1. **Brain-warmer.** Verify the form of the propagator for the massive vector field by plugging in the mode expansion and using the completeness relation for the polarization vectors.

2. **Scalar particle scattering cross sections.**

Let’s consider again the example of a complex scalar field $\Phi$ interacting with a real scalar field $\phi$ with Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \Phi^* \partial^{\mu} \Phi - \frac{1}{2} m^2 \Phi^* \Phi + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} M^2 \phi^2 + \mathcal{L}_I$$

with $\mathcal{L}_I = -g \Phi^* \Phi \phi$.

We can call the $\Phi$ particles are ‘snucleons,’ since they are scalar analogs of nucleons.

What is the leading-order differential cross-section $\frac{d\sigma}{d\Omega}$ for $2 \to 2$ snucleon-snucleon scattering in $d = 3$ space dimensions in the center-of-mass frame? Draw the relevant Feynman diagram(s).

The amplitude is

$$\mathcal{M} = \frac{p_1}{p_3} + \frac{p_2}{p_4} = \frac{g^2}{t - m^2} + \frac{g^2}{u - m^2}$$

where $m$ is the mass of the exchanged meson, and

$$t \equiv (p_1 - p_3)^2 = -2p^2(1 - \cos \theta), \quad u \equiv (p_1 - p_4)^2 = -2p^2(1 + \cos \theta).$$

The differential cross section in the CoM frame in $d = 3$ is

$$\frac{d\sigma}{d\Omega} = |\mathcal{M}|^2 \frac{d\Omega}{|v_A - v_B| 2\omega_A 2\omega_B}$$

$$= |\mathcal{M}|^2 \left| \frac{p_3}{p_1} \right| \frac{1}{64\pi^2 E_{CM}^2} \theta(E_{CM} - 2m)$$
What is the total cross section in the limit that the nucleons are massless?

If the nucleons are massless, then $|p_1| = |p_3| = E_1$, and this is

$$\frac{d\sigma}{d\Omega} = \left| \frac{1}{t-m^2} + \frac{1}{u-m^2} \right|^2 \frac{1}{64\pi^2} \theta(E_{CM} - 2m)$$

The total cross section is

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int_0^\pi d\varphi \int_0^\pi \sin \theta d\theta \frac{d\sigma}{d\Omega}(\cos \theta) = \pi \int_{-1}^1 dy \frac{d\sigma}{d\Omega}(y).$$

Notice that we only integrate the azimuthal angle $\varphi$ from 0 to $\pi$, since the final-state particles are identical. The integrand is proportional to

$$\frac{t + u - 2m^2}{(t-m^2)(u-m^2)} = \frac{-4p^2 - 2m^2}{4p^2(1 - \cos^2 \theta) + 4m^2p^2 + m^4}$$

The integral is doable

$$\int_{-1}^1 \frac{dy}{A + B(1 - y^2)} = -\frac{2 \arctan \left( \frac{B}{\sqrt{A + B}} \right)}{\sqrt{B} \sqrt{-A - B}}$$

where $y \equiv \cos \theta$. Don’t forget to divide the answer for $\sigma$ by 2 to account for the indistinguishability of the two particles coming out.

3. **Decay of a scalar particle.**

Consider the following Lagrangian, involving two real scalar fields $\Phi$ and $\phi$:

$$\mathcal{L} = \frac{1}{2} \left( \partial_\mu \Phi \partial^\mu \Phi - M^2 \Phi^2 + \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) - \mu \Phi \phi^2.$$ 

The last term is an interaction that allows a $\Phi$ particle to decay into two $\phi$s, if the kinematics allow it. Calculate the lifetime of the $\Phi$ particle to lowest order in $\mu$. Draw the relevant Feynman diagram(s). What is the condition on the masses for a finite lifetime?

At leading order, there is just one diagram, which is just a single vertex, so $i\mathcal{M} = -2i\mu$. The factor of two comes from the fact that there are two possible contractions of the $\phi$s with the final-state particles (we should have divided the interaction by 2 to make the notation more consistent with our notation for $\phi^4$ theory).

The decay rate in the center of mass frame is

$$\Gamma = \frac{1}{2} \frac{1}{2M} \int \frac{d^3p_1}{2E_1} \frac{d^3p_2}{2E_2} |\mathcal{M}|^2 \delta^4(p - p_1 - p_2)$$
where \( p^\mu = (M, \vec{0})^\mu \), and the factor of \( \frac{1}{2} \) is because we can’t tell apart the two \( \phi \)s that come out. This quantity reduces to
\[
\Gamma = \frac{\mu^2}{8\pi M^2} \sqrt{M^2 - 4m^2}.
\]

For more detail see these notes section 4.7. The lifetime is \( \infty \) (the decay rate is zero) unless \( M > 2m \); for \( M < 2m \), the decay cannot conserve energy.

4. **Meson scattering.**

Now consider the Yukawa theory with fermions, with
\[
\mathcal{L} = \bar{\Psi} \left( i\partial - m \right) \Psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2 + \mathcal{L}_{\text{int}}
\]
and \( \mathcal{L}_{\text{int}} = g \bar{\Psi} \Psi \phi \).

(a) Draw the Feynman diagram(s) which give(s) the leading contribution to the process \( \phi \phi \to \phi \phi \).

Draw a circle (fermion loop) and attach the meson lines to it. Diagrams which involve exchanging how the meson lines are attached also contribute.

(b) Derive the correct sign of the amplitude by considering the relevant matrix elements of powers of the interaction Hamiltonian. Compare with the Feynman rules for fermions stated in lecture.

(c) Evaluate the diagram in terms of a spinor trace and a momentum integral. Do not do the momentum integral. Suppose that the integral is cutoff at large \( k \) by some cutoff \( \Lambda \). Estimate the dependence on \( \Lambda \).

5. **Ward identity in scalar QED.** We noted in lecture that scalar QED is different from the usual spinor QED in that the coupling to the gauge field is not just \( j^\mu A_\mu \) where \( j^\mu \) is a current independent of \( A \). In this problem we’ll see how this changes the proof of the Ward identity.

(a) Consider a Green’s function in scalar QED of the form \( G \equiv \langle 0 | T \mathcal{O}_1 \cdots \mathcal{O}_n | 0 \rangle \)
where \( \mathcal{O}_i \equiv \mathcal{O}_i(x_i) \) has charge \( Q_i \) under the transformation
\[
\Phi(x) \to e^{i\alpha(x)} \Phi(x), \quad A_\mu \to A_\mu.
\]
(4)

The phrase “\( \mathcal{O} \) has charge \( Q \)” means \( \mathcal{O}(x) \to e^{iQ(x)} \mathcal{O}(x) \). (Notice that (4) is not the gauge transformation, since \( A \) does not transform.) Derive the Schwinger-Dyson equation which follows from demanding that the path
integral is invariant under the change of variables (4) to first order in $\alpha$. (We are supposed to call the result a Ward-Takahashi identity.)

The variation of the action $S[\Phi, A] = \int D_\mu \Phi^* D^\mu \Phi$ under (4) is now

$$\delta S = -i \int \partial_\mu \alpha (\Phi^* D^\mu \Phi - (D^\mu \Phi^*) \Phi) = \int d^4 x \alpha(x) \partial_\mu j_\mu^A(x)$$

where

$$j_\mu^A(x) \equiv i (\Phi^* D^\mu \Phi - (D^\mu \Phi^*) \Phi)$$

is an improved current which depends on the gauge field. It is ‘improved’ in the sense that it is gauge invariant. This is the only difference in the argument relative to spinor QED: in spinor QED, the current is $j_\mu = \bar{\Psi} \gamma^\mu \Psi$, independent of the gauge field. Therefore, the variation of the path integral is

$$0 = -i \int d^4 x \alpha(x) \left( \langle \partial_\mu j_\mu^A(x) \mathcal{O}_1 \cdots \mathcal{O}_n \rangle - \sum_i Q_i \delta^4(x - x_i) G \right)$$

(where $\langle ... \rangle \equiv \langle 0 | T ... | 0 \rangle$). That is, it is $j_\mu^A$ which is conserved up to contact terms.

(b) Consider an amplitude in scalar QED with an external photon of polarization $\epsilon$: $\mathcal{M} = \epsilon^\mu \mathcal{M}_\mu$. Using the LSZ reduction formula, show the result of the previous part implies the Ward identity $p^\mu \mathcal{M}_\mu = 0$. Conclude that the longitudinal photons decouple in scalar QED as well.

This goes through just as in the discussion in the lecture notes of the Ward identity on spinor QED. The only difference is the replacement $j \rightarrow j_A$. 

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