University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2019
Assignment 10 ("Final Exam")

Due 12:30pm Wednesday, March 20, 2019

1. **Brain-warmers.**

   (a) Show that the *adjoint* representation matrices

   $$\left(T^B\right)_{AC} \equiv -\mathbf{i} f_{ABC}$$

   furnish a dim G-dimensional representation of the Lie algebra

   $$[T^A, T^B] = \mathbf{i} f_{ABC} T^C \quad .$$

   Hint: commutators satisfy the Jacobi identity

   $$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0.$$

   (b) From the transformation law for A, show that the non-abelian field strength transforms in the adjoint representation of the gauge group.

   (c) Show that

   $$\mathrm{tr} F \wedge F = d\mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right).$$

   Write out all the indices I've suppressed.

   (d) [Bonus] If you are feeling under-employed, find ω_{2n-1} such that $\mathrm{tr} F^n = d\omega_{2n-1}$.

2. **The field of a magnetic monopole.**

   We saw that $F = dA$ implies (when A is smooth) that $dF = 0$, which means no magnetic charge. If A is singular, dF can be nonzero. Moreover, by a gauge transformation we can move the singularity around and hide it.

   A magnetic monopole of magnetic charge g is defined by the condition that $\int_{S^2} F = g$, where S^2 any sphere surrounding the monopole. If the system is spherically symmetric, we can write

   $$F = \frac{g}{4\pi} d\cos\theta d\varphi.$$

   (In this problem, we'll work on a sphere at fixed distance from the monopole.)

(a) Show that the vector potential

$$A_N = \frac{g}{4\pi}\left(\cos\theta - 1\right) d\varphi$$

gives the correct $F = dA$. Show that it is a well-defined one-form on the sphere except at the south pole $\theta = \pi$.

(b) Show that the one-form

$$A_S = \frac{g}{4\pi}\left(\cos\theta + 1\right) d\varphi$$

also gives the correct $F = dA$. Show that it is well-defined except at the north pole $\theta = 0$.

(c) Near the equator both $A_{N,S}$ are well-defined. Show that *as long as* $eg \in 2\pi\mathbb{Z}$, these two one-forms differ by a gauge transformation

$$A_S - A_N = \frac{1}{\mathbf{i}e} g^{-1}(\theta,\varphi) dg(\theta,\varphi)$$

for $g(\theta,\varphi)$ a $\mathsf{U}(1)$-valued function on the sphere, well-defined away from the poles.

3. **Abrikosov-Nielsen-Oleson vortex string.**

Consider the Abelian Higgs model in $D = 3+1$:

$$\mathcal{L}_h \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}|D_\mu\phi|^2 - V(|\phi|)$$

where ϕ is a scalar field of charge e whose covariant derivative is $D_\mu\phi = \left(\partial_\mu - \mathbf{i}qA_\mu\right)\phi$, and let's take

$$V(|\phi|) = \frac{\kappa}{2}(|\phi|^2 - v^2)^2$$

for some couplings κ, v. Here we are going to do some interesting classical field theory. Set $q = 1$ for a bit.

(a) Consider a configuration which is independent of x^3, one of the spatial coordinates, and static (independent of time). Show that its energy density (energy per unit length in x^3) is

$$U = \int d^2x \left(\frac{1}{2}F_{12}^2 + \frac{1}{2}|D_i\phi|^2 + V(|\phi|)\right).$$

(b) Consider the special case where $\kappa = 1$. Assuming that the integrand falls off sufficiently quickly at large $x^{1,2}$, show that

$$U_{\kappa=1} = \int d^2x \left(\frac{1}{2}\left(F_{12} + |\phi|^2 - v^2\right)^2 + \frac{1}{4}|D_i\phi + \mathbf{i}\epsilon_{ij}D_j\phi|^2 + v^2F_{12} - \frac{1}{2}\mathbf{i}\epsilon_{k\ell}\partial_k\left(\phi^\star D_\ell\phi\right)\right).$$

(c) The first two terms in the energy density of the previous part are squares and hence manifestly positive, so setting to zero the things being squared will minimize the energy density. Show that the resulting first-order equations (they are called BPS equations after people with those initials, Bogolmonyi, Prasad, Sommerfeld)

$$0 = (D_i + \mathbf{i}\epsilon_{ij} D_j)\,\phi, \quad F_{12} = -|\phi|^2 + v^2$$

are solved by ($x^1 + \mathbf{i}x^2 \equiv re^{\mathbf{i}\varphi}$)

$$\phi = e^{\mathbf{i}n\varphi} f(r), \quad A_1 + \mathbf{i}A_2 = -\mathbf{i}e^{\mathbf{i}\varphi}\frac{a(r) - n}{r}$$

if

$$f' = \frac{a}{r}f, \; a' = r(f^2 - v^2)$$

with boundary conditions

$$a \to 0, f \to v + \mathcal{O}\left(e^{-mr}\right), \quad \text{at } r \to \infty \tag{1}$$

$$a \to n + \mathcal{O}(r^2), f \to r^n(1 + \mathcal{O}(r^2)), \quad \text{at } r \to 0.$$

(For other values of κ, the story is not as simple, but there is a solution with the same qualitative properties. See for example §3.3 of E. Weinberg, *Classical solutions in Quantum Field Theory*.)

(d) The second BPS equation and (1) imply that all the action (in particular F_{12}) is localized near $r = 0$. Evaluate the magnetic flux through the $x^1 - x^2$ plane, $\Phi \equiv \int B \cdot da$ in the vortex configuration labelled by n. Show that the energy density is $U = \frac{v^2}{2} \cdot \Phi$.

(e) Show that the previous result for the flux follows from demanding that the two terms in $D_i\phi$ cancel at large r so that

$$D_i\phi \overset{r\to\infty}{\to} 0 \tag{2}$$

faster than $1/r$. Solve (2) for A_i in terms of ϕ and integrate $\int d^2x F_{12}$.

(f) Argue that a single vortex (string) in the ungauged theory (with global $\mathsf{U}(1)$ symmetry)

$$\mathcal{L} = |\partial\phi|^2 + V(|\phi|)$$

does not have finite energy per unit length. By a vortex, I mean a configuration where $\phi \overset{r\to\infty}{\to} ve^{\mathbf{i}\varphi}$, where $x^1 + \mathbf{i}x^2 = re^{\mathbf{i}\varphi}$.

(g) Consider now the case where the scalar field has charge q. (Recall that in a superconductor made by BCS pairing of electrons, the charged field which condenses has electric charge two.) Show that the magnetic flux in the core of the minimal ($n = 1$) vortex is now (restoring units) $\frac{hc}{qe}$.

4. **BPS conditions from supersymmetry.** [bonus!] What's special about $\kappa = 1$? For one thing, it is the boundary between type I and type II superconductors (which are distinguished by the size of the vortex core). More sharply, it means the mass of the scalar equals the mass of the vector, at least classically. Moreover, in the presence of some extra fermionic fields, the model with this coupling has an additional symmetry mixing bosons and fermions, namely supersymmetry. This symmetry underlies the special features we found above. Here is an outline (you can do some parts without doing others) of how this works. The logic in part (c) underlies a lot of the progress in string theory since the mid-1990s. Please do not trust my numerical factors.

   (a) Add to $\mathcal{L}_h$ a charged fermion Ψ (partner of ϕ) and a neutral Majorana fermion λ (partner of A_μ):

   $$\mathcal{L}_f = \frac{1}{2}\mathbf{i}\bar{\Psi}\not{D}\Psi + \mathbf{i}\bar{\lambda}\not{D}\lambda + \bar{\lambda}\Psi\phi + h.c..$$

   Consider the transformation rules

   $$\delta_\epsilon A_\mu = \mathbf{i}\bar{\epsilon}\gamma_\mu\lambda, \delta_\epsilon\Psi = D_\mu\phi\gamma^\mu\epsilon, \quad \delta_\epsilon\phi = -\mathbf{i}\bar{\epsilon}\Psi, \delta_\epsilon\lambda = -\frac{1}{2}\mathbf{i}\sigma^{\mu\nu}F_{\mu\nu}\epsilon + \mathbf{i}(|\phi|^2 - v)\epsilon$$

   where the transformation parameter ϵ is a Majorana spinor (and a grassmann variable). Show that (something like this) is a symmetry of $\mathcal{L} = \mathcal{L}_h + \mathcal{L}_f$. This is $\mathcal{N} = 1$ supersymmetry in $D = 4$.

   (b) Show that the conserved charges associated with these transformations Q_α (they are grassmann objects and spinors, since they generate the transformations, via $\delta_\epsilon$fields $= [\epsilon_\alpha Q_\alpha + h.c., \text{fields}]$), satisfy the algebra

   $$\{Q, \bar{Q}\} = 2\gamma^\mu P_\mu + 2\gamma^\mu\Sigma_\mu \tag{3}$$

   where P_μ is the usual generator of spacetime translations and Σ_μ is the *vortex string charge*, which is nonzero in a state with a vortex string stretching in the μ direction. $\bar{Q} \equiv Q^\dagger\gamma^0$ as usual.

   (c) If we multiply (3) on the right by γ^0, we get the positive operator $\{Q_\alpha, Q_\beta^\dagger\}$. This operator annihilates states which satisfy $Q\,|BPS\rangle = 0$ for some components of Q. Such a state is therefore invariant under some subgroup of the superymmetry, and is called a BPS state. Now look at the right hand side of (3)$\times\gamma^0$ in a configuration where $\Sigma_3 = \pi n v^2$ and show that its energy density is $E \geq \pi|n|v^2$, with the inequality saturated only for BPS states.

   (d) To find BPS configurations, we can simply set to zero the relevant supersymmetry variations of the fields. Since we are going to get rid of the fermion fields anyway, we can set them to zero and consider just the (bosonic) variations of the fermionic fields. Show that this reproduces the BPS equations.

5. **Wilson loops in abelian gauge theory at weak and strong coupling.**

   (a) At weak coupling, the Wilson loop expectation value is a gaussian integral. In $D = 4$, study the continuum limit of a rectangular loop with time extent $T \gg R$, the spatial extent. Show that this reproduces the Coulomb force.

   (b) Consider the weak coupling calculation again for a Wilson loop coupled to a massive vector field. Show that this reproduces an exponentially-decaying force between external static charges.

   (c) [bonus problem] Compute the combinatorial factors in the first few terms of the strong-coupling expansion of the Wilson loop in $\mathsf{U}(1)$ lattice gauge theory.

   (d) [bonus problem] Consider the case of lattice gauge theory in two spacetime dimensions. In this case, show that the plaquette variables are actually independent variables.

6. **Chern-Simons theory, flux attachment, and anyons.** [optional]

   (a) Consider the following action for a $\mathsf{U}(1)$ gauge field in $D = 2 + 1$:

   $$S[A] = \int \left(-\frac{1}{4g^2} F \wedge \star F + \frac{k}{4\pi} A \wedge F \right).$$

   What are the dimensions of g and k? Find the equations of motion for A. Look for plane wave solutions. Show that the resulting particle excitations have a mass which grows with g.

   (b) For the rest of the problem, take $g \to \infty$. Notice that the resulting theory does not require a metric, since the action is made only from exterior derivatives and wedge products of forms. Now add a matter current:

   $$S_j[A] = \int \left(\frac{k}{4\pi} A \wedge F + A \wedge \star j \right).$$

   Find the equations of motion. Show that the Chern-Simons term *attaches k units of flux* to the particles: $F_{12} \propto \rho$.

   (c) Show using the Bohm-Aharonov effect that the particles whose current density is j^μ have anyonic statistics with exchange angle $\frac{\pi}{k}$ (supposing they were bosons before we coupled them to A).

   One way to do this is to consider a configuration of j which describes one particle adiabatically encircling another. Show that its wavefunction acquires a phase $e^{\mathbf{i}2\pi/k}$. This is twice the phase obtained by going halfway around, which (when followed by an innocuous translation) would exchange the particles.