum_{n, \text{3-particle states with momentum } q} \propto \int d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \delta^D(k_1 + k_2 + k_3 - q)
\]

Now instead of an isolated pole, we have a whole collection of poles right next to each other. This is a branch cut. In this example, the branch cut begins at \( q^2 = (3m)^2 \). \( 3m \) is the lowest energy \( q^0 \) at which we can produce three particles of mass \( m \) (they have to be at rest).

Note that in \( \phi^3 \) theory, we would instead find that the particle can decay into two particles, and the sum over two particle states would look like

\[
\sum_{n, \text{2-particle states with momentum } q} \propto \int d\vec{k}_1 d\vec{k}_2 \delta^D(k_1 + k_2 - q)
\]

Now we recall some complex analysis, in the form of the Kramers-Kronig (or dispersion) relations:

\[
\text{Re} G(z) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} d\omega \frac{\text{Im} G(\omega)}{\omega - z}
\]

(valid if \( \text{Im} G(\omega) \) is analytic in the UHP of \( \omega \) and falls off faster than \( 1/\omega \)). These equations, which I think we were supposed to learn in E&M but no one seems to, and which relate the real and imaginary parts of an analytic function by an integral equation, can be interpreted as the statement that the imaginary part of a complex integral comes from the singularities of the integrand, and conversely that those singularities completely determine the function.

An even more dramatic version of these relations (whose imaginary part is the previous eqn) is

\[
f(z) = \frac{1}{\pi} \int dw \frac{\rho(w)}{w - z}, \quad \rho(w) \equiv \text{Im} f(w + i\epsilon).
\]
The imaginary part determines the whole function.

Comments:

- The spectral density \( \text{Im} \mathcal{D}(q) \) determines \( \mathcal{D}(q) \). When people get excited about this it is called the “S-matrix program” or something like that.

- The result we’ve shown protects physics from our caprices in choosing field variables. If someone else uses a different field variable \( \eta \equiv Z^{1/2} \phi + \alpha \phi^3 \), the result above with \( \mathcal{O} = \eta \) shows that

\[
\int d^Dx e^{iqx} \langle T \eta(x)\eta(0) \rangle
\]

still has a pole at \( q^2 = m^2 \) and a cut starting at the three-particle threshold, \( q^2 = (3m)^2 \).

- A sometimes useful fact which we’ve basically already shown:

\[
\text{Im} \mathcal{D}(q) = (2\pi)^D \sum_n \| O_{0n} \|^2 \left( \delta^D(q - p_n) + \delta^D(q + p_n) \right) = \frac{1}{2} \int d^Dx e^{iqx} \langle 0| [\mathcal{O}(x), \mathcal{O}^\dagger(0)] |0 \rangle.
\]

We can summarize what we’ve learned in the Lorentz-invariant case as follows: In a Lorentz invariant theory, the spectral density for a scalar operator \( \phi \) is a scalar function of \( p^\mu \) with

\[
\sum_s \delta^D(p - p_s) \| \langle 0| \phi(0) |s \rangle \|^2 = \frac{\theta(p^0)}{(2\pi)^D-1} \rho(p^2).
\]

The function \( \rho(s) \) is called the spectral density for this Green’s function. Claims:

- \( \rho(s) = \mathcal{N} \text{Im} \mathcal{D} \) for some number \( \mathcal{N} \), when \( s > 0 \).

- \( \rho(s) = 0 \) for \( s < 0 \). There are no states for spacelike momenta.

- \( \rho(s) \geq 0 \) for \( s > 0 \). The density of states for timelike momenta is positive or zero.

- With our assumption about one-particle states, \( \rho(s) \) has a delta-function singularity at \( s = m^2 \), with weight \( Z \). More generally we have shown that

\[
\mathcal{D}(k^2) = \int ds \rho(s) \frac{1}{k^2 - s + i\epsilon}.
\]

This is called the Källen-Lehmann spectral representation of the propagator; it represents it as a sum of free propagators with different masses, determined by the spectral density. One consequence (assuming unitarity and Lorentz symmetry) is
that at large $|k^2|$, the Green’s function must go like $\frac{1}{k^2}$ (or larger), since $\rho(s) \geq 0$ means that there cannot be cancellations between each $\frac{1}{k^2-m^2}$ contribution. This means that if the kinetic term for your scalar field has more derivatives, something must break at short distances (Lorentz is the easiest way out, for example on a lattice).

Taking into account our assumption about single-particle states, this is

$$D(k^2) = \frac{Z}{k^2 - m^2 + i\epsilon} + \int_{(3m)^2}^{\infty} ds \frac{\rho_c(s)}{k^2 - s + i\epsilon},$$

where $\rho_c$ is just the continuum part. The pole at the particle-mass survives interactions, with our assumption. (The value of the mass need not be the same as the bare mass!)

• Finally, suppose that the field $\phi$ in question is a canonical field, in the sense that

$$[\phi(x,t), \partial_t \phi(y,t)] = i\delta^{(3)}(x - y).$$

This is a statement both about the normalization of the field, and that its canonical momentum is its time derivative. Then\textsuperscript{22}

$$1 = \int_0^{\infty} ds \rho(s). \quad (8.5)$$

If we further assume that $\phi$ can create a one-particle state with mass $m$, so that $\rho(s) = Z\delta(s-m^2) + \rho_c(s)$ where $\rho_c(s) \geq 0$ is the contribution from the continuum of $\geq 2$-particle states, then

$$1 = Z + \int_{\text{threshold}}^{\infty} ds \rho_c(s)$$

is a sum rule. It shows that $Z \in [0,1]$ and is just the statement that if the field doesn’t create a single particle, it must do something else. The LHS is the probability that something happens.

\textsuperscript{22} Here’s how to see this. For free fields (chapter 2) we have

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle_{\text{free}} = \Delta_+ (x - y, m^2) - \Delta_+ (y - x, m^2).$$

For an interacting canonical field, we have instead a spectral representation:

$$\langle \Omega | [\phi(x), \phi(y)] | \Omega \rangle = \int d\mu^2 \rho(\mu^2) \left( \Delta_+ (x - y, \mu^2) - \Delta_+ (y - x, \mu^2) \right),$$

where $\rho$ is the same spectral density as above. Now take $\partial_{x^0}|_{x^0=y^0}$ of the BHS and use $\partial_t \Delta_+(x - y; \mu^2)|_{x^0=y^0} = -\frac{1}{2} \delta^3(\vec{x} - \vec{y})$. 

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The idea of spectral representation and spectral density is more general than the Lorentz-invariant case. In particular, the spectral density of a Green’s function is an important concept in the study of condensed matter. For example, the spectral density for the electron 2-point function is the thing that actually gets measured in angle-resolved photoemission experiments (ARPES).

8.2 Cutting rules and optical theorem

[Zee §III.8] So, that may have seemed like some math. What does this mean when we have in our hands a perturbative QFT? Consider the two point function of a relativistic scalar field $\phi$ which has a perturbative cubic interaction:

$$S = \int d^Dx \left( \frac{1}{2} \left( (\partial \phi)^2 + m^2 \phi^2 \right) - \frac{g}{3!} \phi^3 \right).$$

Sum the geometric series of 1PI insertions to get

$$iD_\phi(q) = \frac{i}{q^2 - m^2 - \Sigma(q) + i\epsilon}$$

where $\Sigma(q)$ is the 1PI two point vertex.

The leading contribution to $\Sigma$ comes from the one loop diagram at right and is

$$i\Sigma_{1 \text{ loop}}(q^2) = (ig)^2 \int d^Dk \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(q-k)^2 - m^2 + i\epsilon}.$$  

Consider this function for real $q$, for which there are actual states of the scalar field – timelike $q^\mu$, with $q^0 > m$. The real part of $\Sigma$ shifts the mass. What does it mean if this function has an imaginary part?

---

Claim: $\text{Im} \Sigma/m$ is a decay rate.

It moves the energy of the particle off of the real axis from $m$ to

$$\sqrt{m^2 + \text{Im} \Sigma(m^2)} \approx m + \frac{\text{Im} \Sigma(m^2)}{2m}.$$  

The fourier transform to real time is an amplitude for propagation in time of a state with complex energy $\mathcal{E}$: its wavefunction evolves like $\psi(t) \sim e^{-i\mathcal{E}t}$ and has norm

$$\| \psi(t) \|^2 \sim \| e^{-i(E - \frac{1}{2}\Gamma)t} \|^2 = e^{-\Gamma t}.$$  

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In our case, we have $\Gamma \sim \Im \Sigma(m^2)/m$ (I’ll be more precise below), and we interpret that as the rate of decay of the norm of the single-particle state. There is a nonzero probability that the state turns into something else as a result of time evolution in the QFT: the single particle must decay into some other state – multiple particles. (We will see next how to figure out into what it decays.)

The absolute value of the Fourier transform of this quantity $\psi(t)$ is the kind of thing you would measure in a scattering experiment. This is

$$
\left\| F(\omega) \right\|^2 = \frac{1}{(\omega - M)^2 + \frac{1}{4} \Gamma^2}
$$

is a Lorentzian in $\omega$ with width $\Gamma$. So $\Gamma$ is sometimes called a *width*.

So: what is $\Im\Sigma_{1\text{ loop}}$ in this example?

We will use

$$
\frac{1}{k^2 - m^2 + i\epsilon} = \mathcal{P} \frac{1}{k^2 - m^2} - i\pi \delta(k^2 - m^2) \equiv \mathcal{P} - i\Delta
$$

where $\mathcal{P}$ denotes ‘principal part’. Then

$$
\Im \Sigma_{1\text{ loop}}(q) = -g^2 \int d\Phi (\mathcal{P}_1 \mathcal{P}_2 - \Delta_1 \Delta_2)
$$

with $d\Phi = dk_1 dk_2 (2\pi)^D \delta^D(k_1 + k_2 - q)$.

This next trick, to get rid of the principal part bit, is from Zee’s book (the second edition on p.214; he also does the calculation by brute force in the appendix to that section). We can find a representation for the 1-loop self-energy in terms of real-space propagators: it’s the fourier transform of the amplitude to create two $\phi$ excitations at the origin at time zero with a single $\phi$ field (this is $ig$), to propagate them both from 0 to $x$ (this is $iD(x)^2$) and then destroy them both with a single $\phi$ field (this is $ig$ again). Altogether:

$$
i \Sigma(q) = \int d^D x \, e^{iqx} \, (ig)^2 \, iD(x)iD(x) = g^2 \int d\Phi \frac{1}{k_1^2 - m_1^2 + i\epsilon} \frac{1}{k_2^2 - m_2^2 + i\epsilon}
$$

In the bottom expression, the $i\epsilon$s are designed to produce the *time-ordered* $D(x)$s. Consider instead the strange combination

$$
0 = \int d^D x \, e^{iqx} \, (ig)^2 \, iD_{\text{adv}}(x)iD_{\text{ret}}(x)
$$

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\[ = g^2 \int d \Phi \frac{1}{k_1^2 - m_1^2 - \sigma_1 \epsilon} \frac{1}{k_2^2 - m_2^2 + \sigma_2 \epsilon} \tag{8.7} \]

where \( \sigma_{1,2} \equiv \text{sign}(k_{1,2}^0) \). This expression vanishes because the integrand is identically zero: there is no value of \( t \) for which both the advanced and retarded propagators are nonzero (one has a \( \theta(t) \) and the other has a \( \theta(-t) \), and this is what’s accomplished by the red \( \sigma \)'s). Therefore, we can add the imaginary part of zero

\[-\text{Im}(0) = -g^2 \int d \Phi (p_1 p_2 + \sigma_1 \sigma_2 \Delta_1 \Delta_2)\]

to our expression for \( \text{Im} \Sigma \) to cancel the annoying principal part bits:

\[ \text{Im} \Sigma_{1\text{-loop}} = g^2 \int d \Phi ((1 + \sigma_1 \sigma_2) \Delta_1 \Delta_2). \]

The quantity \( (1 + \sigma_1 \sigma_2) \) is only nonzero when \( k_{1,2}^0 \) have the same sign; but in \( d \Phi \) is a delta function which sets \( q^0 = k_1^0 + k_2^0 \). WLOG we can take \( q^0 > 0 \) since we only care about the propagation of positive-energy states. Therefore both \( k_{1,2}^0 \) must be positive.

The result is that the only values of \( k \) on the RHS that contribute are ones with \textit{positive} energy, which satisfy all the momentum conservation constraints:

\[ \text{Im} \Sigma = g^2 \int d \Phi \theta(k_1^0) \theta(k_2^0) \Delta_1 \Delta_2 \]

\[ = g^2 \int \frac{d^{D-1}k_1}{2 \omega_{\vec{k}_1}} \frac{d^{D-1}k_2}{2 \omega_{\vec{k}_2}} (2\pi)^D \delta^D(k_1 + k_2 - q) . \]

But this is exactly the density of actual final states into which the thing can decay! In summary:

\[ \text{Im} \Sigma = \sum_{\text{actual states } n \text{ of 2 particles into which } \phi \text{ can decay}} \| A_{\phi \rightarrow n} \|^2 \tag{8.8} \]

In this example the decay amplitude \( A \) is just \( ig \).

This result is generalized by the \textit{Cutkosky cutting rules} for finding the imaginary part of a Feynman diagram describing a physical process. The rough rules are the following. Assume the diagram is amputated – leave out the external propagators. Then any line drawn through the diagram which separates initial and final states (as at right)
will ‘cut’ through some number of internal propagators; re-
place each of the cut propagators by \(\theta(p_0)\pi\delta(p^2 - m^2) = \theta(p_0)\frac{\pi\delta(p_2 - \epsilon_p)}{2\epsilon_p}\). As Tony Zee says: the amplitude becomes imaginary when the intermediate particles become real (as opposed to virtual), aka ‘go on-shell’. This is a place where the \(i\)\(\epsilon\)s are crucial.

The general form of (8.8) is a general consequence of unitarity. Recall that the S-matrix is

\[
S_{fi} = \langle f | e^{-iH_T} | i \rangle \equiv (\mathbb{1} + i\mathcal{T})_{fi}.
\]

\[
H = H^\dagger \implies \mathbb{1} = SS^\dagger \implies 2\text{Im} \mathcal{T} \equiv i(T^\dagger - T) \overset{\text{def}}{=} SS^\dagger T^\dagger T.
\]

This is called the optical theorem and it is the same as the one taught in some QM classes. In terms of matrix elements:

\[
2\text{Im} \mathcal{T}_{fi} = \sum_n T^\dagger_{fn} T_{ni}
\]

Here we’ve inserted a resolution of the identity (again on the QFT Hilbert space, the same scary sum) in between the two \(T\) operators. In the one-loop approximation, in the \(\phi^3\) theory here, the intermediate states which can contribute to \(\sum_n\) are two-particle states, so that \(\sum_n \to \int d\vec{k}_1 d\vec{k}_2\), the two-particle density of states.

A bit more explicitly, introducing a basis of scattering states

\[
\langle f | \mathcal{T} | i \rangle = T_{fi} = \delta^4(p_f - p_i)M_{fi}, \quad T^\dagger_{fi} = \delta^4(p_f - p_i)M^*_{if}, \quad (\text{recall that } \delta^d \equiv (2\pi)^d \delta^d)
\]

we have

\[
\langle F | T^\dagger \mathbb{1} | I \rangle = \sum_n \langle F | T^\dagger \sum_n \prod_{f=1}^n \frac{d^3q_f}{2E_f} | \{q_f\} \rangle \langle \{q_f\} | T | I \rangle
\]

\[
= \sum_n \prod_{f=1}^n \frac{d^3q_f}{2E_f} \delta^4(p_F - \sum_f q_f)M^*_{(q_f)F} \delta^4(p_I - \sum_f q_f)M_{(q_f)I}
\]

Now notice that we have a \(\delta^4(p_F - p_I)\) on both sides, and

\[
\int \frac{d^3q_f}{2E_f} \delta^4(p_F - \sum_f q_f) = \int d\Pi_n
\]

is the final-state phase space of the \(n\) particles. Therefore, the optical theorem says

\[
i(M^*_{IF} - M_{FI}) = \sum_n \int d\Pi_n M^*_{(q_f)F} M_{(q_f)I}.
\]

Now consider forward scattering, \(I = F\) (notice that here it is crucial that \(M\) is the transition matrix, \(S = \mathbb{1} + i\mathcal{T} = \mathbb{1} + i\phi(p_T)M\):)

\[
2\text{Im} M_{II} = \sum_n \int d\Pi_n |M_{(q_f)I}|^2.
\]
Recall that for real $x$ the imaginary part of a function of one variable with a branch cut, (like $\text{Im}(x + i\epsilon)^\nu = \frac{1}{2} (\text{Im}(x + i\epsilon)^\nu - (x - i\epsilon)^\nu)$) is equal to (half) the discontinuity of the function $((x)^\nu)$ across the branch cut. In more complicated example (such as a box diagram contributing to 2-2 scattering), there can be more than one way to cut the diagram. Different ways of cutting the diagram correspond to discontinuities in different kinematical variables. To get the whole imaginary part, we have to add these up. A physical cut is a way of separating all initial-state particles from all final-state particles by cutting only internal lines. So for example, a $t$-channel tree-level diagram (like $\text{Im}\left(\right)$ never has any imaginary part; this makes sense because the momentum of the exchanged particle is spacelike.

**Resonances.** A place where this technology is useful is when we want to study short-lived particles. In our formula for transition rates and cross sections we assumed plane waves for our external states. Some particles don’t live long enough for separately producing them: and then watching them decay: ; instead we must find them as resonances in scattering amplitudes of other particles: $\text{Im}\left(\right)$.

So, consider the case $i\mathcal{M} = \langle F| i\mathcal{T} |I \rangle$ where both $I$ and $F$ are one-particle states. A special case of the LSZ formula says

$$\mathcal{M} = -\left(\sqrt{Z}\right)^2 \Sigma = -Z \Sigma$$  \hspace{1cm} (8.9)

where $-i\Sigma$ is the amputated 1-1 amplitude, that is the self-energy, sum of all connected and amputated diagrams with one particle in and one particle out. Let $\Sigma(p) = A(p^2) + iB(p^2)$ (not standard notation), so that near the pole in question, the propagator looks like

$$\tilde{G}^{(2)}(p) = \frac{i}{p^2 - m^2 - \Sigma(p)} \simeq \frac{i}{(p^2 - m^2) \left(1 - \partial_{p^2} A\big|_{m^2}\right)} \frac{-iB}{Z^{-1}} = \frac{iZ}{(p^2 - m^2) - iBZ}. $$

In terms of the particle width $\Gamma_w \equiv -ZB/m$, this is

$$\tilde{G}^{(2)}(p) = \frac{iZ}{(p^2 - m^2) - im\Gamma_w}. $$
So, if we can make the particle whose propagator we’re discussing in the s-channel, the cross-section will be proportional to

\[ \left| \tilde{G}^{(2)}(p) \right|^2 = \left| \frac{1}{(p^2 - m^2) - i m \Gamma_w} \right|^2 = \frac{1}{(p^2 - m^2)^2 + m^2 \Gamma_w^2} \]

a Lorentzian or Breit-Wigner distribution: In the COM frame, \( p^2 = 4E^2 \), and the cross section \( \sigma(E) \) has a resonance peak at \( 2E = m \), with width \( \Gamma_w \). It is the width in the sense that the function is half its maximum when \( E = E_\pm = \sqrt{\frac{m(m \pm \Gamma_w)}{4}} \approx \frac{m}{2} \pm \frac{\Gamma_w}{4} \).

This width is the same as the decay rate, because of the optical theorem:

\[ \Gamma_w = -\frac{BZ}{m} \quad \text{(8.9)} \]

\[ = -\frac{1}{m} (\text{Im} \mathcal{M}_{1\rightarrow 1}) \text{ optical} = \frac{1}{m^2} \sum_n \int d\Pi_n |\mathcal{M}_{\{q_f\}1}|^2 = \Gamma \]

the last equation of which is exactly our formula for the decay rate. If it is not the case that \( \Gamma \ll m \), i.e. if the resonance is too broad, the Taylor expansion of the inverse propagator we did may not be such a good idea.

**Unitarity and high-energy physics.** Two comments: (1) there had better not be any cutoff dependence in the imaginary part. If there is, we’ll have trouble cancelling it by adding counterterms – an imaginary part of the action will destroy unitarity. This is elaborated a bit in Zee’s discussion.

(2) Being bounded by 1, probabilities can’t get too big. Cross sections are also bounded: there exist precise bounds from unitarity on the growth of cross sections with energy, such as the Froissart bound, \( \sigma_{\text{total}}(s) \leq C \ln^2 s \) for a constant \( C \). Xi Yin’s notes describe a proof.

On the other hand, consider an interaction whose coupling \( G \) has mass dimension \( k \). The cross section to which \( G \) contributes has dimensions of area, and comes from squaring an amplitude proportional to \( G \), so comes with at least two powers of \( G \). At \( E \gg \) anything else, these dimensions must be made up with powers of \( E \):

\[ \sigma(E \gg ...) \sim G^2 E^{-2-2k}. \]  

(8.10)

This means that if \( k \leq -1 \), the cross section grows at high energy. In such a case, something else must happen to ‘restore unitarity’. One example is Fermi’s theory of Weak interactions, which involves a 4-fermion coupling \( G_F \sim M_W^{-2} \). Here we know what happens, namely the electroweak theory, about which more soon. In gravity, \( G_N \sim M_{\text{Pl}}^{-2} \), we can’t say yet.
8.3 How to study hadrons with perturbative QCD

[Peskin §18.4] Here is a powerful physics application of both the optical theorem and the spectral representation. Consider the total inclusive cross section for $e^+e^-$ scattering at energies $s = (k + k_\perp)^2 \gg m_e^2$:

$$\sigma_{e^+e^- \text{ optical thm}} = \frac{1}{2s} \text{Im} \mathcal{M}(e^+e^- \leftarrow e^+e^-)$$  \hspace{1cm} (8.11)

where on the RHS, $\mathcal{M}$ is the forward scattering amplitude (meaning that the initial and final electrons have the same momenta). We’ve learned a bit about the contributions of electrons and muons to the BHS of this expression, what about QCD? To leading order in $\alpha$ (small), but to all orders in the strong coupling $\alpha_s$ (big at low energies), the contributions of QCD look like

$$i \mathcal{M}_h = \begin{align*}
(-ie)^2 \bar{u}(k)\gamma_\mu v(k_\perp) &- i \frac{1}{s} i \Pi^{\mu\nu}_h(q) \bar{v}(k_\perp)\gamma_\nu u(k) \\
\text{Ward} &\equiv i (q^2 \eta^{\mu\nu} - q^\mu q^\nu) \Pi_h(q^2)
\end{align*}$$

with

$$\text{Ward} = i \Pi^{\mu\nu}_h(q) \equiv i (q^2 \eta^{\mu\nu} - q^\mu q^\nu) \Pi_h(q^2)$$

the hadronic contribution to the vacuum polarization. We can pick out the contribution of the strong interactions by just keeping these bits on the BHS of (8.11):

$$\sigma_{\text{hadrons } e^+e^-} = \frac{1}{4} \sum_{\text{spins}} \frac{\text{Im} \mathcal{M}_h}{2s} = -\frac{4\pi\alpha}{s} \text{Im} \Pi_h(s).$$

(The initial and final spins are equal and we average over initial spins. We can ignore the longitudinal term $q^\mu q^\nu$ by the Ward identity. The spinor trace is $\sum_{\text{spins}} \bar{u}(k)\gamma_\mu v(k_\perp)\bar{v}(k_\perp)\gamma_\nu u(k) = -2k \cdot k_\perp = -s$.) As a reality check, consider the contribution from one loop of a heavy lepton of mass $M^2 \gg m_e^2$:

$$\text{Im} \Pi_L(s + i\epsilon) = -\frac{\alpha}{3} F(M^2/s)$$

and

$$\sigma_{L^+L^- \leftarrow e^+e^-} = \frac{4\pi \alpha^2}{3} s F(M^2/s)$$

with $F(M^2/s) = \sqrt{1 - \frac{4M^2}{s}} \left(1 + \frac{2M^2}{s}\right) = 1 + \mathcal{O}(M^2/s)$. In perturbative QCD, the leading order result is the same from each quark with small enough mass:

$$\sigma_0^{\text{quarks } e^+e^-} = \sum_{\text{colors flavors}} \frac{3}{2} Q_f^2 \frac{4\pi \alpha^2}{3} s F(M^2/s).$$

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But Q: why is a perturbative analysis of QCD relevant here? You might think asymptotic freedom means QCD perturbation theory is good at high energy or short distances, and that seems to be borne out by noticing that \( \Pi_h \) is a two-point function of the quark contributions to the EM current:

\[
i \Pi_{\mu\nu}^h(q) = -e^2 \int d^4x \ e^{-iq \cdot x} \langle \Omega | T J^\mu(x) J^\nu(0) | \Omega \rangle, \quad J^\mu(x) \equiv \sum_f Q_f \bar{q}_f(x) \gamma^\mu q_f(x).
\]

It looks like we are taking \( x \to 0 \) and studying short distances. But if we are interested in large timelike \( q^\mu \) here, that means that dominant contributions to the \( x \) integral are when the two points are timelike separated, and in the resolution of the identity in between the two \( J_s \) includes physical states of QCD with lots of real hadrons. The limit where we can do (later we will learn how) perturbative QCD is when \( q^2 = -Q_0^2 > 0 \) is spacelike. (Preview: We can use the operator product expansion of the two currents.)

How can we use this knowledge to find the answer in the physical regime of \( q^2 > 0 \)? The fact that \( \Pi_h \) is a two-point function means that it has a spectral representation. It is analytic in the complex \( q^2 \) plane except for a branch cut on the positive real axis coming from production of real intermediate states, exactly where we want to know the answer. One way to encode the information we know is to package it into moments:

\[
I_n \equiv -4\pi \alpha \oint_{C_{Q_0}} dq^2 \frac{\Pi_h(q^2)}{2\pi i (q^2 + Q_0^2)^{n+1}} = -\frac{4\pi \alpha}{n!} (\partial_{q^2})^n \Pi_h|_{q^2 = -Q_0^2}.
\]

On the other hand, we know from the (appropriate generalization to currents of the) spectral representation sum rule (8.5) that \( \Pi_h(q^2) \sim \log(q^2) \), so for \( n \geq 1 \), the contour at infinity can be ignored.

Therefore

\[
I_n = -4\pi \alpha \oint_{\text{Pacman}} dq^2 \frac{\Pi_h(q^2)}{4\pi i (q^2 + Q_0^2)^{n+1}} = \frac{1}{\pi} \int_{s_{\text{threshold}}}^{\infty} ds \frac{\sigma_{\text{hadrons}}}{s (s + Q_0^2)^{n+1}} e^{+e^-}(s).
\]

On the RHS is (moments of) the measurable (indeed, measured) cross-section, and on the LHS is things we can calculate (later). If the convergence of these integrals were uniform in \( n \), we could invert this relation and directly try to predict the cross section to hadrons. But it is not, and the correct cross section varies about the leading QCD answer more and more at lower energies, culminating at various Breit-Wigner resonance peaks at \( \bar{q}q \) boundstates.