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0.1 Introductory remarks

Quantum field theory (QFT) is the quantum mechanics of extensive degrees of freedom. What I mean by this is that at each point of space, there’s some stuff that can wiggle.

It’s not surprising that QFT is so useful, since this situation happens all over the place. Some examples of ‘stuff’ are: the atoms in a solid, or the electrons in those atoms, or the spins of those electrons. A less obvious, but more visible, example is the electromagnetic field, even in vacuum. More examples are provided by other excitations of the vacuum, and it will be our job here to understand those very electrons and atoms that make up a solid in these terms. The vacuum has other less-long-lasting excitations which are described by the Standard Model of particle physics.

Some examples of QFT are Lorentz invariant (‘relativistic’). That’s a nice simplification when it happens. Indeed this seems to happen in particle physics. We’re going to focus on this case for much of this quarter. Still I would like to emphasize: though some of the most successful applications of QFT are in the domain of high energy particle physics, this is not a class on that subject, and I will look for opportunities to emphasize the universality of QFT.

A consequence of relativity is that the number of particles isn’t fixed. That is: there are processes where the number of particles changes in time. This is a crucial point of departure for QFT. It’s a necessary consequence of Lorentz symmetry, but the converse is false: particle production can happen without relativity.

‘Divergences’. Another intrinsic and famous feature of QFT discernible from the definition I gave above is its flirtation with infinity. I said that there is ‘stuff at each point of space’; how much stuff is that? Well, there are two senses in which ‘the number of points of space’ is infinite: (1) space can go on forever (the infrared (IR)), and (2) in the continuum, in between any two points of space are more points (the ultraviolet (UV)). The former may be familiar from statistical mechanics, where it is associated with the thermodynamic limit, which is where interesting things happen. For our own safety, we’ll begin our discussion in a padded room, protected on both sides from the terrors of the infinite.

Prof. Jenkins tells me that 215A ended just as Feynman diagrams were being drawn for the first time. I think therefore that it will be useful to retreat a bit and rederive the diagrammatic expansion from another (in many ways simpler) point of view, namely the path integral.
0.2 Sources and acknowledgement

The material in these notes is collected from many places, among which I should mention in particular the following:

- Peskin and Schroeder, *An introduction to quantum field theory* (Wiley)
- Banks, *Modern Quantum Field Theory: A Concise Introduction* (Cambridge)
- Schwartz, *Quantum field theory and the standard model* (Cambridge)
- David Tong’s lecture notes

Many other bits of wisdom come from the Berkeley QFT courses of Prof. L. Hall and Prof. M. Halpern.
0.3 Conventions

Following most QFT books, I am going to use the + − − − signature convention for the Minkowski metric. I am used to the other convention, where time is the weird one, so I’ll need your help checking my signs. More explicitly, denoting a small spacetime displacement as $dx^\mu \equiv (dt, d\vec{x})^\mu$, the Lorentz-invariant distance is:

$$ds^2 = +dt^2 - d\vec{x} \cdot d\vec{x} = \eta_{\mu\nu} dx^\mu dx^\nu$$

with $\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix}
  +1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & -1 & 0 \\
  0 & 0 & 0 & -1
\end{pmatrix}$.

(spacelike is negative). We will also write $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\partial_t, \vec{\nabla}_x\right)^\mu$, and $\partial^{\mu} \equiv \eta^{\mu\nu} \partial_\nu$. I’ll use $\mu, \nu...$ for Lorentz indices, and $i, k,...$ for spatial indices.

The convention that repeated indices are summed is always in effect unless otherwise indicated.

$D$ is the number of spacetime dimensions, $d$ is the number of space dimensions.

A consequence of the fact that english and math are written from left to right is that time goes to the left.

A useful generalization of the shorthand $\hbar \equiv \frac{\hbar}{2\pi}$ is $d\bar{k} \equiv \frac{dk}{2\pi}$. I will also write $\delta^d(q) \equiv (2\pi)^d \delta^{(d)}(q)$. I will try to be consistent about writing Fourier transforms as

$$\int \frac{d^d k}{(2\pi)^d} e^{i k x} \tilde{f}(k) \equiv \int d^d k \ e^{i k x} \tilde{f}(k) \equiv f(x).$$

IFF $\equiv$ if and only if.
RHS $\equiv$ right-hand side. LHS $\equiv$ left-hand side. BHS $\equiv$ both-hand side.
IBP $\equiv$ integration by parts. WLOG $\equiv$ without loss of generality.
$+\mathcal{O}(x^n) \equiv$ plus terms which go like $x^n$ (and higher powers) when $x$ is small.
$+h.c. \equiv$ plus hermitian conjugate.
$\mathcal{L} \ni \mathcal{O}$ means the object $\mathcal{L}$ contains the term $\mathcal{O}$.

We work in units where $\hbar$ and the speed of light, $c$, are equal to one unless otherwise noted. When I say ‘Peskin’ I usually mean ‘Peskin & Schroeder’.

Please tell me if you find typos or errors or violations of the rules above.
1 The path integral makes some things easy

1.1 From particles to fields to particles again

Here is a way to discover QFT starting with some prosaic ingredients. Consider a linear chain of particles of mass $m$, each connected to its neighbors by springs with spring constant $\kappa$. This is a model of a (one-dimensional) crystalline solid. When in equilibrium, the masses form a regular one-dimensional crystal lattice (equally spaced mass points). Now let $q_n$ denote the displacement of the $n$th mass from its equilibrium position $x_n$ and let $p_n$ be the corresponding momentum. Assume there are $N$ masses and (for simplicity) impose periodic boundary conditions: $q_{n+N} = q_n$. The equilibrium positions themselves are

$$x_n = na, n = 1, 2 \ldots N$$

where $a$ is the lattice spacing. The Hamiltonian for the collection of particles is:

$$H = \sum_{n=1}^{N} \left( \frac{p_n^2}{2m} + \frac{1}{2} \kappa (q_n - q_{n-1})^2 \right) + \lambda q^4. \quad (1.1)$$

Notice that this system is an ordinary QM system, made of particles. In particular, the whole story below will take place within the fixed Hilbert space of the positions of the $N$ particles.

I’ve included a token anharmonic term $\lambda q^4$ to remind us that we are leaving stuff out; for example we might worry whether we could use this model to describe melting.

Set $\lambda = 0$ for a while. With $\lambda = 0$, the hamiltonian above describes a collection of coupled harmonic oscillators, with a matrix of spring constants $V = k_{ab} q_a q_b$. If we diagonalize the matrix of spring constants, we will have a description in terms of decoupled oscillators, called normal modes. Because the chain is translation invariant, the normal modes are labelled by a wavenumber $k$, and the eigenvalues are $\omega_k^2 = \frac{2\kappa}{m} \sin^2 \frac{k a}{2}$. Then we can use our knowledge of the spectrum of a single SHO $H = \hbar \omega (a^\dagger a + \frac{1}{2})$ to construct the whole spectrum of excitations of the chain,

$$H = \sum_k \hbar \omega_k \left( a_k^\dagger a_k + \frac{1}{2} \right) + \frac{p_0^2}{2m}.$$ 

(Here $p_0$ is the center-of-mass momentum of the chain.) The groundstate is $|0\rangle$, the state annihilated by all the annihilation operators $a_k |0\rangle = 0$, and excited states are built like $|k_1, k_2\rangle = a_{k_1}^\dagger a_{k_2}^\dagger |0\rangle$. In the context of an elastic solid, these excitations are called phonons.
Instead, let’s use the path integral.

**Path integral reminder in a box.**

Let’s remind ourselves how the path integral formulation of QM works for a particle in one dimension with \( H = \frac{p^2}{2m} + V(q) \). The basic statement is the following formula for the propagator – the amplitude to propagate from position eigenstate \( |q_0\rangle \) to position eigenstate \( |q\rangle \) during a time interval \( t \) is

\[
\langle q | e^{-iHt} | q_0\rangle = \int_{q(0)=q_0}^{q(t)=q} [dq] e^{\frac{i}{\hbar}\int_0^t dt \left( \frac{1}{2} \dot{q}^2 - V(q) \right)} .
\]

Here \( [dq] \equiv \mathcal{N} \prod_{t=1}^{M_t} dq(t) \) – the path integral measure is defined by a limiting procedure (\( M_t \equiv \frac{1}{\Delta t} \to \infty, \Delta t \to 0, t \) fixed), and \( \mathcal{N} \) is a normalization factor that always drops out of physical quantities so I don’t need to tell you what it is.

Recall that the key step in the derivation of this statement is the evaluation of the propagator for an infinitesimal time step:

\[
\langle q_2 | e^{-iH\Delta t} | q_1\rangle = \langle q_2 | e^{-i\Delta t \frac{p^2}{2m}} e^{-i\Delta t V(q)} | q_2\rangle + O(\Delta t^2) .
\]

An integral expression for this can be obtained by inserting resolutions of the identity

\[
1 = 1^2 = \left( \int dp |p\rangle \langle p| \right) \left( \int dq |q\rangle \langle q| \right)
\]

in between the two exponentials. For a more extensive reminder, please see §2.4 of this document.

**Scalar field theory in one dimension.** [Zee §1.3] The path integral for our collection of oscillators is

\[
Z = \int [dq_1 \cdots dq_N] e^{iS[q]}
\]

with \( S[q] = \int dt \left( \sum_n \frac{1}{2} m_n \dot{q}_n^2 - V(\{q\}) \right) \equiv \int dt L(q, \dot{q}). \) The potential is \( V(\{q\}) = \sum_n \frac{1}{2} \kappa (q_{n+1} - q_n)^2. \) Now suppose we have poor eyesight and can’t resolve the individual atoms in the chain; rather we’re only interested in the long-wavelength (small-wavenumber) physics. So let’s try to take the continuum limit \( a \to 0, N \to \infty. \) Basically the only thing we need is to think of \( q_n = q(x = na) \) as defining a smooth function: [Note that the continuum field is often called \( \phi(x) \) instead of \( q(x) \) for some reason. At least the letters \( q(x) \) and \( \phi(x) \) look similar.]

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We’ll use
\[(q_n - q_{n-1})^2 \simeq a^2 (\partial_x q)^2 |_{x=na}, \quad a \sum f(q_n) \simeq \int dx f(q(x)).\]

The path integral becomes:
\[Z = \int [dq] e^{iS[q]}\]
with \([dq]\) now representing an integral over all configurations \(q(t, x)\) (defined by this limit) and
\[S[q] = \int dt \int dx \frac{1}{2} \left( \mu (\partial_t q)^2 - \mu v_s^2 (\partial_x q)^2 - rq^2 - uq^4 - \ldots \right) \equiv \int dt \int dx \mathcal{L}\]

where I’ve introduced some parameters \(\mu, v_s, r, u\) determined from \(m, \kappa, a\)… in some ways that we needn’t worry about, except to say that they are finite in the continuum limit. The \(\ldots\) includes terms like \(a^4 (\partial_x q)^4\) which are small when \(k \ll \frac{1}{a}\), so we ignore them. \(\mathcal{L}\) is the Lagrangian density whose integral over space is the Lagrangian \(L = \int dx \mathcal{L}\).

The equation of motion is the stationary phase condition,
\[0 = \frac{\delta S}{\delta q(x, t)} = -\mu \ddot{q} + \mu v_s^2 \partial_x^2 q - rq - 2uq^3 - \ldots\]

In this expression I have written a functional derivative; with our lattice regulator, it is simply a(n extremely useful) shorthand notation for the collection of partial derivatives \(\frac{\partial}{\partial q_n}\). \(^1\)

From the phonon problem, we automatically found \(r = u = 0\), and the equation of motion is just the wave equation \((\partial_t^2 - v_s^2 \partial_x^2)q = 0\), where we see that \(v_s\) is the sound speed. This happened because of the symmetry \(q_n \to q_n + \epsilon\). This is the operation

\(^1\) Functional derivatives will be very useful to us. The definition is
\[\frac{\delta \phi(x)}{\delta \phi(y)} = \delta(x - y) \quad (1.2)\]

plus the Liebniz properties (linearity, product rule). More prosaically, they are just partial derivatives, if we define a collection of values of the independent variable \(\{x_n\}\) to regard as grid points, and let
\[\phi_n \equiv \phi(x_n)\]

so that \((1.2)\) is just
\[\frac{\partial \phi_n}{\partial \phi_m} = \delta_{nm}.\]

If you are not yet comfortable with the machinery of functional derivatives, please work through pages 2-28 through 2-30 of this document now.
that translates the whole crystal. It guarantees low-energy phonons near $k = 0$ because it means $q(x)$ can only appear in $S$ via its derivatives. (This is a general property of Goldstone modes; more on this later.)

We can construct a hamiltonian from this action by defining a canonical field-momentum density $\pi(x) = \frac{\partial L}{\partial \dot{q}} = \mu \dot{q}$ and doing the Legendre transformation:

$$H = \sum_n (p_n \dot{q}_n - L_n) = \int dx (\pi(x) \dot{q}(x) - \mathcal{L}) = \int dx \left( \frac{(\pi(x))^2}{2\mu} + \mu v_s^2 (\partial_x q(x))^2 + rq^2 + uq^4 + \ldots \right).$$

(1.3)

Note that I suppress the dependence of all the fields on $t$ just so it doesn’t get ugly, not because it isn’t there. Also, I emphasize that the position along the chain $x$ here is just a label on the fields, not a degree of freedom or a quantum operator.

The field $q$ is called a scalar field because it doesn’t have any indices decorating it. This is to be distinguished from e.g. the Maxwell field, which is a vector field, and which we’ll discuss soon. (Note that vibrations of a crystal in three dimensions actually do involve vector indices! We omit this complication.)

The lattice spacing $a$ and the size of the box $Na$ in the discussion above are playing very specific roles in regularizing our 1-dimensional scalar field theory. The lattice spacing $a$ implies a maximum wavenumber or shortest wavelength and so is called an “ultraviolet (UV) cutoff”, because the UV is the short-wavelength end of the visible light spectrum. The size of the box $Na$ implies a maximum wavelength mode which fits in the box and so is called an “infrared (IR) cutoff”.

If (in addition to the continuum limit) we also take the infinite volume limit, then the sums over $k$ become integrals. In this limit we can make the replacement

$$\frac{1}{L^d} \sum_k \sim \int d^dk, \quad L^d \delta_{kk'} \sim (2\pi)^d \delta^{(d)}(k - k').$$

A check of the normalization factors comes from combining these two rules

$$1 = \sum_k \delta_{k,k'} = \int d^d k (2\pi)^d \delta^{(d)}(k - k').$$

Continuum (free) scalar field theory in $d + 1$ dimensions. These continuum expressions are easy to generalize to scalar field theory in any number of dimensions. Let’s do this directly in infinite volume and set $\mu = 1$ by rescaling fields. The action is

$$S[\phi] = \int d^d x dt \left( \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} v_s^2 \vec{\nabla} \phi \cdot \vec{\nabla} \phi \right).$$

(1.4)
This is almost what we would have found for the long-wavelength \((ka \ll 1)\) description of a \(d\)-dimensional lattice of masses on springs, like a mattress (except that there would have been one \(\phi\) for each direction in which the atoms can wiggle). The equation of motion is

\[
0 = \frac{\delta S(\phi)}{\delta \phi(x)} = -\partial_t^2 \phi + v_s^2 \nabla^2 \phi - V'(\phi). \tag{1.5}
\]

For the harmonic case \(V(\phi) = \frac{1}{2}m^2\phi^2\) we know what we’re doing, and (1.5) is called the Klein-Gordon equation,

\[
0 = \left( \partial_\mu \partial^\mu + m^2 \right) \phi. \tag{1.6}
\]

(Notice that I’ve set \(v_s = c = 1\) here, and this is where we have committed to a choice of signature convention; take a look at the conventions page §0.3.) In relativistic notation, the Lagrangian density is just \(L = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right)\). This describes free continuum real massive relativistic scalar quantum field theory. (Match the adjectives to the associated features of the lagrangian; collect them all!)

The canonical momentum is \(p = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}\) and the Hamiltonian (which we can instantly promote to a quantum operator by using boldface symbols) is then

\[
H = \int d^d x \left( \frac{\pi(x)^2}{2} + \frac{1}{2} v_s^2 \left( \nabla \cdot \nabla \phi \right) + \frac{1}{2} m^2 \phi^2 \right). 
\]

Note that all these terms are positive.

A translation invariant linear problem is solved by Fourier transforms: \(\phi(x) = \int d^d k \ e^{-i \vec{k} \cdot \vec{x}} \phi_k\), and \(\pi(x) = \int d^d k \ e^{-i \vec{k} \cdot \vec{x}} \pi_k\), this is

\[
H = \int d^d k \ \left( \frac{1}{2} \pi_k \pi_{-k} + \frac{1}{2} \left( v_s^2 k^2 + m^2 \right) \phi_k \phi_{-k} \right) 
\]

where \(k^2 = (-i \vec{k}) \cdot (i \vec{k}) = \vec{k} \cdot \vec{k}\). This is merely a sum of decoupled oscillators, except for the coupling between wavenumbers \(k\) and \(-k\). We can read off the normal mode frequencies, aka the dispersion relation:

\[
\omega_k^2 = v_s^2 k^2 + m^2.
\]

Notice that this is also the condition for a Fourier mode \(e^{i \vec{k} \cdot \vec{x} - \omega t}\) to solve the Klein-Gordon equation (1.6).

We can decouple the modes with wavenumber \(k\) and \(-k\) by introducing the ladder operators\(^2\)

\[
\phi_k \equiv \sqrt{\frac{\hbar}{2 \omega_k}} \left( a_k + a_k^\dagger \right), \quad \pi_k \equiv \frac{1}{\sqrt{2 \omega_k}} \left( a_k - a_k^\dagger \right), \quad [a_k, a_{k'}^\dagger] = (2\pi)^d \delta(d)(k - k'). 
\]

\(^2\)Beware that the mode operators \(a_k\) defined here differ by powers of \(2\pi/L\) from the finite-volume objects in the previous discussion. These agree with Peskin’s conventions.
Their commutator follows from \([\phi(x), \pi(y)] = i\delta^{(d)}(x - y)\). In terms of the ladder operators,

\[
H = \int d^d k \, \hbar \omega_k \left( a_k^\dagger a_k + \frac{1}{2} \right).
\]

The field operators

\[
\begin{align*}
\phi(\vec{x}) &= \int d^d k \, \sqrt{\frac{\hbar}{2\omega_k}} \left( e^{i\vec{k} \cdot \vec{x}} a_k + e^{-i\vec{k} \cdot \vec{x}} a_k^\dagger \right), \\
\pi(\vec{x}) &= \frac{1}{i} \int d^d k \, \sqrt{\frac{\hbar \omega_k}{2}} \left( e^{i\vec{k} \cdot \vec{x}} a_k - e^{-i\vec{k} \cdot \vec{x}} a_k^\dagger \right),
\end{align*}
\]

(1.7)
satisfy the canonical commutation relation

\[
[\phi(\vec{x}), \pi(\vec{x}')] = i\hbar \delta^d(\vec{x} - \vec{x}').
\]

I emphasize that this is really the same equation as our starting point for each ball on springs:

\[
[q_n, p_{n'}] = i\hbar \delta_{nn'}.
\]

The mode expansions (1.7) contain a great deal of information. First notice that \(\phi\) is manifestly hermitian. Next, notice that from \(\phi(\vec{x}) \equiv \phi(\vec{x}, 0)\) by itself we cannot disentangle \(a_k\) and \(a_k^\dagger\), since only the combination \(a_k + a_k^\dagger\) multiplies \(e^{i\vec{k} \cdot \vec{x}}\). The momentum \(\pi\) contains the other linear combination. However, if we evolve the field operator in time using the Heisenberg equation we find

\[
\phi(\vec{x}, t) \equiv e^{iHt} \phi(\vec{x}) e^{-iHt} = \int d^d k \, \sqrt{\frac{\hbar}{2\omega_k}} \left( e^{i\vec{k} \cdot \vec{x} - i\omega_k t} a_k + e^{-i\vec{k} \cdot \vec{x} + i\omega_k t} a_k^\dagger \right).
\]

(1.8)

Indeed we can check that the relation \(\pi = \dot{\phi}\) holds.

Notice that the dependence on spacetime is via a sum of terms of the form:

\[
e^{i\vec{k} \cdot \vec{x} - i\omega_k t} = e^{ik_\mu x^\mu} |_{k^0 = \omega_k}
\]

and their complex conjugates. These are precisely all the solutions to the wave equation (1.6). For each \(\vec{k}\), there are two solutions, one with positive frequency and one with negative frequency. You might have worried that solutions with both signs of the frequency mean that the world might explode or something (like it would if we tried to replace the Schrödinger equation for the wavefunction with a Klein-Gordon equation). This danger is evaded in a beautiful way: the coefficient of the positive frequency solution with wavenumber \(\vec{k}\) is the destruction operator for the mode; the associated negative frequency term comes with the creation operator for the same mode, as a consequence of reality of the field.
1.2 Fields mediate forces

[Zee §1.3] Consider again our chain of balls on springs. Suppose a giant hand reaches in and pushes the atom at position \( x_n \) a little bit. This can be described by adding to the hamiltonian a term

\[
\delta V(q) = -J_n(t)q_n(t)
\]

which applies a force \( J_n(t) \) to the \( n \)th atom. We can ask, in the presence of such a force, what is the amplitude to go from state \( I \) to state \( F \) in time \( T \):

\[
\langle F \mid e^{-\frac{i}{\hbar} \int_0^T dt H(t)} \mid I \rangle = \int [D\phi] e^{\frac{i}{\hbar} \int dt \left( \frac{1}{2} (\partial \phi)^2 - V(\phi) + J(x)\phi(x) \right)}.
\]

As you see, this is a quantity for which we have a path integral representation. Here’s a reason we might care about this quantity: take the initial and final states to be the groundstates:

\[
\langle 0 \mid e^{-\frac{i}{\hbar} \int_0^T dt H(t)} \mid 0 \rangle \simeq e^{-\frac{i}{\hbar} \int_0^T dt E_{gs}(J)}.
\]

If the time-dependence is slow enough, the answer is obtained by the adiabatic approximation: just add up the instantaneous groundstate energy at each time step.

[End of Lecture 1]

Let’s retreat to the case where the action is quadratic in \( \phi \), so that we can actually do the path integral:

\[
\mathcal{L}(\phi) = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) \text{H.P.} - \frac{1}{2} \phi \left( \partial^2 + m^2 \right) \phi + \text{total derivative.}
\]

(1.9)

Going back to the lattice to make the integrals slightly less scary, we have

\[
e^{iW[J]} \equiv \int [D\phi] e^{\frac{i}{\hbar} \int (\mathcal{L} + J\phi)} = \int_{-\infty}^{\infty} \prod_{n,t} dq_{n,t} e^{\frac{i}{\hbar} q_x A_{xy}q_y + iJ_x q_x} = \sqrt{\frac{(2\pi i)^{NM_t}}{\det A}} e^{-\frac{i}{\hbar} J_x A_{xy}^{-1} J_y}.
\]

Here repeated indices are summed as usual: \( q_x A_{xy}q_y = \int dxdy \phi(x) A_{xy} \phi(y) \), etc... So you can see that the matrix \( A \) multiplying the quadratic term in this gaussian integral is \( A_{xy} = -\delta^{d+1}(x-y)(\partial_x^2 + m^2) \). It is an \( NM_t \times NM_t \) matrix. Its inverse \( A^{-1} \) satisfies by definition \( A_{xz} A^{-1}_{zy} = \delta_{xy} \), which is the differential equation

\[
-(\partial^2 + m^2)D(x-y) = \delta(x-y).
\]

(1.10)

This equation says that \( D \) is a Green’s function for the operator \(-(\partial^2 + m^2)\). The fact that there is no special point in spacetime says \( A_{xy}^{-1} = D(x-y) \) only depends on the difference of its arguments.

Does this integral actually converge? On the homework you saw an integral of the form \( \int dq e^{-\frac{1}{2}q^2} \), which surely converges if \( A \) is a positive matrix. Actually, this is

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overkill – it is enough to replace $m^2 \to m^2 - i\epsilon$ to make all the integrals converge. Here $\epsilon$ is an infinitesimal, which means $\epsilon^2 = 0$ and $c\epsilon = \epsilon$ for any positive $c$. Then each $\int dq_{nt}$ will have a factor of $e^{-\epsilon \int q_{nt}^2}$ which suppresses the integrand in the dangerous large-field region.

The equation (1.10) is translation-invariant and linear so you should not be surprised that it is solved by going to Fourier space (in space and time):

$$D(x) = \int d^{d+1}k \ e^{ik_{\mu}x^\mu} D_k, \quad \delta^{d+1}(x) = \int d^{d+1}k \ e^{ik_{\mu}x^\mu}. $$

in terms of which (1.10) becomes the algebraic equation 1 = $(k^2 - m^2 + i\epsilon)D_k$. Hence

$$D(x) = \int d^{d+1}k \ \frac{e^{ikx}}{k^2 - m^2 + i\epsilon}. $$

Notice that the shift by $\epsilon$ saves the day here: it keeps the integration contour from running right over the pole at $k^2 = m^2$, by moving slightly in the imaginary direction. More explicitly,

$$k^2 - m^2 + i\epsilon = \omega^2 - \bar{k}^2 - m^2 + i\epsilon$$

is zero when

$$\omega = \pm \sqrt{\bar{k}^2 + m^2 - i\epsilon} \ 	ext{Taylor} \quad \omega_k = \sqrt{\bar{k}^2 + m^2}. $$

In the second step I Taylor expanded $\sqrt{\omega^2 - i\epsilon} = \sqrt{\omega_k^2 - \frac{i\epsilon}{\omega_k}} + \mathcal{O}(\epsilon^2)$ and used the facts that $\omega_k > 0$, and that anything positive times an infinitesimal is an infinitesimal.

We can then do the $\omega$ integral by contours: if $t > 0$ ($t < 0$), we can close the contour in the UHP (LHP) since the integrand goes like $e^{-i\omega t}$, and the integral equals the residue of the pole at $\omega = \omega_k = i\epsilon$ (times $2\pi i$):

$$D(x) = -i \int d^d k \left( \theta(t) \frac{e^{-i(\omega_k t - \bar{k} \cdot \bar{x})}}{2\omega_k} + \theta(-t) \frac{e^{i(\omega_k t - \bar{k} \cdot \bar{x})}}{2\omega_k} \right). \quad (1.11)$$

We’ll learn to call this time-ordered in a moment.

---

3Here I have shown you one way to make the integral well-defined. You might worry that there could be others (there are). Another thing you might be bothered by is the boundary conditions on the fields and their relation to the initial and final states. These issues are closely related! In the next subsection, we’ll say more.

4We are using the Cauchy residue theorem $\oint_C dz f(z) = 2\pi i \sum_{z_j} \text{Res}_{z=z_j} f$ where $z_j$ are the poles of $f$. To remember the sign, consider a small circle $C_0$ counterclockwise around the origin and $f(z) = 1/z$, so $\oint_{C_0} \frac{dz}{z} = \int_0^{2\pi} d\theta = 2\pi i$. 

14
**The propagator.** Who is $D(x)$, besides some quantity in terms of which we did a Gaussian integral? Recall from the homework that the inverse matrix can be extracted via a two-point correlation function:

$$A^{-1} = \int dq q^2 e^{-\frac{i}{2} q A q} / Z.$$ 

Putting back all the labels, the same manipulations show that

$$D(x-y) \equiv \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

(1.12)

– the amplitude to propagate an excitation created from the vacuum by $\phi(x)$ to be annihilated by $\phi(y)$. The propagator, for short.

(Notice that if the system is Lorentz invariant (which starting from (1.9) it is) then since $D(x)$ is a scalar quantity, it can only depend on $x$ through Lorentz invariants made from $x^\mu$, namely the proper distance $x^2 = t^2 - \vec{x}^2$, and the sign of $t$.)

Why the '?' in (1.12)? For one thing, $\phi(x)$ and $\phi(y)$ are operators – the order matters. How do I know which order in which to write them? To reproduce (1.11) the thing to do is to time-order them:

$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle \equiv \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle.$$ 

To see this, plug in the mode expansion (1.7) to see e.g.

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \bar{d}k \bar{d}q \frac{\sqrt{\omega_k \omega_q}}{2 \omega_k} e^{-i k x + i q y} \langle 0 | a_k a_q^\dagger | 0 \rangle = \int \bar{d}k \frac{\sqrt{\omega_k}}{2 \omega_k} e^{-i k (x-y)}$$

(where $k^0 = \omega_k$, $q^0 = \omega_q$ to satisfy the KG equation), which reproduces the first term in (1.11).\(^5\)

Now why should we care about the propagator? Look again at $W[J]$. We’ve learned that (up to terms independent of $J$),

$$W[J] = -\frac{1}{2} \int d^{d+1} x d^{d+1} y J(x) D(x-y) J(y) = -\frac{1}{2} \int d^{d+1} k J_k^* \frac{1}{k^2 - m^2 + i\epsilon} J_k$$

Here $J(x) = \int d^{d+1} k e^{i k x} J_k$, $J_k^* = J_{-k}$ (since $J(x)$ is real).

We get to pick $J(x)$. Let’s choose $J = J_1 + J_2$ to describe (in Zee’s words) two lumps sitting still on the mattress: $J_a(x) = \delta^3(x - x_a)$, $a = 1, 2$. Then $J_k = J_1 = J_2$.

\(^5\)The other ways of making the path integral well-defined correspond to other ways of ordering the $\phi$s, and other initial and final states.

\(^6\)In comparing to (1.11), it helps to notice that we can redefine the $k$ integration variable to reverse the sign of the exponent of the spatial part, $\int d^d f(\vec{k}^2) e^{i \vec{k} \cdot \vec{x}} = \int d^d k f(\vec{k}^2) e^{-i \vec{k} \cdot \vec{x}}$. (Thanks to Hung-Hwa Lin for help during lecture.)
\[
\int dx^0 e^{-ik^0 x^0} \left( e^{i\vec{k} \cdot \vec{x}_1} + e^{i\vec{k} \cdot \vec{x}_2} \right).
\]
The interaction between the two lumps mediated by the mattress field \( \phi \) will then be described by the \( J_1 J_2 \) cross-terms in \( W[J] \):

\[
W[J] = -\frac{2}{2} \int dx^0 \int dy^0 \int dk^0 \left( e^{i\vec{k} \cdot \vec{x}_1 - y^0} \right) \int d^3 k \frac{e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{k^2 - m^2 + i\epsilon} + \ldots
\]

(1.13)

\[
= -\int dx^0 \left( \int dh^0 2\pi \delta(k^0) \right) \int d^3 k \frac{e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{k^2 - m^2 + i\epsilon} + \ldots
\]

(1.14)

\[
= +\int dx^0 \int d^3 k \frac{e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{k^2 + m^2 - i\epsilon} + \ldots
\]

(1.15)

(The \( \ldots \) indicate terms which don’t depend on \( x_1, x_2 \), so let’s ignore them.)

For this choice of \( J \), the Hamiltonian is time-independent, and \( e^{iW} = \langle 0 | e^{-iHT} | 0 \rangle = e^{-iE_{gs}(J)T} \), so \( W = -E_{gs}(J)T \). We learn that

\[
E_{gs}(J) = -\int d^d k \frac{e^{i\vec{k} \cdot \vec{x}_{12}}}{k^2 + m^2} + \text{const.}
\]

Notice that we can drop the \( i\epsilon \) now, because this integrand is nonsingular for real \( \vec{k} \). In \( d = 1 \), there are poles at \( k = \pm im \), and we can close the contour in the UHP for free to get\(^7\)

\[
E_{gs}(J) = -\frac{2\pi i}{2\pi} \frac{e^{-mx}}{2im} = -\frac{e^{-mx}}{2m}.
\]

Since \( x \) is the separation between the lumps, this means that our field has produced an \textit{attractive} force between the lumps

\[
F = -\partial_x E_{gs}(J) = +\frac{1}{2} e^{-mx}
\]

which falls off exponentially with the separation between the lumps. The range of the potential goes inversely with the mass of the ‘force carrier’ \( \phi \). The 3d version of this potential \( e^{-mr}/r \) (see footnote 7) is called the Yukawa potential.

\(^7\)For convenience, here’s the integral in 3d:

\[
\int d^3 k \frac{e^{i\vec{k} \cdot \vec{x}}}{k^2 + M^2} \approx \cos \theta = \frac{1}{(2\pi)^2} \int_0^\infty \frac{k^2 dk}{k^2 + M^2} \int_{-1}^{1} dy^0 e^{iky} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{dk k \sin kr}{k^2 + M^2}
\]

\[
= \frac{1}{(2\pi)^2} \left( \frac{1}{2i} \int_{-\infty}^{\infty} \frac{dk k e^{ikr}}{k^2 + M^2} + \text{h.c.} \right) \quad \text{close contour in UHP for free}
\]

\[
= \frac{1}{(2\pi)^2} \frac{1}{2i} \frac{1}{2M} \frac{e^{-iMr}}{2iM} \cdot 2 = \frac{e^{-Mr}}{4\pi r}.
\]
1.3 Euclidean path integral and Wick rotation

Here is a route to defining the path integral (actually the same as the replacement \( m^2 \rightarrow m^2 - i\epsilon \)) which makes clearer what is going on with the initial and final states.

The whole point here can be made for a single mode of the field – a single harmonic oscillator – with action

\[ S[q] = \frac{1}{2} \int dt \left( (\partial_t q)^2 - \Omega^2 q^2 \right) - \int Jq \]

(where \( \Omega^2 = \vec{k}^2 + m^2 \) if you like). Consider the replacement \( \tau = it \) in the action:

\[ S[q] = -\frac{1}{2} i \int d\tau \left( -(\partial_\tau q)^2 - \Omega^2 q^2 \right) + i \int d\tau Jq = i \int d\tau \left( \frac{1}{2} ((\partial_\tau q)^2 + \Omega^2 q^2) + Jq \right). \]

With this replacement, the path integral becomes

\[ \int [Dq] e^{-\int d\tau \left( \frac{1}{2} ((\partial_\tau q)^2 + \Omega^2 q^2) + Jq \right)} \equiv \int [Dq] e^{-S_E[q]}. \]

This integrand suppresses configurations with large \( q \), and large \( \partial_\tau q \), and the integral is therefore totally well-defined. The euclidean action is

\[ S_E[q] = \int d\tau \left( \frac{1}{2} ((\partial_\tau q)^2 + \Omega^2 q^2) + Jq \right) = \int d\tau \left( \frac{1}{2} q (-\partial_\tau^2 + \Omega^2) q + Jq \right) \]

where \((-\partial_\tau^2 + \Omega^2)\) is a positive operator (meaning all of its eigenvalues are positive). Call its inverse \( G \), which then, by definition, satisfies

\[ (-\partial_\sigma^2 + \Omega^2) G(\sigma, \tau) = \delta(\sigma - \tau) \]

The fact that our system is time-translation invariant means \( G(\sigma, \tau) = G(\sigma - \tau) \). We can solve this equation in fourier space: \( G(s) = \int d\omega e^{i\omega s} G_\omega \) makes it algebraic:

\[ G_\omega = \frac{1}{\omega^2 + \Omega^2} \]

and we have

\[ G(\tau) = \int d\omega \frac{e^{i\omega\tau}}{\omega^2 + \Omega^2} = e^{-\Omega |\tau|} \frac{1}{2\Omega}. \]  

(Do it by residues: the integrand has poles at \( \omega = \pm i\Omega \) (see the figure 1 below). The absolute value of \(|\tau|\) is crucial, and comes from the fact that the contour at infinity converges in the upper (lower) half plane for \( \tau < 0 \) (\( \tau > 0 \)).)

\[ ^8 \text{It is called euclidean because the} \ (\partial_\tau q)^2 \ \text{has the same sign as the spatial derivatives} \ (\partial_x q)^2, \ \text{so this is the action we get in euclidean spacetime with metric} \ \delta_{\mu\nu}, \ \text{rather than} \ \eta_{\mu\nu}. \ \text{Exercise: put back the spatial derivative terms and check that this is the case.} \]
I claim that the real-time calculation which keeps the oscillator in its groundstate is the analytic continuation of the one we did above, where we replace

$$\omega_{\text{Mink}} = e^{-i(\pi/2-\epsilon)}\omega_{\text{above}}$$  \hspace{1cm} (1.17)

where $\epsilon$ is (a familiar) infinitesimal. In the picture of the euclidean frequency plane in Fig. 1, this is a rotation by nearly 90 degrees. We don’t want to go all the way to 90 degrees, because then we would hit the poles at $\pm i\Omega$. The replacement (1.17) just means that if we integrate over real $\omega_{\text{Mink}}$, we rotate the contour in the integral over $\omega$ as follows:

$$\omega_{\text{eucl}} = -i\omega_{\text{Mink}}, \quad \omega_{\text{eucl}}t_{\text{eucl}} = \omega_{\text{Mink}}t_{\text{Mink}}, \quad t_{\text{eucl}} = +it_{\text{Mink}}.$$  \hspace{1cm} (1.18)

So this is giving us a contour prescription – a prescription for negotiating the poles – for the real-frequency integral. The result is the *Feynman* contour, and it is the same as
what we got from \( m^2 \to m^2 - i\epsilon \): depending on the sign of the (real) time separation of the two operators (recall that \( t \) is the difference), we close the contour around one pole or the other, giving the *time-ordered* propagator. For the case of a free scalar field, the replacement \( m^2 \to m^2 - i\epsilon \) had the same effect of rotating the real-frequency contour away from the poles. It is also the same as shifting the frequency by \( \Omega \to \Omega - i\epsilon \), as indicated in the right part of Fig. 2. This prescription works in a case where there is no \( m^2 \) term.

Notice for future reference that the euclidean action and real-time action are related by

\[
S_{\text{eucl}}[Q] = \int dt_{\text{eucl}} \frac{1}{2} \left( \left( \frac{\partial Q}{\partial t_{\text{eucl}}} \right)^2 + \Omega^2 Q^2 \right) = -i S_{\text{Mink}}[Q] = -i \int dt_{\text{Mink}} \frac{1}{2} \left( \left( \frac{\partial Q}{\partial t_{\text{Mink}}} \right)^2 - \Omega^2 Q^2 \right).
\]

because of (1.18). This means the path integrand is \( e^{-S_{\text{eucl}}} = e^{iS_{\text{Mink}}} \).

---

**Euclidean evolution.** Now, why does the contour coming from the euclidean path integral put the oscillator into its groundstate? The point in life of the euclidean time evolution to prepare the groundstate from an arbitrary state:

\[
e^{-\beta H} |\text{any}\rangle = \sum_n e^{-\beta E_n} |n\rangle \langle n|\text{any}\rangle \propto |\text{gs}\rangle + \mathcal{O}\left( e^{-\beta(E_1-E_{\text{gs}})} \right) \tag{1.19}
\]

– the euclidean-time propagator \( e^{-\beta H} \) beats down the amplitude of any excited state relative to the groundstate, for large enough \( \beta \).

And the euclidean path integral gives a formula for this euclidean propagation amplitude. Recall that the path integral representation for the real-time propagation
amplitude is
\[ \langle f | e^{-iHt} | i \rangle = \int_{f \leftarrow -i} [dq] e^{i \int_0^t dt L} \, . \]

On the RHS here, we sum over all paths between \( i \) and \( f \) in time \( t \) (i.e. \( q(0) = q_i, q(t) = q_f \)), weighted by a phase \( e^{i \int f dt L} \). But that means you also know a representation for
\[ \sum_f \langle f | e^{-\beta H} | f \rangle \equiv \text{tr} e^{-\beta H} \]

- namely, you sum over all periodic paths \( q_i = q_f \) in imaginary time \( t = -i \beta \). So:
\[ Z(\beta) = \text{tr} e^{-\beta H} = \oint [dq] e^{-\int_0^\beta d\tau L} = \oint [dq] e^{-S_{\text{eucl}}[q]} \]

The LHS is the partition function in quantum statistical mechanics. The RHS is the euclidean functional integral we’ve been using. [For more on this, see Zee §V.2]

The period of imaginary time, \( \beta \equiv 1/T \), is the inverse temperature. We’ve been studying the limit as \( \beta \to \infty \). Taking \( \beta \to \infty \) means \( T \to 0 \), and you’ll agree that at \( T = 0 \) we project onto the groundstate (if there’s more than one groundstate we have to think more).

**Time-ordering.** To summarize the previous discussion: in real time, we must choose a state, and this means that there are many Green’s functions, not just one: \( \langle \psi | q(t)q(s) | \psi \rangle \) depends on \( |\psi\rangle \), unsurprisingly.

But we found a special one which arises by analytic continuation from the euclidean Green’s function, which is unique\(^9\). It is
\[ G(s, t) = \langle \mathcal{T} (q(s)q(t)) \rangle \, , \]

the time-ordered, or Feynman, Green’s function, and I write the time-ordering symbol \( \mathcal{T} \) to emphasize this. I emphasize that from our starting point above, the time ordering arose because we have to close the contour in the UHP (LHP) for \( t < 0 \ (t > 0) \).

---

\(^9\) Another important perspective on the uniqueness of the euclidean Green’s function and the non-uniqueness in real time: in euclidean time, we are inverting an operator of the form \( -\partial_t^2 + \Omega^2 \) which is positive (⇐ all its eigenvalues are positive) – recall that \( -\partial_t^2 = \hat{p}^2 \) is the square of a hermitian operator. If all the eigenvalues are positive, the operator has no kernel, so it is completely and unambiguously invertible. This is why there are no poles on the axis of the (euclidean) \( \omega \) integral in (1.16). In real time, in contrast, we are inverting something like \( +\partial_t^2 + \Omega^2 \) which annihilates modes with \( \partial_t = i\Omega \) (if we were doing QFT in \( d > 0 + 1 \) this equation would be the familiar \( p^2 - m^2 = 0 \)). These are called ‘on-shell states’, they are actual states in the spectrum of the Hamiltonian of the system. So the operator we are trying to invert has a kernel and this is the source of the ambiguity. In frequency space, this is reflected in the presence of poles of the integrand on the contour of integration; the choice of how to negotiate them encodes the choice of Green’s function.
Let’s pursue this one more step. The same argument tells us that the generating functional for real-time, time-ordered correlation functions of $q$ is

$$Z[J] = \langle T e^{i \int dt q(t) J(t)} \rangle = \langle 0 | T e^{i \int J_q} | 0 \rangle ,$$  \hspace{1cm} (1.20)

in the sense that

$$\langle T q(t_1) q(t_2) \ldots \rangle = \frac{1}{Z} \frac{\delta}{\delta J(t_1)} \frac{\delta}{\delta J(t_2)} \ldots Z[J] \big|_{J=0} .$$

In the second step of (1.20) I just emphasized that the real time expectation value here is really a **vacuum** expectation value, as long as we use the $i \epsilon$ prescription above to define the integrals. This quantity has the picturesque interpretation as the **vacuum persistence amplitude**, in the presence of the source $J$.\(^{10}\)

So we see that in general, the correlation functions that are computed by this “$i \epsilon$ prescription” of Wick rotating from Euclidean spacetime are time-ordered vacuum expectation values:

$$\frac{1}{Z} \int [D \phi] e^{i \mathcal{S}_{m^2 \rightarrow m^2 - \epsilon} f(\phi)} = \langle 0 | T f(\phi) | 0 \rangle .$$

**Causality.** In other treatments of this subject, you will see the Feynman contour motivated by ideas about causality. This was not the logic of our discussion here but it is reassuring that we end up in the same place. Note that even in 0+1 dimensions there is a useful notion of causality: effects should come after their causes.

### 1.4 Feynman diagrams from the path integral

Subsection §1.3 was a sophisticated discussion of QFT in 0+1 dimensions (*i.e.* ordinary quantum mechanics of a single particle), since we focussed on a single mode. To attempt to demystify some more of the structure we’ll discover in QFT, let’s regress even further, and consider the case of QFT in 0+0 dimensions. By the path-integral representation, this means ordinary integrals. If everything is positive, this is probability theory.

Suppose we want to do the integral

$$Z(J) = \int_{-\infty}^{\infty} dq \ e^{-\frac{1}{2} m^2 q^2 - \frac{\phi}{4} q^4 + J_q} \equiv \int dq \ e^{-S(q)} .$$ \hspace{1cm} (1.21)\(^{10}\)

\(^{10}\)Actually, more useful is the generating function of **connected** correlation functions:

$$\langle T q(t_1) q(t_2) \ldots \rangle_c = \frac{\delta}{\delta J(t_1)} \frac{\delta}{\delta J(t_2)} \ldots \log Z[J] \big|_{J=0} .$$

where $\langle q_1 q_2 \rangle_c \equiv \langle q_1 q_2 \rangle - (q_1) \langle q_2 \rangle$. Higher-point connected correlation functions are defined by subtracting the gaussian answer. Connected correlation functions are well-named because they are computed by connected Feynman diagrams, as we’ll discuss more next.
It is the path integral for $\phi^4$ theory with fewer labels. For $g = 0$, this is a gaussian
integral which we know how to do. For $g \neq 0$ it’s not an elementary function of its
arguments. We can develop a (non-convergent!) series expansion in $g$ by writing it as

$$Z(J) = \int dq \, e^{-\frac{1}{2}m^2q^2+Jq} \left( 1 - \frac{g}{4!}q^4 + \frac{1}{2!} \left( -\frac{g}{4!}q^4 \right)^2 + \cdots \right)$$

and integrating term by term. And the term with $q^{4n}$ (that is, the coefficient of $rac{1}{n!} \left( \frac{-g}{4!} \right)^n$) is

$$\int dq \, e^{-\frac{1}{2}m^2q^2+Jq} q^{4n} = \left( \frac{\partial}{\partial J} \right)^4 \int dq \, e^{-\frac{1}{2}m^2q^2+Jq} = \left( \frac{\partial}{\partial J} \right)^4 e^{\frac{1}{2}J \frac{1}{m^2} J} \frac{2\pi}{m^2}.$$

So:

$$Z(J) = \sqrt{\frac{2\pi}{m^2}} e^{-\frac{g}{4!} \left( \frac{\partial}{\partial J} \right)^4 J}.$$

This is a double expansion in powers of $J$ and powers of $g$. The process of computing
the coefficient of $J^n g^m$ can be described usefully in terms of diagrams. There is a factor
of $1/m^2$ for each line (the propagator), and a factor of $(-g)$ for each 4-point vertex
(the coupling), and a factor of $J$ for each external line (the source). For example, the
coefficient of $g J^4$ comes from:

$$\sim \left( \frac{1}{m^2} \right)^4 g J^4.$$

There is a symmetry factor which comes from expanding the exponential: if the
diagram has some symmetry preserving the external labels, the multiplicity of diagrams
does not completely cancel the $1/n!$.

As another example, consider the analog of the two-point function:

$$G \equiv \langle q^2 \rangle \left|_{J=0} \right. = \frac{\int dq \, q^2 \, e^{-S(q)}}{\int dq \, e^{-S(q)}} = -2 \frac{\partial}{\partial m^2} \log Z(J = 0). \quad (1.22)$$

In perturbation theory this is:

$$G \approx \underbrace{\quad + \quad + \quad + \quad + \quad }_{\mathcal{O}(g^3)}$$
\[ = m^{-2} \left( 1 - \frac{1}{2} g m^{-4} + \frac{2}{3} g^2 m^{-8} + \mathcal{O}(g^3) \right) \quad (1.23) \]

To get the numerical coefficients note that Wick’s theorem for this simple case is

\[ \langle q^k \rangle_0 = \begin{cases} 0, & k \text{ odd} \\ (k-1)!!, & k \text{ even} \end{cases} \quad (1.24) \]

– the number of ways of pairing \( k \) objects. Here \((k-1)!! \equiv (k-1)(k-3)(k-5)\ldots\cdot3\cdot1\). This is because there are \( k-1 \) choices of partner for the first \( q \), after which there are \( k-3 \) choices of partner for the next one, etc.

Some important structural comments: A diagram contributing to \( G \) which has any part not connected to the external legs is cancelled by the expansion of the denominator \( Z = \int dq e^{-S(q)} \). The contributions to \( Z \) are called ‘vacuum diagrams’ (since they have no external lines, so they are like an amplitude for nothing to turn back into nothing). \( Z \) is a sum over all diagrams with no external lines, including disconnected ones. As you saw in 215A, this sum exponentiates: \( Z = e^{\sum \text{connected diagrams}} \).

**Some labels.** Some of these points are clearer if we put back some of the labels. So consider the slightly more complicated case

\[ Z = \int \prod_{i=1}^N dq_i e^{-S(q)}, \quad S(q) = \frac{1}{2} q_i A_{ij} q_j + \frac{g}{4!} \sum_i q_i^4 \equiv S_0 + \frac{g}{4!} \sum_i q_i^4. \]

(Think of \( i \) as like a position index, and \( A \) as a difference operator, so this is a discretization of \( \phi^4 \) theory.) Then we can develop a perturbative expansion by writing

\[ \langle q_1 \cdots q_k \rangle = \frac{\int \prod_{i=1}^N dq_i e^{-S(q)} q_1 \cdots q_k}{Z} \]

\[ = \frac{\int \prod_{i=1}^N dq_i e^{-S_0(q)} e^{-\frac{g}{4!} \sum_i q_i^4} q_1 \cdots q_k}{\int \prod_{i=1}^N dq_i e^{-S_0(q)} e^{-\frac{g}{4!} \sum_i q_i^4}} \]

\[ \approx \frac{\sum_{n=0} \left(-\frac{g}{4!}\right)^n / n! \int dq e^{-S_0} \sum_{i_1,\ldots,i_n} \prod_{i=1}^n q_i^4 q_{i_1} \cdots q_k}{\sum_{n=0} \left(-\frac{g}{4!}\right)^n / n! \int dq e^{-S_0} \sum_{i_1,\ldots,i_n} \prod_{i=1}^n q_i^4}. \quad (1.27) \]

(Note the step with the \( \approx \) is where we exchange the sum over \( n \) with the integral over \( q \).)

Now the general statement of Wick’s theorem is:

\[ \langle q_1 \cdots q_k \rangle_0 = \begin{cases} 0, & k \text{ odd} \\ \sum_{\text{contractions}} A_{i_{1/2}}^{-1} \cdots A_{i_{k/2-1/2}}^{-1}, & k \text{ even} \end{cases} \]

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Here the sum is over all ways of pairing up the $k$ fields. (Note that this reduces to (1.24) if we remove all the labels.)

Let’s think about a particular term, e.g. an $\mathcal{O}(g^2)$ contribution to $\langle q_iaq_j \rangle$:

$$
\begin{align*}
\langle q_i q_j \rangle & \approx -g \sum_i \left( A_{i1}^{-1} A_{23}^{-1} + A_{i2}^{-1} A_{34}^{-1} + A_{i3}^{-1} A_{42}^{-1} \right) \\
& \quad + \frac{g^2}{2!} \sum_{ij} \left( \frac{1}{2} A_{i1}^{-1} A_{i4}^{-1} A_{i2}^{-1} A_{i3}^{-1} + \cdots \right) + \cdots
\end{align*}
$$

The factor of $1/6$ is called a symmetry factor. Instead of by explicit combinatorics, we could have gotten this number by dividing by the order of the automorphism group of the diagram. An automorphism of the diagram is a map from the diagram to itself which preserves the external lines and the connectivity. In this diagram, we can permute the three internal lines amongst themselves, giving $|S_3| = 3! = 6$ in the denominator. Don’t get hung up on the symmetry factors.

As a final example for now, here is the expansion of the four-point function:

$$
\langle q_1 q_2 q_3 q_4 \rangle \approx + \frac{1}{2} m^2 q_1^2 + i \epsilon q_1^2 + \cdots
$$

All the labels. The Feynman diagrams we’re going to draw all the time are the same but with more labels. Notice that each of the $q$s in our integral could come with a label, $q \rightarrow q_a$. Then each line in our diagram would be associated with a matrix $(m^{-2})_{ab}$ which is the inverse of the quadratic term $q_a m^2 q_b$ in the action. If our diagrams have loops we get free sums over the label. If that label is conserved by the interactions, the vertices will have some delta functions. In the case of translation-invariant field theories we can label lines by the conserved momentum $k$. Each comes with a factor of the free propagator $\frac{1}{k^2 + m^2 + i \epsilon}$, each vertex conserves momentum, so

[End of Lecture 3]
comes with $i g \delta^D(\sum k)(2\pi)^D$, and we must integrate over momenta on internal lines $\int d^D k$.

**Brief comments about large orders of perturbation theory.**

- The perturbation series about $g = 0$ does not converge. How do I know? One way to see this is to notice that if I made $g$ even infinitesimally negative, the integral itself would not converge (the potential would be unbounded below), and $Z_{g=-|\epsilon|}$ is not defined. Therefore $Z_g$ as a function of $g$ cannot be analytic in a neighborhood of $g = 0$. This argument is due to Dyson, and applies also in most QFTs. This means there is more to QFT than perturbation theory: the perturbation series does not define the field theory amplitudes.

- The expansion of the exponential in the integrand is clearly convergent for each $q$. The place where we went wrong is exchanging the order of integration over $q$ and summation over $n$.

- In this case, the perturbation expansion can be given a closed form expression:

$$Z(0) \simeq \sqrt{\frac{2\pi}{m^2}} \sum_n \frac{(-1)^n}{n!} \frac{2^{2n+\frac{1}{2}}}{(4!)^n} \Gamma\left(2n + \frac{1}{2}\right) \left(\frac{g}{m^4}\right)^n.$$  

(1.28)

- The expansion for $G$ is of the form

$$G \simeq m^{-2} \sum_{n=0}^{\infty} c_n \left(\frac{g}{m^4}\right)^n.$$  

When $n$ is large, the coefficients satisfy $c_{n+1} \lesssim -\frac{2}{3} nc_n$ (you can see this by looking at the coefficients in (1.28)) so that $|c_n| \sim n!$. This factorial growth of the number of diagrams is general in QFT and is another way to see that the series does not converge.

- The fact that the coefficients $c_n$ grow means that there is a best number of orders to keep. The errors start getting bigger when $c_{n+1} \left(\frac{g}{m^4}\right) \sim c_n$, that is, at order $n \sim \frac{3m^2}{2g}$. So if you want to evaluate $G$ at this value of the coupling, you should stop at that order of $n$. An estimate of the error by the next term left out gives something that goes like $e^{-\# / g}$.

- I said above that the fact that the perturbation series doesn’t converge means that it doesn’t define the field theory amplitudes. What does it miss? To answer
this, consider trying to do the integral (1.21) by saddle point (at \( J = 0 \) for simplicity):

\[
0 = S'(q_*) = m^2 q_* + \frac{g}{3!} q_*^3.
\]

(Note the resemblance to the equations of motion.) This has three solutions:

\[
q_* = 0, \quad q_* = \pm i \sqrt{\frac{3! m^2}{g}}.
\]

The expansion about the ‘trivial’ saddle at \( q_* \) (where the action is \( S(q_* = 0) = 0 \)) reproduces the perturbation series. At the other saddles,

\[
S \left( q_* = \pm i \sqrt{\frac{3! m^2}{g}} \right) = - \frac{3 m^4}{2 g},
\]

which means their contribution would go like \( e^{\frac{3 m^4}{2 g}} \), which actually would blow up at weak coupling, \( g \to 0 \). These saddles are not on the contour and don’t contribute for small positive \( g \), but more generally (as for example when \( m^2 < 0 \)), there will be effects that go like \( e^{\frac{a}{g}} \). This is a function whose series expansion in \( g \) at \( g = 0 \) is identically zero. You can never find it by doing perturbation theory in \( g \) about \( g = 0 \).

- A technique called Borel resummation can sometimes produce a well-defined function of \( g \) from an asymptotic series whose coefficients diverge like \( n! \). The idea is to make a new series

\[
B(z) \equiv \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n
\]

whose coefficients are ensmallened by \( n! \). Then to get back \( Z(g) \) we use the identity

\[
1 = \frac{1}{n!} \int_{0}^{\infty} dze^{-z} z^n
\]

and do the Laplace transform of \( B(z) \):

\[
\int_{0}^{\infty} dz B(z) e^{-z/g} = \sum_{m=0}^{\infty} c_m \frac{\int_{0}^{\infty} dze^{-z/g} z^m}{m!} = g \sum_{m=0}^{\infty} c_m g^m = gZ(g).
\]

This procedure requires both that the series in \( B(z) \) converges and that the Laplace transform can be done. In fact this procedure works in this case.

The fact that the number of diagrams at large order grows like \( n! \) is correlated with the existence of saddle-point contributions to \( Z(g) \) which go like \( e^{-a/g} \).
This is because they are associated with singularities of $B(z)$ at $z = a$; such a singularity means the sum of $\frac{c_n}{n!} z^n$ must diverge at $z = a$. (More generally, non-perturbative effects which go like $e^{-a/g^{1/p}}$ (larger if $p > 1$) are associated with (faster) growth like $(pn)!$. In string theory, $p = 2$. See this classic work.)

• In fact in this case, we know the whole function. The integral actually does have a name – it’s a Bessel function:

$$Z(J = 0) = \frac{2}{\sqrt{m^2}} \sqrt{\rho} e^{\rho} K_\frac{1}{4}(\rho), \quad \rho \equiv \frac{3m^4}{4g}$$

(for $\text{Re} \sqrt{\rho} > 0$), as Mathematica will tell you. Because we know about Bessel functions, in this case we can actually figure out what happens at strong coupling, when $g \gg m^4$, using the asymptotics of the Bessel function.

• The functions $G(g)$ and $Z(g)$ can be analytically continued in $g$ away from the real axis, and can in fact be defined on the whole complex $g$ plane. It has a branch cut on the negative real axis, across which its discontinuity is related to its imaginary part. The imaginary part goes like $e^{-a|g|}$ near the origin and can be computed by a tunneling calculation like (1.29).

How did we know $Z$ has a branch cut? One way is from the asymptotics of the Bessel function. But, better, why does $Z$ satisfy the Bessel differential equation as a function of the couplings? The answer, as you’ll check on the homework, is that the Bessel equation is a Schwinger-Dyson equation,

$$0 = \int_{-\infty}^{\infty} \frac{\partial}{\partial q} \left( \text{something } e^{-S(q)} \right)$$

which results from demanding that we can change integration variables in the path integral.

For a bit more about this, you might look at sections 3 and 4 of this recent paper from which I got some of the details here. See also the giant book by Zinn-Justin. There is a deep connection between the large-order behavior of the perturbation series about the trivial saddle point and the contributions of non-trivial saddle points. The keywords for this connection are resurgence and trans-series and a starting reference is here.

The class of equations (1.30) is very important: it shows that the equations of motion are true in Green’s functions, up to contact terms, the contributions where the
\frac{\partial}{\partial q} \text{ hits the 'something'. For example, in scalar field theory}

\[ 0 = \int [D\phi] \frac{\delta}{\delta \phi(y)} (\phi(x)e^{iS[\phi]}) = \left\langle \phi(x)i \frac{\delta S}{\delta \phi(y)} \right\rangle + \delta^{d+1}(x - y). \]

In the special case where $S$ is quadratic, $S = \int \phi A \phi$ , this shows that the two-point function is a Green’s function for the quadratic operator $A$. 
1.5 Lagrangian field theory

[Here we fill in the bits of Peskin §2.2 that we missed above.] Let’s consider a classical field theory in the Lagrangian description. This means that the degrees of freedom are a set of fields $\phi_r(x)$, where $r$ is a discrete index (for maybe spin or polarization or flavor), and we specify the dynamics by the classical action. If the world is kind to us (in this class we assume this), the action is an integral over space and time of a Lagrangian density

$$S[\phi] \equiv \int d^{d+1}x \mathcal{L}(\phi, \partial^\mu \phi).$$

This important assumption is an implementation of locality.

This central object encodes the field equations, the canonical structure on the phase space, the Hamiltonian, the symmetries of the theory.

I’ve sneakily implied that we are going to assume Lorentz invariance, so that $\mathcal{L}$ depends on the 4-vector $\partial^\mu \phi$, and not its components separately. I am also going to assume that the action $S$ is real.

Two examples to keep in mind are the Klein-Gordon Lagrangian:

$$\mathcal{L}_{KG} = \frac{1}{2} \partial^\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

and the Maxwell Lagrangian:

$$\mathcal{L}_{EM} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} = \frac{1}{4e^2} (E^2 - B^2)$$

with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $A_\mu$ regarded as the independent degrees of freedom.

A word about units: in units with $\hbar = c = 1$, everything has units of mass to some power, called its mass dimension. Energy and momentum $p_\mu = \hbar k_\mu$ have mass dimension +1. The space and time coordinates $x^\mu$ have mass dimension -1. The action goes in the exponential of the path integral measure $\int [D\phi] e^{i \pi}$ and so must be dimensionless. So the Lagrangian density has mass dimension $d+1$. This means that the KG field has mass dimension $\frac{d-1}{2}$ (and the mass $m$ has mass dimension 1 (yay!)). In $d+1 = 3+1$ dimensions, $E \sim \dot{A}, B \sim \nabla A$ have mass dimension 2 and $A$ has mass dimension one (and $e$ is dimensionless). This is nice because then the covariant derivative $\partial_\mu + A_\mu$ has mass dimension one. Notice that $E^2 + B^2$ has dimension 4 which is good for an energy per unit volume.

The equation of motion is

$$0 = \frac{\delta S}{\delta \phi_r(x)}.$$
Note the functional derivative. You can check that in the case when $L$ depends only on $\phi$ and $\partial_\mu \phi$, this is the same as the Lagrange EOM

$$0 = \frac{\partial L}{\partial \phi_r} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi_r)}$$

(for each $r$) which I can’t remember. Note that since we are interested here in the bulk equations of motion, we ignore boundary terms unless we are interested in field theory on a space with boundary. That is a worthy subject but an unnecessary complication for now.

By redefining the field by e.g. $\phi \equiv \frac{1}{D} (\chi - B/C)$, we can make the KG theory uglier

$$L = A + B\chi + \frac{1}{2} C\chi^2 + \frac{1}{2} D\partial^\mu \chi \partial_\mu \chi.$$ 

From the path integral point of view, the field is just an integration variable. Sometimes, its normalization is meaningful, like in the phonon example where it began its life as the displacement of the atoms from their equilibrium. So you see that relative to the most general possible Lagrange density for a scalar field, we are not losing generality except in our neglect of interactions, and in our neglect of terms with more derivatives. The former neglect we will repair little by little in this course, by doing perturbation theory. The latter is justified well by the renormalization group philosophy, which is a subject for later.

**Canonical field momentum and Hamiltonian.** The Hamiltonian viewpoint in field theory has the great virtue of bringing out the physical degrees of freedom. It has the great shortcoming that it picks out the time coordinate as special and obscures Lorentz symmetry.

The canonical field momentum is defined to be

$$\pi(x) = \frac{\partial L}{\partial (\partial_t \phi(x))}.$$ 

Notice that this expression assumes a local Lagrangian density. $\pi$ is actually a ‘field momentum density’ in the sense that the literal canonical momentum is $\frac{\partial}{\partial \phi(x)} L = d^d x \pi(x)$ (as opposed to $L$). I will often forget to say ‘density’ here.

The hamiltonian is then

$$H = \sum_n p_n q_n - L = \int d^d x \left( \pi(x) \dot{\phi}(x) - L \right) \equiv \int d^d x \ h.$$ 

**Noether’s theorem and the Noether method.** Yay, symmetries. Why do physicists love symmetries so much? One reason is that they offer possible resting
places along our never-ending chains of ‘why?’ questions. For example, one answer to
the question “Why QFT?” is (certainly this is the one given in Weinberg’s text, but
just as certainly it is not the only one): quantum mechanics plus Poincaré symmetry.

They are also helpful for solving physical systems: Continuous symmetries are
associated with conserved currents. Suppose the action is invariant under a continuous
transformation of the fields \( \phi, \phi(x) \mapsto \phi'(x) \). (The invariance of the action is what
makes the transformation a symmetry.) ‘continuous’ here means we can do the trans-
formation just a little bit, so that \( \phi(x) \mapsto \phi(x) + \epsilon \Delta \phi(x) \) where \( \epsilon \) is an infinitesimal parameter.

If the transformation with constant \( \epsilon \) (independent of space and time) is a symmetry,
then the variation of the action with \( \epsilon = \epsilon(x,t) \) must be proportional to \( \partial_\mu \epsilon \) (at least
assuming some smoothness properties of the action), and so that it vanishes \( \forall \phi \) when
\( \epsilon \) is constant:

\[
S[\phi + \epsilon(x) \Delta \phi] - S[\phi] = \int d^d x dt \partial_\mu \epsilon(x) j^\mu \text{IBP} = - \int d^d x dt \epsilon(x) \partial_\mu j^\mu.
\]

But if the equations of motion are obeyed, then the action is invariant under any
variation of \( \phi \), including this one, for arbitrary \( \epsilon(x) \). But this means that \( \partial_\mu j^\mu = 0 \), the
current is conserved. These words are an accurate description of the equation because
they mean that the charge

\[
Q_R \equiv \int_R d^d x \ j^0
\]
in some region of space \( R \) can only change by leaving the region (assume the definition
of \( R \) is independent of time):

\[
\partial_t Q_R = \int_R d^d x \ \partial_t j^0 = - \int_R d^d x \ \nabla \cdot \vec{j} = - \int_{\partial R} d^{d-1} x \hat{n} \cdot \vec{j}
\]
where in the last step we used Stokes’ theorem.

This trick with pretending the parameter depends on space is called the Noether
method. More prosaically, the condition that the action is invariant means that the
Lagrangian density changes by a total derivative (we assume boundary terms in the
action can be ignored):

\[
\mathcal{L}(\phi', \partial_\mu \phi') \overset{\text{symmetry}}{=} \mathcal{L}(\phi, \partial_\mu \phi) + \epsilon \partial_\mu \mathcal{J}^\mu
\]
but on the other hand, by Taylor expansion,

\[
\mathcal{L}(\phi', \partial_\mu \phi') \overset{\text{calculus}}{=} \mathcal{L}(\phi, \partial_\mu \phi) + \epsilon \left( \frac{\partial \mathcal{L}}{\partial \phi} \Delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\mu \Delta \phi \right)
\]
\[
\begin{align*}
\text{IBP} & = \mathcal{L}(\phi, \partial_\mu \phi) + \epsilon \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \Delta \phi + \epsilon \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Delta \phi \right).
\end{align*}
\]

By combining the previous two equations for \( \mathcal{L}(\phi') \), we see that on configurations which satisfy the EOM, \( 0 = \partial_\mu j^\mu \) with
\[ j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_r)} \Delta \phi_r - J^\mu. \] (1.31)

Notice that I stuck the index back in at the last step.

There is a converse to the Noether theorem, which is easier to discuss directly in quantum mechanics. Given a conserved charge \( Q \), that is, a hermitian operator with \( [H, Q] = 0 \), we can make a symmetry transformation of the fields \( \phi \) by
\[ \delta \phi \equiv i \epsilon [Q, \phi]. \] (1.32)

We’ll say that \( Q \) \textit{generates} the symmetry, for the following reason. (1.32) is the infinitesimal version of the finite transformation
\[ \phi \to \phi' \equiv e^{i\epsilon Q} \phi e^{-i\epsilon Q}. \]

The object \( U \equiv e^{i\epsilon Q} \) is a unitary operator (since \( Q \) is hermitian) which represents the action of the symmetry on the Hilbert space of the QFT. It is a symmetry in the sense that it commutes with the time evolution operator \( e^{-\nabla Ht} \).

Some examples will be useful:

- For example, suppose \( S[\phi] \) only depends on \( \phi \) through its derivatives, for example, \( S[\phi] = \int \frac{1}{2} \partial_\mu \phi \partial_\mu \phi \). Then there is a shift symmetry \( \phi \to \phi' \equiv \phi + \epsilon \). Letting \( \epsilon \) depend on spacetime, the variation of the action is \( S[\phi + \epsilon(x)] - S[\phi] = -\int \epsilon \partial_\mu \partial_\mu \phi \), so the current is \( j_\mu = \partial_\mu \phi \). Let’s check the converse: Indeed, the charge \( Q = \int_{\text{space}} j_0 \) generates the symmetry in the sense that for small \( \epsilon \), the variation in the field is
\[ \delta \phi \equiv \phi' - \phi = \epsilon = i \epsilon [Q, \phi] \]
(if we were doing classical mechanics, we should replace \( i [Q, \phi] \) with the Poisson bracket). Using our expression for the current this is
\[ \delta \phi = i \epsilon \left[ \int d^d y \dot{\phi}(y), \phi(x) \right] = \epsilon \]
which is indeed true by the canonical commutation relations. In this case the finite transformation is again \( \phi \to \phi + \epsilon \).
Consider a complex scalar $\Phi$, and suppose $S[\Phi, \Phi^*]$ is invariant under $\Phi \rightarrow e^{i\epsilon} \Phi = \Phi + i\epsilon \Phi + \mathcal{O}(\epsilon^2)$, such as $S = \int (\partial \Phi^* \partial \Phi - V(\Phi^* \Phi))$. This $U(1)$ phase transformation can be rewritten in terms of the real and imaginary parts as an $SO(2)$ rotation. The charge can be written as

$$Q = \int d^d x j^0 = \int d^d p (a_p^\dagger a_p - b_p^\dagger b_p)$$

where the two sets of creation and annihilation operators are associated with excitations of $\Phi$ and $\Phi^\dagger$ respectively. (That is, quantize $\phi_{1,2}$ as we did for a single real scalar field, in terms of mode operators $a_{1,2}$ respectively. Then let $a \equiv a_1 + i a_2, b \equiv a_1 - i a_2$, up to numerical prefactors.) So the particles created by $a$ and $b$ have opposite charge (this follows given the mode expansion $\Phi_k \sim a_k^\dagger + b_{-k}^\dagger$) and can be interpreted as each others’ antiparticles: there can be symmetry-respecting processes where an $a$ particle and $b$ particle take each other out.

The previous two examples are related. Consider the case where $V(\Phi^* \Phi) = \lambda(\Phi^* \Phi - v^2)^2$ Changing variables to polar coordinates in field space, $\Phi = \rho e^{i\phi}$, the Lagrangian is

$$\mathcal{L} = \rho^2 (\partial \phi)^2 + (\partial \rho)^2 - \lambda(\rho^2 - v^2)^2.$$  

If $\lambda$ is big, the potential forces $\rho = v$, and its fluctuations are heavy, and we are left with $\mathcal{L} = v^2 (\partial \phi)^2$, where $\phi \rightarrow \phi + \epsilon$ is a symmetry. Notice that $\phi \equiv \phi + 2\pi$ is periodic.

Consider spacetime translations, $x^\mu \rightarrow x^\mu - a^\mu$. We can think of this as a transformation of the fields by

$$\phi(x) \mapsto \phi(x + a) = \phi(x) + a^\nu \partial_\nu \phi + \mathcal{O}(a^2).$$

Our transformation parameter is now itself a four-vector, so we’ll get a four-vector of currents $T_\nu^\mu$. This will be a symmetry as long as the lagrangian doesn’t depend explicitly on space and time (so $\partial_\mu \mathcal{L} = 0$) but rather depends on space and time only via the fields (so $0 \neq \frac{d}{dx^\mu} \mathcal{L} \overset{\text{chain rule}}{=} \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial \phi} + \partial_\mu \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)}$). Let’s use the prosaic method for this one: the shift in the Lagrangian density also can be found by Taylor expansion

$$\mathcal{L} \mapsto \mathcal{L} + a^\mu \frac{d}{dx^\mu} \mathcal{L} = \mathcal{L} + a^\nu \partial_\nu (\delta_\nu^\mu \mathcal{L}).$$
So the formula (1.31) gives

\[ T^{\mu}_{\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_{\nu} \phi)} \partial_{\nu} \phi \Delta_{\lambda}^{\mu} - \mathcal{L} \delta^{\mu}_{\nu}. \]

For the time translation, the conserved charge \( T^{0}_{0} \) gives back the hamiltonian density \( \mathfrak{h} = \pi \dot{\phi} - \mathcal{L} \) obtained by Legendre transformation. The conserved quantity from spatial translations is the momentum carried by the field, which for the KG field is

\[ \mathbf{P}_{i} = \int d^{d}x \ T^{0}_{i} = - \int d^{d}x \ \pi \partial_{i} \phi. \]

For the Maxwell field, this gives the Poynting vector.

There is some ambiguity in the definition of the stress tensor (associated with the possibility of adding total derivatives to \( \mathcal{L} \)).

Let’s check that the expression above for the conserved momentum agrees with our expectations. In particular, in free field theory the total momentum of the state \(|\mathbf{k}_{1}, \cdots, \mathbf{k}_{n}\rangle\) should be just the sum of the momenta of the particles, \( \mathbf{P} = \sum_{i=1}^{n} \hbar \mathbf{k}_{i} \) (with interactions the story can be more complicated). Indeed

\[ \mathbf{P}_{i} = - \int d^{d}x \ \pi \partial_{i} \phi = \int d^{d}k \mathbf{k}_{i} \mathbf{a}^{\dagger}_{k} \mathbf{a}_{k} \]

agrees with this. (Notice that I used rotation invariance of the vacuum to not worry about a possible constant term.)

- I have the impression that you learned all about the rest of the Poincaré group already in 215A.
2 From correlation functions to the $S$ matrix

We’ve derived an expression for correlation functions, such as (1.22), in terms of a sum of diagrams connected to the external lines, ordered by the number of powers of the coupling constant. Our next goal is to organize this sum.

First let’s make contact with the way the diagrammatic expansion was (I think) introduced in 215A. A time-ordered real-time Green’s function has path integral representation (with $i\epsilon$ prescription implicit, and $\phi_i \equiv \phi(x_i)$)

$$G^n \equiv \langle \Omega | T \phi_1 \cdots \phi_n | \Omega \rangle = Z^{-1} \int [D\phi] \phi_1 \cdots \phi_n e^{iS[\phi]}$$

$$= \frac{\int [D\phi] \phi_1 \cdots \phi_n e^{-i \int V(\phi)} e^{iS_0[\phi]}}{\int [D\phi] e^{iS_0[\phi]} e^{-i \int V(\phi)}}$$

$$= \frac{\langle 0 | T \phi_1 \cdots \phi_n e^{-i \int V(\phi)} | 0 \rangle}{\langle 0 | Te^{-i \int V(\phi)} | 0 \rangle}. \quad (2.3)$$

Here we’ve written $S = S_0 - \int V$ where $S_0$ is gaussian. The last object here is a time-ordered expectation value in the free theory, which we know how to compute by Wick contraction.

Two comments about this formula: (1) It must be admitted that in (2.1) the variable ‘$\phi$’ is seriously overloaded: on the LHS it is used to represent a (Heisenberg-picture) operator, while on the RHS it is used to represent a (functional) integration variable. (2) This formula (or a related one) is sometimes called the Dyson formula for interaction-picture time evolution.

Taylor expanding the exponential $e^{-i \int V}$ in (2.3) reproduces the diagrammatic expansion. (Notice that in real-time quantities, the interaction vertex comes with a factor of $-ig$.) The denominator is the sum of bubble diagrams. In both numerator and denominator, the disconnected diagrams exponentiate, and therefore cancel. Here is a reminder of why this is true:

The exponentiation of the disconnected diagrams. [Peskin page 96] There are some patterns in these sums of diagrams to which it behooves us to attend. (The
following discussion transcends the $\phi^4$ example.) The general diagram has the form:

Only some of the components are attached to the external legs; for a given diagram $A$, call the factor associated with these components $A_c$ (note that $A_c$ need not be fully connected). The rest of the diagram is made of a pile of ‘bubbles’ of various types $V_i$ (each one internally connected, but disconnected from the external lines) and multiplicities $n_i$ (e.g. $V_1$ could be a figure eight, and there could be $n_1 = 2$ of them. These bubbles (or ‘vacuum bubbles’) would be there even if we didn’t have any external lines, and they would have the same value; they are describing the fluctuations intrinsic to the vacuum. The amplitude associated with the general diagram is then

$$
\mathcal{M}_A = \mathcal{M}_{A_c} \cdot \frac{V_1^{n_1}}{n_1!} \cdot \frac{V_2^{n_2}}{n_2!} \cdots \frac{V_\alpha^{n_\alpha}}{n_\alpha!}
$$

where the $n_i!$ factors are the most important appearance of symmetry factors: they count the number of ways to permute the identical copies of $V_i$ amongst themselves.

The numerator of $G^{(n)}$ is then

$$
G^{(n)}_{\text{numerator}} = \langle 0 | \mathcal{T} (\phi_1 \cdots \phi_n e^{-i \int V}) | 0 \rangle = \sum_{A} \mathcal{M}_A = \sum_{A_c} \mathcal{M}_{A_c} \sum_{\{n_i=0\}} \frac{V_1^{n_1}}{n_1!} \cdot \frac{V_2^{n_2}}{n_2!} \cdots \frac{V_\alpha^{n_\alpha}}{n_\alpha!} = \sum_{A_c} \mathcal{M}_{A_c} \cdot e^{\sum_i V_i} = \sum_{A_c} \mathcal{M}_{A_c} e^{\sum_i V_i} \tag{2.4}
$$

– the bubbles always exponentiate to give the same factor of $e^{\sum_i V_i}$, independent of the external data in $G$. In particular, consider the case of $n = 0$, where there are no external lines and hence no $A_c$:

$$
G^{(0)}_{\text{numerator}} = \langle 0 | \mathcal{T} e^{-i \int V} | 0 \rangle = 1 \cdot e^{\sum_i V_i}
$$

But we care about this because it is the denominator of the actual Green’s function:

$$
G^{(n)} = \frac{\langle 0 | \mathcal{T} (\phi_1 \cdots \phi_n e^{-i \int V}) | 0 \rangle}{\langle 0 | \mathcal{T} e^{-i \int V} | 0 \rangle} = \frac{G^{(n)}_{\text{numerator}}}{G^{(0)}_{\text{numerator}}} = \sum_{A_c} \mathcal{M}_{A_c}. \tag{2.5}
$$
And with that we can forget all about the bubbles. So for example,

\[ G^{(2)} = \quad + \quad + \quad + \quad + \]

\[ G^{(4)} = \quad + \quad + \quad + \quad + \quad + \quad + \quad + \]

Notice that in this manipulation (2.5) we are adding terms of many orders in perturbation theory in the coupling \( g \). If we want an answer to a fixed order in \( g \), we can regard anything of higher order as zero, so for example, it makes perfect sense to write

\[ G^{(2)} = e V e + O(g) = e V e + O(g). \]

(I only drew one kind of bubble in the previous expression since that one was easy to type.)

**Momentum space Green’s functions from Feynman diagrams.** In translation-invariant problems, things are usually a little nicer in momentum space. In \( \phi^4 \) theory in \( d + 1 \) dimensions, let’s think about

\[ \tilde{G}^{(n)}(p_1 \cdots p_n) \equiv \prod_{i=1}^{n} \int d^{d+1}x_i e^{-ip_i x_i} G^{(n)}(x_1 \cdots x_n). \]

This an off-shell Green’s function, a function of general \( p \), not necessarily \( p^2 = m^2 \). It will, however, vanish unless \( \sum_i p_i^\mu = 0 \) by translation invariance. Consider a fully-connected contribution to it, at order \( g^N \). (We’ll get the others by multiplying these bits.)

In \( \phi^4 \) theory, we need to make a diagram by connecting \( n \) external position vertices \( x_i \) to \( N \) 4-valent vertices \( z_a \) using Feynman propagators \( \Delta_F(y_A-y_B) = \int d^{d+1}q e^{i(q y_A - q y_B)} \delta(q^2 - m^2 + i\epsilon) \)

where \( \{y_A\} = \{z_a, x_i\} \). All of the position dependence is in these exponentials.

Since each propagator has two ends, the number of lines (by the fully-connected assumption) is

\[ N_I = \frac{\# \text{ of ends of lines}}{2} = \frac{n + 4N}{2} = \frac{n}{2} + 2N. \]

This is the number of \( q \) integrals, a priori.

The integral over the external positions \( x_i \) (in the Fourier transform) gives \( \int d^{d+1}x_i e^{i p_i (q_i - p_i)} = \delta^{d+1}(q_i - p_i) \) and so we can label the external lines by \( p_i \) (and we lose \( n \) \( q \) integrals).

The integral over the position of each internal vertex is of the form \( \int d^{d+1}z e^{i (\Sigma v q_v)} \)

where \( q_v \) are the momenta associated to the lines coming into the vertex. So each
internal vertex decreases the number of $q$ integrals by 1. One combination of the momenta is fixed by overall momentum conservation so we have left

$$N_f - n - (N - 1) = N - \frac{n}{2} + 1 \equiv N_L$$

momentum integrals. This number is $\geq 0$ for fully connected diagrams, and it is the number of loops in the diagram. (This counting is the same as in a Kirchoff’s law resistor network problem.)*

For example, consider a particular contribution to $G^{(4)}$ ($n = 4$ external legs) and $N = 2$ interaction vertices

\[ \begin{array}{c}
\text{which has } N_f = \frac{4 + 2 \cdot 4}{2} = 6.
\end{array} \]

In the example, $N_L = 2 - 2 + 1 = 1$ which agrees with one undetermined momentum integral. This gives the amplitude

\[ \mathcal{M}_{FC}(p_1 \cdots p_n) = (-ig)^N \cdot s(FC) \phi^{(d+1)}(\sum p_i) \int \prod_{\text{loops}, \alpha = 1}^{N_L} d^{d+1}k_\alpha \prod_{\text{lines}, r} \frac{i}{q_r^2 - m^2 + i\epsilon} \]

\[ = \frac{(-ig)^2}{2!} \phi^{d+1} \left( \sum_{i=1}^{4} p_i \right) \prod_{i=1}^{n=4} \frac{i}{p_i^2 - m^2 - i\epsilon} \int d^{d+1}k \frac{i}{k^2 - m^2 + i\epsilon} \int d^{d+1}k \frac{i}{(p_1 + p_2 + k)^2 - m^2 + i\epsilon} \]

(You might notice that the integral over $k$ is in fact formally infinite, since at large $k$ it goes like $\int A \frac{d^dk}{k^2} \sim \log(A)$. Try to postpone that worry.) The propagators for the external lines just factor out, and can be brought outside the momentum integrals. Notice that here $p$ is general, and this function has poles when the external particles go on-shell, $p_i^2 = m^2$.

So here are the momentum space Feynman rules for Green’s function in $\phi^4$ theory:

- Every line gives a factor of $\frac{i}{p^2 - m^2 + i\epsilon} = \tilde{\Delta}_F(p)$. Notice that since $\Delta_F(x - y) = \Delta_F(y - x)$, the choice of how we orient the momenta is not so fateful.

- An internal vertex gives

\[ \begin{array}{c}
\text{momentum conservation at each vertex. So, set } \sum_i p_i = 0 \text{ at each vertex (I've assumed the arrows are all pointing toward the vertex). After imposing momentum}
\end{array} \]

\[11\text{ Here’s a proof that (2.6) is the number of loops in the diagram: place the } N + n \text{ internal and external vertices on the page. Add the propagators one at a time. You must add } N + n - 1 \text{ just to make the diagram fully connected. After that, each line you add makes a new loop.} \]
conservation, the remaining consequence of the vertex is

\[ \times = -i g. \]

- Integrate over the loop momenta \( \prod_{\alpha=1}^{N_L} d^{d+1} q_\alpha \) for each undetermined momentum variable. There is one for each loop in the diagram. You should think of these integrals as just like the Feynman path integral: if there is more than one way to get from here to there, we should sum over the amplitudes.

- Multiply by the wretched symmetry factor \( s(A) \).

- For \( \bar{G}(p) \), multiply by an overall \( \phi^{d+1} (\sum p) \) in each diagram.

- An external vertex at fixed position, \( s \rightarrow t = e^{-ipx} \). (Such vertices would arise if we wanted to compute \( G(x) \) using momentun-space feynman rules.) More generally, external vertices are associated with the wavefunctions of the states we are inserting; here they are plane waves.

Here is another perspective on the exponentiation of the vacuum bubbles. Consider the diagram:

\[
\begin{align*}
\begin{tikzpicture}
\filldraw[black] (0,0) circle (2pt);
\draw[thick,->] (-1,0) -- (1,0);
\end{tikzpicture}
\end{align*}
\]  

\[ = (-ig)^2 \prod_{i=1}^{4} \int d^{d+1} p_i \phi^{d+1} (p_1 + p_2) \phi^{d+1} (p_1 + p_2) \ldots \]

The two delta functions come from the integrals over \( z_{1,2} \), and we can restore sense by remembering this:

\[
\left( \phi^{d+1} (p_1 + p_2) \right)^2 = \phi^{d+1} (p_1 + p_2) \int d^{d+1} z_2 = \phi^{d+1} (p_1 + p_2) VT
\]

where \( VT \) is the volume of spacetime. This factor arises because this process can happen anywhere, anytime. There is one such factor for each connected component of a collection of vacuum bubbles, so for example the diagram \( \left( \begin{smallmatrix} \phi & \phi \end{smallmatrix} \right) \) is proportional to \( (VT)^2 \). But the free energy \( \propto \log Z = \log G^{(0)} \) should be extensive, \( \propto VT \). Therefore, the vacuum bubbles must exponentiate.

The whole two point function in momentum space is then (through order \( g^2 \)):

\[
\bar{G}^{(2)} = \frac{\lambda^4}{N_L + 1} + \frac{\lambda^2}{N_L + 2} + \frac{\lambda}{N_L + 1} + \frac{4}{N_L + 1} + \mathcal{O}(g^3)
\]

(2.7)

39
I draw the blue dots to emphasize the external propagators. Notice that for the two-point function, the number of loops is \( N_L = N - \frac{n}{2} + 1 = N \), the same as the number of powers of \( g \). More generally, for \( n \neq 2 \), there is an additive shift: \( N_L = \text{constant plus number of powers of } g \).

**Organizing the propagator.** We would like to unpack the physics contained in the correlation functions which we’ve learned to compute in perturbation theory. The first interesting one is the two-point function aka the propagator. Let’s factor out the overall delta function by writing:

\[
\bar{G}^{(2)}(p_1, p_2) \equiv \phi^{d+1}(p_1 + p_2) \bar{G}^{(2)}(p_1).
\]

It will be useful to re-organize this sum, in the following way:

Here’s the pattern: we define a diagram to be *one-particle irreducible* (1PI) if it cannot be disconnected by cutting through a single internal propagator. So for example, \( \begin{array}{c}
\end{array} \) is 1PI, but \( \begin{array}{c}
\end{array} \) is not; rather, the latter contributes to the bit with two 1PI insertions. Then

\[
\bar{G}^{(2)}(p) = \begin{array}{c}
\end{array} \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \ldots
\]

So that we may write equations without pictures, let

\[
-i \Sigma(p) \equiv \begin{array}{c}
\end{array}
\]

denote the 1PI two-point function. \( \Sigma \) being 1PI means that the external lines sticking out of it are ‘nubbins,’ placeholders where propagators may be attached. That’s why there are no blue dots at the ends.
Now suppose we know \(\Sigma\). It is known as the self-energy, for reasons we will see next. Then we can write

\[
\tilde{G}^{(2)}(p) = \frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2} (-i\Sigma(p)) \frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2} (-i\Sigma(p)) \frac{i}{p^2 - m_0^2} (-i\Sigma(p)) \frac{i}{p^2 - m_0^2} + \cdots
\]

\[
= \frac{i}{p^2 - m_0^2} \left(1 + \frac{\Sigma}{p^2 - m_0^2} + \left(\frac{\Sigma}{p^2 - m_0^2}\right)^2 + \cdots\right)
\]

\[
= \frac{i}{p^2 - m_0^2} \frac{1}{1 - \frac{\Sigma}{p^2 - m_0^2}} = \frac{i}{p^2 - m_0^2 - \Sigma(p)}.
\]

(2.8)

[End of Lecture 5]

We see that the self-energy shifts the \(m^2\) of the particle – it moves the location of the pole in the propagator. In the interacting theory, \(m_0^2 + \Sigma(p)|_{\text{pole}}\) is the physical mass, while \(m_0\) (what we’ve been calling \(m\) until just now) is deprecatingly called the ‘bare mass’. For \(p^2 \sim m^2\), we will write

\[
\tilde{G}^{(2)}(p) \equiv \left(\frac{iZ}{p^2 - m^2} + \text{regular bits}\right)
\]

(2.9)

This equation defines the residue \(Z\) which is called the ‘wavefunction renormalization factor’. It is 1 in the free theory, and represents the amplitude for the field to create a particle, and the other terms, which are not singular at \(p^2 = m^2\), represent the amplitude for the field to do something else (such as create multiparticle states), and are absent in the free theory. Later we will see that unitarity requires \(Z \leq 1\). Notice that if we know \(\Sigma\) only to some order in perturbation theory, then (2.8) is still true, up to corrections at higher order.

The notion of 1PI extends to diagrams for \(\tilde{G}^{(n>2)}(p_1 \cdots p_n)\). Let

\[
\tilde{G}^{(n)}_{1PI}(p_1 \cdots p_n) \equiv \text{1PI}
\]

where the blob indicates the sum over all 1PI diagrams with \(n\) external nubbins (notice that these do not have the blue circles that were present before). This means \(\tilde{G}_{1PI}\) does not include diagrams like:

\[
\text{or } \quad \text{ or } \quad \text{ or } \quad \text{ or }.
\]
Notice that 1PI diagrams are *amputated* – their external limbs have been cut off.

**LSZ reduction formula.** This is almost what we need to make $S$-matrix elements. If we multiply the $n$-point function by $\prod_{i=1}^{n} p_i^2 - m^2$, we cancel out the propagators from the external legs, near the mass shell. This object is naturally called the *amputated* $n$-point function. (It differs from the 1PI $n$-point Green’s function because of diagrams like this one which is amputated but not 1PI.) If we then take $p_i^2 \rightarrow m^2$, we keep only the part of $\tilde{G}$ which is singular on the mass-shell. And here’s why we care about that:

\[
\text{Claim (the LSZ reduction formula):}
\]

\[
S_{fi} \equiv \langle \vec{p}_1 \cdots \vec{p}_n | S | \vec{k}_1 \cdots \vec{k}_m \rangle = \prod_{a=1}^{n+m} \left( \lim_{P_a \to E_{P_a}} \frac{P_a^2 - m^2}{i\sqrt{Z}} \right) \tilde{G}^{(n+m)}(k_1 \cdots k_m, -p_1 \cdots -p_n)
\tag{2.10}
\]

where $P_a \in \{p_i, k_i\}$. In words: the $S$-matrix elements are obtained from Green’s functions by amputating the external legs, and putting the momenta on-shell. Notice that choosing all the final momenta $p_i$ different from all the initial momenta $k_i$ goes a long way towards eliminating diagrams which are not fully connected.

This formula provides the bridge from time-ordered Green’s functions (which we know how to compute in perturbation theory now) and the $S$-matrix, which collects probability amplitudes for things to happen to particles, in terms of which we may compute cross sections and lifetimes. Let us spend just another moment inspecting the construction of this fine conveyance.

**Why is LSZ true?** Here’s the argument I’ve found which best combines concision and truthiness. [It is mainly from the nice book by Maggiore §5.2; I also like Schwartz’ chapter 6; Peskin’s argument is in section 4.6.] The argument has several steps. The field operators in this discussion are all in Heisenberg picture.

1. First, for a free field, the mode expansion implies that we can extract the ladder operators by:

\[
\sqrt{2\omega_k}a_k = i \int d^dx \ e^{ikx} \left( -i\omega_k + \partial_0 \right) \phi_{\text{free}}(x)
\]
\[ \sqrt{2\omega_k} a_k^\dagger = -i \int d^d x \ e^{-ikx} ( + i \omega_k + \partial_0 ) \phi_{\text{free}}(x) \]  
(2.11)

Notice that the LHS is independent of time, but the integrand of the RHS is not.

2. Now let’s pretend that we can turn the interactions off at \( t = \pm \infty \), so that the asymptotic states we are scattering are free particles.\(^{12}\) This allows us to write the field in terms of some pretend free fields of mass \( m \) (not \( m_0 \))

\[ \phi(x) \left\{ \begin{array}{l} t \rightarrow -\infty \quad Z^\frac{1}{2} \phi_{\text{in}}(x) \\ t \rightarrow +\infty \quad Z^\frac{1}{2} \phi_{\text{out}}(x) \end{array} \right. \]

The factors of \( \sqrt{Z} \) are required to get the correct two point functions (2.9) near the mass shell. The mode operators for \( \phi_{\text{in}} \) are called \( a_{\text{in}}^\dagger \), etc. \( \phi_{\text{in, out}} \) are free fields: their full hamiltonian is \( H_0 \). They are in Heisenberg picture, and the reference time for \( \phi_{\text{in, out}} \) is \( \pm \infty \) respectively. Since they are free fields, we can use (2.11) to write

\[ \sqrt{2\omega_k} a_{\text{in}}^\dagger = -i \int d^d x \ e^{-ikx} ( + i \omega_k + \partial_0 ) \phi_{\text{in}}(x) = -i Z^{-1/2} \int d^d x \ e^{-ikx} ( + i \omega_k + \partial_0 ) \phi(x) | t \rightarrow -\infty \]

where in the second step we used the independence on time in (2.11), even though \( \phi(x) \) is not a free field. An expression for \( a_{\text{out}}^\dagger \) obtains if we take \( t \rightarrow +\infty \) instead.

3. Now make this expression manifestly covariant using the fundamental theorem of calculus:

\[ \sqrt{2\omega_k} (a_{\text{in}}^\dagger - a_{\text{out}}^\dagger) = i Z^{-1/2} \int_{-\infty}^{\infty} dt \partial_t \left( \int d^d x \ e^{-ikx} ( i \omega_k + \partial_0 ) \phi(x) \right) \]

\( \text{IBP in time} \quad i Z^{-1/2} \int d^{d+1} x \left( e^{-ikx} \partial_0^2 \phi - \phi \cdot \partial_0^2 e^{-ikx} \right) \)

\( \text{IBP in space} \quad i Z^{-1/2} \int d^{d+1} x e^{-ikx} ( \Box + m^2 ) \phi(x) \)  
(2.12)

In the last step we made a promise to only use wavepackets for external states, so that we can do IBP in space.

4. Now, here’s where the \( S \)-matrix enters. Assume none of the incoming momenta \( k_i \) is the same as any outgoing momentum \( p_j \).

\[ \langle p_1 \cdots p_n | S | k_1 \cdots k_m \rangle \]

\(^{12}\)Here’s why this is really bad: nearly everything we might scatter is a boundstate. For example: atoms, nuclei, nucleons etc... But if there are no interactions there are no boundstates.
\[
\begin{align*}
  &= \prod_{p,k} \sqrt{2\omega} \langle \Omega | \prod_p a_p^{\text{out}} S \prod_k a_k^{\text{in}\dagger} | \Omega \rangle \\
  &= \prod_{p,k} \sqrt{2\omega} \langle \Omega | \mathcal{T} \left( \prod_p a_p^{\text{out}} S \prod_k a_k^{\text{in}\dagger} \right) | \Omega \rangle \\
  &= \prod_{p,k} \sqrt{2\omega} \langle \Omega | \mathcal{T} \left( \prod_p a_p^{\text{out}} S \left( a_{k_1}^{\text{in}\dagger} - a_{k_1}^{\text{out}\dagger} \right) \prod_m \sqrt{\omega_k} a_k^{\text{in}\dagger} \right) | \Omega \rangle \\
  \end{align*}
\]

\[(2.12)\quad iZ^{-1/2} \int d^{d+1} x_1 e^{-ik_1 x_1} \langle \Omega | \mathcal{T} \left( \prod_p \sqrt{\omega_p} a_p^{\text{out}} S \left( \Box + m^2 \right) \phi(x_1) \prod_m \sqrt{\omega_k} a_k^{\text{in}\dagger} \right) | \Omega \rangle = iZ^{-1/2} \int d^{d+1} x_1 e^{-ik_1 x_1} \left( \Box + m^2 \right) \langle \Omega | \mathcal{T} \phi(x_1) \prod_m \sqrt{\omega_k} a_k^{\text{in}\dagger} | \Omega \rangle + X
\]

In the last step, \(X\) comes from where the \(\Box x_1\) hits the time ordering symbol. This gives terms which will not matter when we take \(k^2 \to m^2\), I promise.

5. Now do this for every particle to get

\[
\langle p_1 \cdots p_n | S | k_1 \cdots k_m \rangle = \prod_{j=1}^m \int d^{d+1} y_j \ e^{-i p_j y_j} i Z^{-1/2} \left( \Box_j + m^2 \right) \langle \Omega | \mathcal{T} \phi(x_1) \cdots \phi(y_j) S | \Omega \rangle + X
\]

The \(x\) and \(y\) integrals are just Fourier transforms, and this says that near the mass shell,

\[
\tilde{G}^{(n+m)}(k_1 \cdots k_m, -p_1 \cdots -p_n) = \prod_{a} \frac{i \sqrt{Z}}{P_a^2 - m^2} \langle p_1 \cdots p_n | S | k_1 \cdots k_m \rangle + \text{regular}
\]

(where \(P_a \in \{p_j, k_i\}\)) which is the same as (2.10).

---

Comment: In our discussion of QFT, a special role has been played by fields called \(\phi\). Suppose we have some other (say hermitian) local operator \(\mathcal{O}\) such that

\[
\langle p | \mathcal{O}(x) | \Omega \rangle = Z_{\mathcal{O}} e^{i p x}
\]

where \(\langle p \rangle\) is a one-particle state made by our friend \(\phi\) (we could put some labels, \(e.g.\) for spin or polarization or flavor, on both the operator and the state, but let’s not). Such an \(\mathcal{O}\) is called an ‘interpolating field’ or ‘interpolating operator’. And suppose we have information about the correlation functions of \(\mathcal{O}\):

\[
G^{(n)}_{\mathcal{O}}(1 \cdots n) \equiv \langle \Omega | \mathcal{T}\left( \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \right) | \Omega \rangle.
\]

In this case, there is a more general statement of LSZ:

\[
\prod_{a \in i} \left( Z_a^{-1/2} i \int d^{d+1} x_a e^{-i p_a x_a} (\Box_a + m_a^2) \right)
\]
\[ \prod_{b \in f} \left( Z_b^{-1/2} i \int d^{d+1} x_b e^{ip_b x_b} (\Box_b + m_b^2) \right) G^{(n)}_O (1 \cdots n) = \langle \{ p_f \} | S | \{ p_a \} \rangle \quad (2.13) \]

This more general statement follows as above if we can write \( O_a \to \infty \sqrt{Z_a} \phi_{\text{in}} \). This more general formula allows us to scatter particles that are not ‘elementary’ in the sense that they are made by the fields in terms of which we write our Lagrangian.

Here is a summary of the long logical route connecting Feynman diagrams to measurable quantities in particle physics:

The final step was covered in 215A.
**S-matrix from Feynman diagrams.** The end result of the previous discussion is a prescription to compute S-matrix elements from Feynman diagrams. In a translation-invariant system, the S matrix always has a delta function outside of it. Also we are not so interested in the diagonal elements of the S matrix where nothing happens. So more useful than the S matrix itself are the scattering amplitudes M defined by

\[
\langle f | (S - \mathbb{1}) | i \rangle \equiv (2\pi)^{d+1}\delta^{(d+1)} \left( \sum_f p_f - \sum_i p_i \right) iM_{fi}.
\]  

(2.14)

(The object \(iM\delta^{d+1}(\sum p)\) is sometimes called the transfer matrix. The \(i\) is a convention.)

The rules for the Feynman diagram calculation of \(\mathcal{M}\) (for \(\phi^4\) theory, as a representative example) are:

1. Draw all amputated diagrams with appropriate external nubbins for the initial and final states. For a diagram with \(N_L\) loops think of \(N_L\) letters that are like \(k\) or \(q\) or \(p\) to call the undetermined loop momenta.

2. For each vertex, impose momentum conservation and multiply by the coupling \((-i\lambda)\).

3. For each internal line, put a propagator.

4. For each loop, integrate over the associated momentum \(\int d^{d+1}k\).

A comment about rule 1: For tree-level diagrams (diagrams with no loops), ‘amputate’ just means leave off the propagators for the external lines. More generally, it means leave off the resummed propagator (2.8). In particular, a diagram like \(\times\) is already included by using the correct \(Z\) and the correct \(m\).

---

I skipped the example in lecture. Please read through it and make sure you are happy about it. Please ask questions if you are not. We will do some examples in QED soon.

**Example: snucleon scattering.** [Here we follow Tong §3.5 very closely] Let’s consider an example with a complex scalar field \(\Phi\) interacting with a real scalar field \(\phi\) with Lagrangian

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \Phi^{\ast} \partial^{\mu} \Phi - \frac{1}{2} m^2 \Phi^{\ast} \Phi + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} M^2 \phi^2 + \mathcal{L}_I
\]

(2.15)

with \(\mathcal{L}_I = -g\Phi^{\ast} \Phi \phi\).
In specifying initial states below, I will need names for the mode operators of the two fields:

$$\phi = \int \frac{d^dp}{\sqrt{2\omega_p}} (a_p e^{-ipx} + a^+_p e^{ipx}) |p^\rho = \omega_p\rangle$$

$$\Phi = \int \frac{d^dp}{\sqrt{2E_p}} (b_p e^{-ipx} + c^+_p e^{ipx}) |p^\rho = E_p\rangle$$

where I’ve written $$\omega_p \equiv \sqrt{M^2 + p^2}, E_q \equiv \sqrt{m^2 + q^2}$$. Notice that the $$\Phi \to e^{-i\alpha}\Phi$$ symmetry is conserved; the charge is

$$q = N_c - N_b.$$ 

But the $$\phi$$ particles are not conserved.\(^{13}\) Relative to $$\phi^4$$ theory, the differences in the Feynman rules are: we have two kinds of propagators, one of which is oriented (to keep track of the flow of the conserved $$\Phi$$ number through the diagrams), and instead of a 4-point vertex which costs $$-i\,g$$, we have a 3-point vertex for $$\phi\Phi^*\Phi$$ which costs $$-i\,g$$.

Let’s consider 2 → 2 scattering of $$\Phi$$ particles, so

$$|i\rangle = |\vec{p}_1, \vec{p}_2\rangle, |f\rangle = |\vec{p}_3, \vec{p}_4\rangle \quad \text{with} \quad |\vec{p}_i, \vec{p}_j\rangle \equiv \sqrt{2E_{\vec{p}_i}} \sqrt{2E_{\vec{p}_j}} b^\dagger_{\vec{p}_i} b^\dagger_{\vec{p}_j} |0\rangle.$$ 

(To appreciate some of the beauty of the diagram technique, see Tong §3.3.3 for the artisanal version of this calculation.) The Feynman rules above give, to leading nonzero order,

$$iM = \left(-i\,g\right)^2 \left( \frac{i}{(p_1 - p_3)^2 - M^2 + i\epsilon} + \frac{i}{(p_1 - p_4)^2 - M^2 + i\epsilon} \right). \quad (2.16)$$

The diagrams depict two ‘snucleons’ $$\Phi$$ (solid lines with arrows indicating snucleons versus antinucleons) exchanging a meson $$\phi$$ (double gray line, with no arrow) with momentum $$k \equiv p_1 - p_3 = p_4 - p_2$$ (first term) or $$k \equiv p_4 - p_1 = p_2 - p_3$$ (second term). Time goes to the left as always. Notice that here I am being careful about using arrows on the lines to indicate flow of particle number through the diagram, while the extra (light blue) arrows indicate momentum flow.

\(^{13}\)You might notice a possible problem with this theory: what happens to the quadratic term for $$\Phi$$ when $$\phi$$ is very negative? Let’s not take it too seriously.
The meson in these diagrams is virtual, or off-shell, in the sense that it does not satisfy its equation of motion $k^2 \neq M^2$. In fact, each of these diagrams is actually the sum of retarded and advanced exchange of real on-shell particles. (For more on this statement, see Schwartz chapter 4 or §4.5 of the notes here.) The two diagrams included in (2.16) make the amplitude symmetric under interchanging the two particles in the initial or final state, as it must be because they are indistinguishable bosons.

Two more examples with the same ingredients are useful for comparison. If we instead scatter a snucleon and an anti-snucleon, so $|i\rangle = \sqrt{2E_{\vec{p}_1}} \sqrt{2E_{\vec{p}_2}} b_{\vec{p}_1}^\dagger c_{\vec{p}_2}^\dagger |0\rangle$, then the leading diagrams are

$$iM = \left( (-ig)^2 \left( \frac{i}{(p_1 + p_2)^2 - M^2 + i\epsilon} + \frac{i}{(p_1 - p_3)^2 - M^2 + i\epsilon} \right) \right).$$

This one has a new ingredient: in the first diagram, the meson momentum is $k = p_1 + p_2$, which can be on-shell, and the $i\epsilon$ matters. This will produce a big bump, a resonance, in the answer as a function of the incoming center-of-mass energy $\sqrt{s} \equiv \sqrt{(p_1 + p_2)^2}$.

Finally, we can scatter a meson and a snucleon:

$$iM = \left( (-ig)^2 \left( \frac{i}{(p + k)^2 - m^2 + i\epsilon} + \frac{i}{(p - k')^2 - m^2 + i\epsilon} \right) \right).$$

Now the intermediate state is a snucleon.

There is a common notation for the Lorentz-invariant combinations of the momenta appearing in these various processes, called Mandelstam variables, of which $s$ is one. A concise summary appears in §3.5.1 of Tong’s notes.
3 QED

3.1 Quantum light: Photons

I skipped this subsection in lecture. Please read through it and make sure you are happy about it. Please ask questions if you are not.

The quantization of the Maxwell field is logically very similar to the case of a harmonic chain. There are just a few complications from its several polarizations, and from the fact that quantum mechanics means that the vector potential is real and necessary (whereas classically it is just a convenience). This is a quick-and-dirty version of the story. I mention it here to emphasize that the machinery we are developing applies to a system you have already thought a lot about!

Maxwell’s equations (with \( c = 1 \)) are:

\[
\begin{align*}
\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} &= 0, & \nabla \cdot \vec{B} &= 0, & \vec{\nabla} \times \vec{E} &= -\partial_t \vec{B}, \\
\partial_\mu F_{\mu\nu} &= 4\pi j_\nu, & \nabla \cdot \vec{E} &= 4\pi \rho, & \vec{\nabla} \times \vec{B} &= \partial_t \vec{E} + 4\pi \vec{j},
\end{align*}
\]

(3.1)

(3.2)

(where the familiar electric and magnetic fields are \( E^i = -F^{0i} \) and \( \epsilon^{ijk} B^k = -F^{ij} \)). The first two equations (3.1) are constraints on \( \vec{E} \) and \( \vec{B} \) which mean that their components are not independent. This is annoying for trying to treat them quantumly. To get around this we introduce potentials \( A_\mu = (\Phi, \vec{A}) \) which determine the fields by taking derivatives and which automatically solve the constraints (3.1):

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
\]

aka

\[
\vec{E} = -\vec{\nabla} \Phi - \partial_t \vec{A}, \quad \vec{B} = \vec{\nabla} \times \vec{A}.
\]

Potentials related by a gauge transformation

\[
\vec{A} \rightarrow \vec{A}^\lambda = \vec{A} - \vec{\nabla} \lambda, \quad \Phi \rightarrow \Phi^\lambda = \Phi + \partial_\lambda
\]

for any function \( \lambda(\vec{r},t) \), give the same \( \vec{E}, \vec{B} \). The Bohm-Aharonov effect is proof that (some of the information in) the potential is real and useful, despite this redundancy. We can partially remove this redundancy by choosing our potentials to satisfy Coulomb gauge

\[
\nabla \cdot \vec{A} = 0.
\]

In the absence of sources \( \rho = 0 = \vec{j} \), we can also set \( \Phi = 0 \). In this gauge, Ampere’s law becomes

\[
c^2 \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = c^2 \vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{A}) - c^2 \nabla^2 \vec{A} = -\partial_t^2 \vec{A} \quad \text{i.e.} \quad \Box \vec{A} = 0.
\]

This wave equation is different from the scalar wave equation \( \Box \phi = 0 \) in three ways:
• we’re in three spatial dimensions,
• the speed of sound $v_s$ has been replaced by the speed of light $c$,
• the field $\vec A$ is a vector field obeying the constraint $\vec \nabla \cdot \vec A = 0$. In fourier space $\vec A(x) = \sum_k e^{i\vec k \cdot \vec x} \vec A(k)$ this condition is
  \[ 0 = \vec k \cdot \vec A(k) \]
  – the vector field is transverse.

An action which gives rise to Maxwell’s equations is
\[ S[A] = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) = \int d^4x \mathcal{L}_{\text{Maxwell}}. \]
\[ \mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (E^2 - B^2). \]

Note that we must regard $A$ as the dynamical variable to obtain (3.2) by $0 = \frac{\delta S}{\delta A(x)}$. The canonical momentum of $A$ is then $\Pi_A = \frac{\partial \mathcal{L}_{\text{Maxwell}}}{\partial \dot{A}_i} = E^i$. So the Hamiltonian is\(^{14}\):
\[ H = \frac{1}{2} \int d^3x \left( \vec E^2 + c^2 \vec B^2 \right). \tag{3.3} \]

Here $\vec E = -\partial_t \vec A$ plays the role of field momentum $\pi(x)$ in (1.3), and $\vec B = \vec \nabla \times \vec A$ plays the role of the spatial derivative $\partial_x q$. We immediately see that we can quantize this system just like for the scalar case, with the canonical commutator
\[ [\phi(x), \pi(x')] = i\hbar \delta(x - x') \quad \leadsto \quad [A_i(\vec r), E_j(\vec r')] = -i\hbar \delta^3(\vec r - \vec r')\delta_{ij} \]
where $i, j = 1..3$ are spatial indices\(^\text{15}\). So we can immediately write down an expression for the quantum Maxwell field in terms of independent creation and annihilation operators:
\[ \vec A(\vec r) = \int d^3k \frac{1}{\sqrt{2\omega_k}} \sum_{s=1,2} \left( a_{k,s} \vec e_s(\vec k) e^{i\vec k \cdot \vec r} + a_{k,s}^\dagger \vec e_s^*(\vec k) e^{-i\vec k \cdot \vec r} \right) \]

\(^{14}\)You may also recall that the energy density of a configuration of Maxwell fields is $u = \frac{1}{2} \left( \vec E^2 + \vec B^2 \right)$. This result can be obtained either by Legendre transformation of $\mathcal{L}_{\text{Maxwell}}$, or from $T^{0\,0}$, the energy momentum tensor.

\(^\text{15}\)As a check, note that using this Hamiltonian and the canonical commutator, we can reproduce Maxwell’s equations using Ehrenfest’s theorem:
\[ \langle \partial_t^2 \vec A \rangle = \partial_t \langle \vec E \rangle = -\frac{i}{\hbar} \langle [H, \vec E] \rangle = \langle c^2 \vec \nabla^2 \vec A \rangle. \]
The field momentum is \( \vec{E} = -\partial_t \vec{A} \):

\[
\vec{E}(\vec{r}) = i \frac{\omega_k}{2} \sum_{s=1,2} \left( a_{\vec{k},s} \vec{e}_s(\hat{k}) e^{i \vec{k} \cdot \vec{r}} - a^\dagger_{\vec{k},s} \vec{e}^*_{s}(\hat{k}) e^{-i \vec{k} \cdot \vec{r}} \right)
\]

Also, the magnetic field operator is

\[
\vec{B} = \vec{\nabla} \times \vec{A} = \sum_{\vec{k}} \sum_s \sqrt{\frac{\hbar}{2 \epsilon_0 \omega_k L^3}} \vec{k} \times \left( a_{\vec{k},s} \vec{e}_s(\hat{k}) e^{i \vec{k} \cdot \vec{r}} - a^\dagger_{\vec{k},s} \vec{e}^*_{s}(\hat{k}) e^{-i \vec{k} \cdot \vec{r}} \right);
\]

the magnetic field is analogous to \( \vec{\nabla} \phi \) in the scalar field theory\(^{16}\). Plugging these expressions into the Hamiltonian (3.3), we can write it in terms of these oscillator modes (which create and annihilate photons). As for the scalar field, the definitions of these modes were designed to make this simple: It is:

\[
H = \sum_{\vec{k},s} \hbar \omega_k \left( a^\dagger_{\vec{k},s} a_{\vec{k},s} + \frac{1}{2} \right).
\]

Notice that in this case we began our story in the continuum, rather than with microscopic particles connected by springs. (However, if you read Maxwell’s papers you’ll see that he had in mind a particular UV completion involving gears and cogs. I actually don’t understand it; if you do please explain it to me.)

The vacuum energy is

\[
E_0 = \frac{1}{2} \sum_{\vec{k},s} \hbar \omega_k = \frac{L^3}{(2\pi)^3} \int d^3 k \hbar c k.
\]

The fact that \( \sum_k \) is no longer a finite sum might be something to worry about. This vacuum energy has physical consequences, since it can depend on boundary conditions placed on the field by conducting objects, as we’ll discuss in §4.1.

### 3.2 More on vector fields

A few things we did not do yet for vector fields: study the propagator, figure out the data on external states, and understand the relation of between the masslessness of the photon and gauge invariance.

\(^{16}\)I should say a little more about the polarization vectors, \( \vec{e}_s \). They conspire to make it so that there are only two independent states for each \( \vec{k} \) and they are transverse \( \vec{k} \cdot \vec{e}_s(\hat{k}) = 0 \), so \( s = 1, 2 \). The polarization vectors of a given \( \vec{k} \) can be chosen to satisfy the following completeness relation:

\[
\sum_s e_{si}(\hat{k}) e^*_{sj}(\hat{k}) = \delta_{ij} - \hat{k}_i \hat{k}_j.
\]

This says that they span the plane perpendicular to \( \hat{k} \).
Consider the following Lagrangian for a vector field \( A_\mu \) (which I claim is the most general quadratic Poincaré-invariant Lagrangian with at most two derivatives):

\[
\mathcal{L} = -\frac{1}{2} \left( \partial_\mu A_\nu \partial^\mu A_\nu + a \partial_\mu A_\nu \partial_\nu A^\mu + b A_\mu A^\mu + c \epsilon^{\mu \nu \rho \sigma} \partial_\mu A_\nu \partial_\rho A_\sigma \right). 
\]

The sign is chosen so that spatial derivatives are suppressed, and the normalization of the first term is fixed by rescaling \( A \). (Another possible-seeming term, \( \partial_\mu A_\nu \partial_\nu A^\mu \), is related to the second term by two IBPs.) The last term is a total derivative, \( \epsilon^{\mu \nu \rho \sigma} \partial_\mu A_\nu \partial_\rho A_\sigma \), and will not affect the EoM or anything at all in perturbation theory; it is called a \( \theta \) term.

The EoM are

\[
0 = \frac{\delta}{\delta A_\nu(x)} \int \mathcal{L} = -\partial^2 A_\nu - a \partial_\nu (\partial \cdot A) + b A_\nu 
\]

which (like any translation-invariant linear equation) is solved by Fourier transforms \( A_\mu(x) = \epsilon_\mu e^{-ikx} \), if

\[
k^2 \epsilon_\mu + ak_\mu (k \cdot \epsilon) + b \epsilon_\mu = 0.
\]

There are two kinds of solutions: longitudinal ones with \( \epsilon_\mu \propto k_\mu \) (for which the dispersion relation is \( k^2 = -\frac{b}{1+a} \)), and transverse solutions \( \epsilon \cdot k = 0 \) with dispersion \( k^2 = -b \).

The longitudinal mode may be removed by taking \( b \neq 0 \) and \( a \to -1 \), which we will do from now on. This gives the Proca Lagrangian:

\[
\mathcal{L}_{a=-1,b=-\mu^2} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} \mu^2 A_\mu A^\mu,
\]

where as usual \( F_{\mu \nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \). Note that the EOM (Proca equation) \( 0 = \partial F_\nu + \mu^2 A_\nu \) implies \( 0 = \partial^\nu A_\nu \) by \( 0 = \partial^\mu \partial_\nu F_{\mu \nu} \). So each component of \( A_\mu \) satisfies (by (3.5)) the KG equation, \( k^2 = \mu^2 \), and the transverse condition \( \epsilon \cdot k = 0 \). In the rest frame, \( k^\mu = (k^0, \vec{0})^\mu \), we can choose a basis of plane-wave transverse solutions which are eigenstates of the vector rotation generator

\[
J^z = i \begin{pmatrix} -1 & +1 \\ 1 & -1 \end{pmatrix}, \text{ namely, } \epsilon^{(\pm)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mp i \end{pmatrix}, \epsilon^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]

They are normalized so that \( \epsilon^{(r)} \cdot \epsilon^{(s)} = +\delta^{rs} \) and \( \sum_{r=\pm 1,0} \epsilon^{(r)} \epsilon^{(r)*} = -\eta_{\mu \nu} + \frac{k_\mu k_\nu}{\mu^2} \) so that they project out \( \epsilon \propto k \). Notice that in the massless case, only two of these three polarization states will be transverse to \( k^\mu \). If \( \vec{k} \propto \hat{z} \) (for example in the massless case with \( k^\mu = (E, 0, 0, E)^\mu \)) then these \( \epsilon \) are also all helicity eigenstates: \( h = \vec{J} \cdot \vec{k} = J^z \).
Canonical stuff: The canonical momenta are \( \pi^i = \frac{\partial L}{\partial \dot{A}^i} = -F^{0i} = E^i \) (as for electrodynamics in §3.1) and \( \pi^0 = \frac{\partial L}{\partial \dot{A}^0} = 0 \). This last bit is a little awkward, but it just means we can solve the equations of motion for \( A_0 \) algebraically in terms of the other (real) dofs:

\[
0 = \frac{\delta S}{\delta A_0} = \nabla \cdot A_0 - \mu^2 A_0 = (- \nabla^2 + \mu^2) A_0 + \nabla \cdot \dot{A} \implies A_0(\vec{x}) = \int d^3 y e^{-\mu |\vec{x} - \vec{y}|} \frac{(\nabla \cdot \dot{A})}{4\pi |\vec{x} - \vec{y}|} .
\]

(3.6)

So at each moment \( A_0 \) is determined by \( A_i \). (Notice that this is still true for \( \mu \to 0 \).) The hamiltonian density is (after using \( \pi^i = F^{0i} \), integration by parts, and the equations of motion for \( A_0 \))

\[
b = +\frac{1}{2} \left( \frac{F^2_{0i}}{F_{0i}} + \frac{1}{2} F_{ij}^2 + \mu^2 A_i^2 + \mu^2 A_0^2 \right) = \frac{1}{2} \left( \vec{E}^2 + \vec{B}^2 + \mu^2 \vec{A}^2 + \mu^2 A_0^2 \right) \geq 0,
\]

where positivity follows from the fact that it is a sum of squares of real things.

The canonical equal time commutators are then

\[
[A_i(t, \vec{x}), F^0_{j}(t, \vec{y})] = i\delta^j_i \delta^{(3)}(\vec{x} - \vec{y})
\]

which if we add up the plane wave solutions as

\[
A_{\mu}(x) = \sum_{r=1,2,3} \int \frac{d^3 k}{\sqrt{2\omega_k}} \left( e^{-ikx} a_{\mu}^{(r)} + e^{ikx} a_{\mu}^{(r)*} \right)
\]

give the bosonic ladder algebra for each mode

\[
[a_{\mu}^{(r)}, a_{\mu}^{(s)*}] = \delta^{(3)}(\vec{k} - \vec{p}) \delta^{rs}.
\]

The normal-ordered hamiltonian is

\[
: H := \sum_{r} \int d^3 k \omega_k a_{\mu}^{(r)} a_{\mu}^{(r)*}.
\]

Using the mode expansion above, the propagator for the \( A_{\mu}(x) \) field is found to be

\[
\langle 0 | T A_{\mu}(x) A_{\nu}(y) | 0 \rangle = \int d^4 k e^{-ik(x-y)} \left[ -\frac{i(\eta_{\mu\nu} - k_\mu k_\nu / \mu^2)}{k^2 - \mu^2 + i\epsilon} \right].
\]

(3.7)

Notice that like in the spinor case the polarization sum \( \sum_r \epsilon^{(r)*}_\mu \epsilon^{(r)}_\nu = -(\eta_{\mu\nu} - k_\mu k_\nu / \mu^2) \) appears in the numerator of the propagator. (Note that there are 3 orthonormal polarizations, so this is a rank-3 matrix; its kernel is the longitudinal direction, \( k^\mu \).) The quantity in square brackets is then the momentum-space propagator. Since \( \langle 0 | A_{\mu}(x) | k, r \rangle = \)

53
\( \epsilon^r(k) e^{-ikx} \), a vector in the initial state produces a factor of \( \epsilon^r_\mu(k) \), and in the final state gives \( \epsilon^* \).

**Massless case.** In the limit \( \mu \to 0 \) some weird stuff happens. If we couple \( A_\mu \) to some object \( j^\mu \) made of other matter, by adding \( \Delta \mathcal{L} = j^\mu A_\mu \), then we learn that \( \partial_\mu A^\mu = \mu^{-2} \partial_\mu j^\mu \). This means that in order to take \( \mu \to 0 \), it will be best if the current is conserved \( \partial_\mu j^\mu \).

One example is the QED coupling, \( j^\mu = \bar{\Psi} \gamma^\mu \Psi \). Here \( j^\mu \) is the Noether current for the symmetry \( \Psi \to e^{i\alpha} \Psi \) of the Dirac Lagrangian. This coupling \( A_\mu j^\mu \) arises from the 'minimal coupling' prescription of replacing \( \partial_\mu \to D_\mu = \partial_\mu + ieqA_\mu \) in the Dirac Lagrangian. In this case, the model, with Lagrangian

\[
\mathcal{L} = \bar{\Psi} \left( i\gamma^\mu \partial_\mu - m \right) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\mu^2}{2} A_\mu A^\mu \big|_{\mu^2 = 0},
\]

has a local invariance under \( A_\mu \to A_\mu + \partial_\mu \lambda(x)/e, \Psi(x) \to e^{iq\lambda(x)} \Psi(x) \). For \( \lambda \) non-constant (and going to zero far away), this is a redundancy of our description rather than a symmetry (for example, they have the same configuration of \( \mathbf{E}, \mathbf{B}, \oint \mathbf{A} \)). That is, configurations related by this gauge transformation should be regarded as equivalent.

**[End of Lecture 6]**

Another example can be obtained by taking a complex scalar and doing the same replacement: \( \mathcal{L} = D_\mu \Phi^* D^\mu \Phi + ... \). Notice that in this case the vertex involves a derivative, so it comes with a factor of \( \frac{1}{e} \). Also, there is a \( A_\mu A_\nu \Phi^* \Phi \) coupling, which gives a vertex \( -ie^2 q^2 \eta_{\mu\nu} \).

How do I know that configurations related by a gauge transformation should be regarded as equivalent? If not, the kinetic operator for the massless vector field \( (\eta_{\mu\nu} (\partial^\mu \partial_\rho) - \partial_\mu \partial_\nu) A^\nu = 0 \) is not invertible (even in Euclidean section!), since it annihilates \( A_\nu = \partial_\nu \lambda \).

What’s the propagator for a massless vector field, then? One strategy is to simply ignore the gauge equivalence and use the same propagator (3.7) that we found in the massive case with \( \mu \to 0 \). Since the dynamics are gauge invariant, it will never make gauge-variant stuff, and the longitudinal bits \( \propto k_\mu k_\nu \) in (3.7) (which depend on \( \mu \)) will just drop out, and we can take \( \mu \to 0 \) in the denominator at the end. This actually works. The guarantee that it works is the QED Ward identity: any amplitude with an
external vector $\epsilon(k)_\mu$ is of the form

$$M = iM^\mu(k)\epsilon_\mu(k)$$

and if all external fermion lines are on-shell then

$$M^\mu(k)k_\mu = 0.$$  

There is a complicated diagrammatic proof of this statement in Peskin; Schwartz §8.4 argues that it is a necessary condition for Lorentz invariance of $M = \epsilon_\mu \cdot M^\mu$; and we will see some illustrations of it below (I also recommend Zee §II.7). But it is basically a statement of current conservation: such an amplitude is made (by LSZ and the photon Schwinger-Dyson equation) from a correlation function involving an insertion of the electromagnetic current $j^\mu(k) = \int d^4x \ e^{-ikx} j^\mu(x)$, in the form, $M^\mu \sim \langle \Omega|...j^\mu(k)...|\Omega \rangle$, and $k_\mu j^\mu(k) = 0$ is current conservation.\(^\text{17}\)

This property guarantees that we will not emit either of the unphysical polarizations of massless photons, since the amplitude to do so is either $A(\text{emit} \epsilon_\lambda \propto k_\lambda) = \epsilon_\mu M^\mu \propto k_\mu M^\mu \overset{\text{Ward}}{=} 0$, or the $\mu \to 0$ limit of

$$A \left( \text{emit} \epsilon_\lambda^L = \tfrac{1}{\mu}(k, 0, 0, -\omega)_\lambda \right) \propto \epsilon_\mu L^\mu = \frac{1}{\mu} \left( k_\mu M^0 - \omega M^3 \right) = \frac{1}{\mu} \left( k_\mu M^0 - \sqrt{k^2 + \mu^2 M^3} + \cdots \right)$$

$$= \frac{1}{\mu} \underbrace{k_\mu M^0}_{=0, \text{by Ward}} - \mu \frac{M^2}{2k} + \mathcal{O}(\mu^3) \overset{\mu \to 0}{\to} 0.$$  

For the same reason, in summing over photon final-states (for example in computing a cross section), we’ll have

$$\sum_\epsilon |M|^2 = \sum_\epsilon \epsilon(k)_\mu \epsilon(k)_\nu^* M^\mu(k)M^\nu(k)^*. $$

\(^{17}\) Current conservation $\partial_\mu j^\mu$ is a statement which requires the equations of motion (recall the proof of Noether’s theorem). Recall that equations of motion are true in correlation functions, up to contact terms, using the independence of the path integral on choice of integration variables. By contact terms, I mean terms which are only nonzero when two operators are at the same point. So you can worry about the contact terms in the argument for the Ward identity. The reason they do not contribute is that all the operators in the correlation function (using the LSZ formula) correspond to external states. A collision between the operators creating the external particles would lead to a disconnected amplitude, which could only contribute for degenerate kinematical configurations, and we can ignore them. If you would like to read more words about this, look at Schwartz §14.8, or §3.3.
This sum is only over the two physical polarizations of the massless photon. If we choose a frame with $k^\mu = (k, 0, 0, k)^\mu$, the Ward identity says $0 = k_\mu M^\mu = k (M^0 - M^3)$, so

$$
\sum_\epsilon \epsilon(k)_\mu \epsilon(k)_\nu M^\mu(k) M^\nu(k)^* = |M^1|^2 + |M^2|^2 = |M^1|^2 + |M^2|^2 + |M^3|^2 - |M^0|^2 = -\eta_{\mu\nu} M^\mu(k) M^\nu(k)^*,
$$

that is, just like in the numerator of the propagator, we can replace

$$
\sum_\epsilon \epsilon(k)_\mu \epsilon(k)_\nu \sim -\eta_{\mu\nu}
$$

since they differ by stuff proportional to $k^\mu$ which vanishes when contracted with the rest of the amplitude. In (3.8) we see explicitly that the crazy timelike polarization (which looks like negative probability) cancels the longitudinal polarization $\vec{\epsilon} \propto \vec{k}$.

**Gauge fixing.** You might not be happy with the accounting procedure I’ve advocated above, where unphysical degrees of freedom are floating around in intermediate states and only drop out at the end by some formal trick. In that case, a whole zoo of formal tricks called **gauge fixing** has been prepared for you. Here’s a brief summary to hold you over until we really need it for the non-Abelian case.

At the price of Lorentz invariance, we can make manifest the physical dofs, by choosing **Coulomb gauge**. That means we restrict $\partial_\mu A^\mu = 0$ (so far, so Lorentz invariant) and also $\vec{\nabla} \cdot \vec{A} = 0$. Looking at (3.6), we see that this kills off the bit of $A_0$ that depended on $\vec{A}$. We also lose the helicity-zero polarization $\vec{\nabla} \cdot \vec{A} \propto \epsilon^{(0)}$. But the Coulomb interaction is instantaneous action at a distance.

To keep Lorentz invariance, we can instead merely discourage configurations with $\partial \cdot \vec{A} \neq 0$ by adding a term to the action

$$
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial \cdot \vec{A})^2
$$

for some arbitrary number $\xi$. Physics should not depend on $\xi$, and this is a check on calculations. The propagator is

$$
\langle T A_\mu(x) A_\nu(y) \rangle_\alpha = \int d^4 k \ e^{-ik(x-y)} \left[ \frac{-i(\eta_{\mu\nu} - (1 - \xi) k_\mu k_\nu / \mu^2)}{k^2 - \mu^2 + i\epsilon} \right]
$$

and again the bit with $k_\mu k_\nu$ must drop out. $\xi = 1$ is called Feynman gauge and makes this explicit. $\xi = 0$ is called Landau gauge and makes the propagator into a projector onto $k_\perp$.

It becomes much more important to be careful about this business in non-Abelian gauge theory.
3.3 On the non-perturbative proof of the Ward identity

[Schwartz §14.8] First, consider a Green’s function from which we might make an S-matrix element by LSZ,

\[ G \equiv \langle \Omega | T \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) | \Omega \rangle = \int [D\Psi] e^{iS} \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \]

where the operators \( \mathcal{O}_1(x_1) \mapsto e^{-iQ_\alpha \mathcal{O}_1(x_1)} \) have charge \( Q_\alpha \) under a global \( U(1) \) symmetry. For example the \( \mathcal{O}(x) \) could be just the elementary field \( \Psi(x) \).\(^{18}\)

Now change variables in the path integral so that \( \mathcal{O}_i(x_i) \mapsto e^{-iQ_\alpha \mathcal{O}_i(x_i)} \); the action will shift by \( S \mapsto S - \int \alpha \partial \mu j^\mu \) where \( j^\mu \) is the Noether current. The path integral doesn’t change at all, so its infinitesimal variation is

\[
0 = \delta G = \int [D\Psi] \left( -\int i\alpha \partial \mu j^\mu e^{iS} \mathcal{O}_1 \cdots \mathcal{O}_n - i \sum_i Q_i \alpha(x_i) e^{iS} \mathcal{O}_1 \cdots \mathcal{O}_n \right) \]

\[
= \int d^Dx \alpha(x) \left[ i\partial \mu \langle j^\mu(x) \mathcal{O}_1 \cdots \mathcal{O}_n \rangle - \sum_i Q_i \delta^D(x - x_i)G \right].
\]

Since this is true for any \( \alpha(x) \), we learn that the thing in square brackets is zero: \( \partial \mu j^\mu = 0 \) up to contact terms. This is called the Ward-Takahashi identity.

Now suppose we do this same manipulation in a gauge theory, like QED. The additional terms in \( S \) are \(-\frac{i}{4} F_{\mu\nu} F^{\mu\nu} + i A_\mu \bar{\Psi} \gamma^\mu \Psi \), which are invariant under the transformation, so don’t change these statements. Notice that the transformation we’re doing here is not the gauge transformation, since \( A_\mu \) doesn’t transform – we’re only doing the gauge transformation on the matter fields here, so their kinetic terms actually shift and produce the \( \alpha \partial^\mu j_\mu \) term above. Photon field insertions in \( G \) don’t contribute, since they have charge zero here.

Next, think about the LSZ formula for an S-matrix element with (say) two external photons:

\[ \mathcal{M} = \langle \epsilon, \cdots \epsilon_k | S | \cdots \rangle^{\text{LSZ}} = e^{i\epsilon \bar{\epsilon}} i^{n} \int d^4xe^{ikx} \square_{\mu\nu} \int d^4x_1 e^{i\rho k x_1} \square^{\rho\sigma} \int \cdots \langle A^\nu(x) \cdots A^\sigma(x_k) \cdots \rangle \]

(3.11)

where \( \square_{\mu\nu} \) is shorthand for the photon kinetic operator \( \square_{\mu\nu} \equiv \square \eta_{\mu\nu} - \partial_\mu \partial_\nu / \mu^2 \). The Schwinger-Dyson equation for \( A_\mu \) then implies that

\[
\square^{\rho\sigma} \square_{\mu\nu} \langle A^\nu(x) \cdots A^\sigma(x_k) \cdots \rangle = \square^{\rho\sigma} \left( \langle j^\mu(x) \cdots A^\sigma(x_k) \cdots \rangle - i \delta^4(x - x_k) \eta_{\mu\sigma} \langle \cdots \rangle \right) \]

\[
= \langle j_\mu(x) \cdots j_\sigma(x_k) \rangle - i \delta(x - x_k) \eta_{\mu\sigma} \langle \cdots \rangle \]

(3.12)

\[
(3.13)\]

\(^{18}\)You’ll have to trust me for now that the path integral for fermionic fields exists. That’s the only information about it we’ll need here. Also I’ve absorbed the factor of \( Z^{-1} \) into \([D\Psi]\).
First of all, this is why I said we could get the S-matrix elements with photons from correlators with currents. But notice that this is only true up to the contact terms. But those are disconnected amplitudes which we can ignore.

Finally, set the polarization of one of the photons equal to its momentum $\epsilon = p$. Then

$$p^\mu \mathcal{M}_\mu = \epsilon_\mu^i \int d^4x e^{ipx} \int d^4x_1 e^{ipkx_k} \int dy e^{iqyu} (i \partial_y + m_1) \cdots (i \partial_y \mu \cdots \mu \rho (x_1) \cdots \Psi(y) \cdots)$$

$$= (q_1 - m_1)(q_2 - m_2) \cdots \sum_j Q_j \tilde{G}(\cdots, q_j \pm p, \cdots) \quad (3.14)$$

where the $\pm$ depends on whether particle $j$ is incoming or outgoing. At the last step we used the Fourier transform of (3.10).

Now here’s the punchline: The $\tilde{G}$ on the RHS of (3.15) has poles at $(q_j \pm p)^2 = m_j^2$, and not at $q_j^2 = m_j^2$. So when it’s multiplied by $\frac{i}{q_j^2 - m_j^2}$ it will vanish. End of story. Notice that no use of perturbation theory was made here.

### 3.4 Feynman rules for QED

First, Feynman rules for Dirac fermion fields, more generally\(^1\). As always in these notes, time goes to the left, so I draw the initial state on the right (like the ket) and the final state on the left (like the bra).

1. An internal fermion line gives

\[
\begin{array}{c}
\text{L} \\
\end{array} 
= \frac{i}{\not{k} - m_\Psi}
\]

\(^1\)Another good example of a QFT with interacting fermions is the Yukawa theory theory of a Dirac fermion field plus a scalar $\phi$ and an interaction

$$V = g \phi \bar{\Psi} \Psi \quad \Rightarrow
\begin{array}{c}
\text{p'} \\
\end{array} 
= -ig \delta^{rr'} \quad (3.16)$$

Notice that in $3 + 1$ dimensions, $[g] = +4 - [\phi] - 2[\Psi] = 4 - 1 - 2\frac{3}{2} = 0$, the coupling is dimensionless. This describes more realistically the interactions between nucleons (which are fermions, as opposed to nucleons) and scalar pions, which hold together nuclei. It also is a crude sketch of the Higgs coupling to matter; notice that if $\phi$ is some nonzero constant $\langle \phi \rangle$, then there is a contribution to the mass of the fermions, $g \langle \phi \rangle$. 
which is a matrix on the spinor indices.

There are four possibilities for an external fermion line of definite momentum. Here \(u, v\) are respectively the positive- and negative-energy solutions of the Dirac equation, 
\[
(\not{p} - m) \, u^r(k) = 0, \quad (\not{p} + m) \, v^r(k) = 0.
\]
(For a reminder, see e.g. §5.4 of my 215A lecture notes.)

2. \[\begin{array}{c}
\hline
\text{\tiny \(k\)}
\end{array}\] \[\begin{array}{c}
\hline
\text{\tiny \(r\)}
\end{array}\] = \[\Psi | k, r \rangle = u^r(k)

3. \[\begin{array}{c}
\hline
\text{\tiny \(k\)}
\end{array}\] \[\begin{array}{c}
\hline
\text{\tiny \(r\)}
\end{array}\] = \langle k, r | \bar{\Psi} = \bar{u}^r(k)

4. \[\begin{array}{c}
\hline
\text{\tiny \(k\)}
\end{array}\] \[\begin{array}{c}
\hline
\text{\tiny \(r\)}
\end{array}\] = \[\bar{\Psi} | k, r \rangle = \bar{v}^r(k)

5. \[\begin{array}{c}
\hline
\text{\tiny \(k\)}
\end{array}\] \[\begin{array}{c}
\hline
\text{\tiny \(r\)}
\end{array}\] = \langle k, r | \bar{\Psi} = v^r(k).

6. Some advice: When evaluating a Feynman diagram with spinor particles, always begin at the head of the particle-number arrows on the fermion lines, and keep going along the fermion line until you can’t anymore. This will keep the spinor indices in the form of matrix multiplication. Why: every Lagrangian you’ll ever encounter has fermion parity symmetry, under which every fermionic field gets a minus sign; this means fermion lines cannot end, except on external legs. The result is always of the form of a scalar function (not a matrix or a spinor) made by sandwiching gamma matrices between external spinors:

\[r'p' \begin{array}{c}
\hline
\text{\tiny \(k\)}
\end{array}\] \[\begin{array}{c}
\hline
\text{\tiny \(r\)}
\end{array}\] = \[\sum_{a,b,...=1,4} \bar{u}^r(p')_a (\text{pile of gamma matrices})_{ab} \, u^r(p)_b \]

Furthermore, in S-matrix elements the external spinors \(u(p), v(p)\) satisfy the equations of motion \((\not{p} - m)u(p) = 0\), a fact which can be used to our advantage to shrink the pile of gammas.

There can also be fermion lines which form internal loops (though not at tree level, by definition). In this case, the spinor indices form a trace,

\[\sum_a (\text{pile of gamma matrices})_{aa} \equiv \text{tr} (\text{pile of gamma matrices}).\]

We’ll learn to compute such traces below (around (3.18)); in fact, traces appear even in the case with external fermions if we do not measure the spins.
7. Diagrams related by exchanging external fermions have a relative minus sign.

8. Diagrams with an odd number of fermion loops have an extra minus sign.

The last two rules are best understood by looking at an example in detail.

To understand rule 8 consider the following amplitude in the Yukawa theory with interaction (3.16):

\[ \sum_{abcd} \bar{\Psi}_a(x) \Psi_b(x) \bar{\Psi}_c(y) \Psi_d(y) = (-1) \text{tr}(\Psi(x) \bar{\Psi}(y) \Psi(x) \bar{\Psi}(y)) = (-1) \text{tr}S_F(x - y)S_F(x - y) \]

[Peskin page 119] To understand rule 7 consider $\Psi \Psi$ → $\Psi \Psi$ (nucleon) scattering in the Yukawa theory:

The blob represents the matrix element

\[ \langle p_3 r_3; p_4 r_4 | \mathcal{T} e^{-i \int V d^4z} | p_1 r_1; p_2 r_2 \rangle_0 \]

where the initial state is

\[ | p_1 r_1; p_2 r_2 \rangle_0 \propto a_{r_1}^{a_1} a_{r_2}^{a_2} | 0 \rangle \]

and the final state is

\[ \langle p_3 r_3; p_4 r_4 \rangle = (\langle p_3 r_3; p_4 r_4 \rangle_0^\dagger )^\dagger \propto (0) a_{p_3}^{r_3} a_{p_4}^{r_4} = - \langle 0 | a_{p_3}^{r_3} a_{p_4}^{r_4} \]

where note that the dagger reverses the order.

The leading contribution comes at second order in $V$:

\[ \langle p_3 r_3; p_4 r_4 | \mathcal{T} \left( \frac{1}{2!}(ig)^2 \int d^4z_1 \int d^4z_2 (\bar{\Psi} \Psi \phi)_1 (\bar{\Psi} \Psi \phi)_2 \right) | p_1 r_1; p_2 r_2 \rangle_0 \]

To get something nonzero we must contract the $\phi$s with each other. The diagrams at right indicate best the possible ways to contract the fermions. Exchanging the roles of $z_1$ and $z_2$ interchanges two pairs of fermions so costs no signs and cancels the $1/2!$.

The overall sign is annoying but can be fixed by demanding that the diagonal bit of the $S$-matrix give

\[ \langle p_3 p_4 | (1 + ...) | p_1 p_2 \rangle = +\delta(p_1 - p_3)\delta(p_2 - p_4) + \cdots \]

The relative sign is what we’re after, and it comes by comparing the locations of fermion
operators in the contractions in the two diagrams at right. In terms of the contractions, these \( t- \) and \( u- \) channel diagrams are related by leaving the annihilation operators alone and switching the contractions between the creation operators and the final state. Denoting by \( a_{1,2}^\dagger \) the fermion creation operators coming from the vertex at \( z_{1,2} \),

\[
\langle 0 | a_{p_4} a_{p_2} a_{1}^\dagger a_{2}^\dagger ... + \langle 0 | a_{p_4} a_{p_3} a_{1}^\dagger a_{2}^\dagger ... = \langle 0 | a_{p_4} a_{p_2} a_{1}^\dagger a_{2}^\dagger ... - \langle 0 | a_{p_3} a_{p_2} a_{1}^\dagger a_{2}^\dagger … \]

In the last expression the fermion operators to be contracted are all right next to each other and we see the relative minus sign.

While we’re at it, let’s evaluate this whole amplitude to check the Feynman rules I’ve claimed and get some physics out. It is

\[
S_{fi} = -g^2 \int d^4q \int dz_1 dz_2 \int d^4q \frac{e^{-i\phi(z_1-z_2)}\mathbf{i}}{q^2 - m^2 + i\epsilon} \left( e^{-i\phi(p_1 - p_3)} \bar{u}^{r_3}(p_3) u^{r_1}(p_1) \cdot e^{-i\phi(p_2 - p_4)} \bar{u}^{r_4}(p_4) u^{r_2}(p_2) - (3 \leftrightarrow 4) \right) .
\]

In the first \( (t-\)channel) term, the integrals over \( z_{1,2} \) gives \( \delta(p_1 - p_3 - q)\delta(p_2 - p_4 - q) \), and the \( q \) integral then gives \( \delta(p_1 + p_2 - p_3 - p_4) \), overall momentum conservation. In the second \( (u-\)channel) term, \( q = p_1 - p_4 = p_3 - p_2 \). Altogether,

\[
S_{fi} = \mathbb{1} + \frac{1}{2} \phi(\mathcal{P}_T) i\mathcal{M}
\]

with, to leading order,

\[
i\mathcal{M} = -ig^2 \left( \frac{1}{t - m^2} (\bar{u}_3 u_1)(\bar{u}_4 u_2) - \frac{1}{u - m^2} (\bar{u}_4 u_1)(\bar{u}_3 u_2) \right)
\]

(3.17)

with \( t \equiv (p_1 - p_3)^2 \), \( u \equiv (p_1 - p_4)^2 \). This minus sign implements Fermi statistics.

**Yukawa force revisited.** In the non-relativistic limit, we can again relate this amplitude to the force between particles, this time with the actual spin and statistics of nucleons. In the COM frame, \( p_1 = (m, \mathbf{p}) \), \( p_2 = (m, -\mathbf{p}) \) and \( p_3 = (m, \mathbf{p}') \), \( p_4 = (m, -\mathbf{p}') \). In the non-relativistic limit, the spinors become \( u_\rho = \left( \frac{\sqrt{\sigma \cdot \mathbf{p}}\xi^\rho}{\sqrt{\sigma \cdot \mathbf{p}}\xi^\rho} \right) \rightarrow \sqrt{m} \left( \xi^\rho \right) \) so that \( \bar{u}_3 u_1 \equiv \bar{u}(p_3)^{r_3} u(p_1)^{r_1} = 2m\xi_{r_3}^1 \xi_{r_1}^1 = 2m\delta_{r_3 r_1} \). Let’s simplify our lives and take two distinguishable fermions (poetically, they could be proton and neutron, but let’s just add a label to our fermion fields; they could have different masses, for example, or different couplings to \( \phi \), call them \( g_1, g_2 \)). Then we only get the \( t-\)channel diagram. The intermediate scalar momentum is \( q = p_1 - p_3 = (0, \mathbf{p} - \mathbf{p}') \) so \( t = (p_1 - p_3)^2 = -q^2 = - (\mathbf{p} - \mathbf{p}')^2 \) and

\[
i\mathcal{M}_{NR,COM} = ig_1 g_2 \frac{1}{q^2 + m^2} 4m^2 \delta^{r_3 r_1} \delta^{r_2 r_4}.
\]

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Compare this to the NR Born approximation matrix element

\[
2\pi\delta(E_p - E_{p'}) \left( -i\tilde{V}(\vec{q}) \right) = \langle \vec{p}' | S | \vec{p} \rangle_{NR} \\
= \sum_{r_4} \int d^3p_4 V \prod_{i=1}^{4} \frac{1}{\sqrt{2E_i}} S \langle 34 \leftarrow 12 \rangle \\
= \frac{1}{\sqrt{2m'}} \delta^{r_1 r_3} \frac{ig_1g_2}{q^2 + m^2_\phi} \\
\]

where in the second line we summed over possible final states of the second (target) particle, and corrected the relativistic normalization, so that \( \langle \vec{p}' | \vec{p} \rangle_{NR} = \delta^3(p - p') \). This is completely independent of the properties of the second particle. We infer that the scalar mediates a force with potential \( U(x) = -\frac{g_1g_2e^{-m_\phi r}}{4\pi r} \). It is attractive if \( g_1g_2 > 0 \).

[End of Lecture 7]

**Back to QED.** The new ingredients in QED are the propagating vectors, and the interaction hamiltonian \( V = e\bar{\Psi}\gamma^\mu\Psi A_\mu \). The rest of the Feynman rules are

9. The interaction vertex gets a

\[
\begin{align*}
\begin{array}{c}
\text{Interaction Vertex}
\end{array}
\end{align*}
\]

\[ = -ie\gamma^\mu \]

10. An external photon in the initial state gets a \( e^\mu(p) \), and in the final state gets a \( e^{\mu*}(p) \).

11. An internal photon line gets a

\[
\begin{align*}
\begin{array}{c}
\text{Internal Photon Line}
\end{array}
\end{align*}
\]

\[ = \frac{i}{k^2 - m_\gamma^2} \left( -\eta^{\mu\nu} + (1 - \xi)k^\mu k^\nu/k^2 \right) \]

where \( m_\gamma = 0 \) (it’s sometimes useful to keep it in there for a while as an IR regulator) and the value of \( \xi \) is up to you (meaning that your answers for physical quantities should be independent of \( \xi \)).

**Spinor trace ninjutsu.**

The trace is cyclic: \( \text{tr} (AB \cdots C) = \text{tr} (CAB \cdots) \). \( (3.18) \)
Our gamma matrices are $4 \times 4$, so $\text{tr}1 = 4$.

$$\text{tr}\gamma^\mu = \text{tr}(\gamma^5)^2 \gamma^\mu \overset{(3.18)}{=} \text{tr}\gamma^5 \gamma^\mu \gamma^5 \overset{(3.18)}{=} 0 \implies -\text{tr}\gamma^\mu = 0. \quad (3.19)$$

The same trick shows that the trace of any odd number of gammas vanishes. The idea is that an odd number of gammas is a map between the $L$ and $R$ subspaces, so it has only off-diagonal terms in the Weyl basis.

$$\text{tr}\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = 4 (\eta^\mu\nu \eta^\rho\sigma + \eta^\sigma\mu \eta^\rho\nu - \eta^\mu\rho \eta^\nu\sigma). \quad (3.21)$$

Why is this? The completely antisymmetric bit vanishes because it is proportional to $\gamma^5$ which is traceless (by the same argument as (3.19)). If any pair of indices is the same then the other two must be too by (3.20). If adjacent pairs are the same they can just square to one and we get +1; if alternating pairs are the same (and different from each other) then we must move them through each other with the anticommutator. If they are all the same we get 4.

$$\text{tr}\gamma^\mu \gamma^\nu = -4i\epsilon^{\mu\nu\rho\sigma}.$$  

### 3.5 QED processes at leading order

Now we are ready to do lots of examples, nearly all of which (when pushed to the end) predict cross sections which are verified by experiments to about one part in 137.\footnote{I guess it is this overabundance of scientific victory in this area that leads to the intrusion of so many names of physicists in the following discussion.} Here $\frac{1}{137} \approx \alpha \equiv \frac{e^2}{4\pi}$ is the small number by which the next order corrections are suppressed.\footnote{This statement is true naively (in the sense that the next diagrams which are nonzero come with two more powers of $e$), and also true in fact, but in between naiveté and the truth is a long road of renormalization, which begins in the next section.}

Did I mention that the antiparticle of the electron, predicted by the quantum Dirac theory (i.e. by Dirac), is the positron? It has the same mass as the electron and the opposite electromagnetic charge, since the charge density is the 0 component of the electromagnetic current, $j^\mu = \bar{\Psi} \gamma^\mu \Psi$, so the charge is

$$\int d^3x j^0(x) = \int \bar{\Psi} \gamma^0 \Psi = \int \Psi^\dagger \Psi = \int d^3p \sum_s (a^\dagger_{p,s} a_{ps} - b^\dagger_{p,s} b_{ps}).$$
So \( b^+ \) creates a positron.

[Schwarz §13.3, Peskin §5.1] Perhaps the simplest to start with is scattering of electrons and positrons. We can make things even simpler (one diagram instead of two) by including also the muon, which is a heavy version of the electron\(^{22}\), and asking about the process \( \mu^+ \mu^- \leftarrow e^+ e^- \). At leading order in \( e \), this comes from

\[
M_{\mu^+ \mu^- \leftarrow e^+ e^-} = (-\text{i} e \bar{u}_3 (p_3) \gamma^\mu v^4 (p_4))_{\text{muons}} \left( -\text{i} \eta_{\mu\nu} - \frac{(1-\xi)k_{\mu}k_{\nu}}{k^2} \right) \left( -\text{i} e \bar{v}_2 (p_2) \gamma^\nu u^1 (p_1) \right)_{\text{electrons}} \tag{3.22}
\]

with \( k \equiv p_1 + p_2 = p_3 + p_4 \) by momentum conservation at each vertex. I’ve labelled the spinors according to the particle types, since they depend on the mass.

**Ward identity in action.** What about the \( k_\nu k_\mu \) term in the photon propagator? The spinors satisfy their equations of motion, \( \gamma_1 u_1 = m_e u_1 \) (where \( u_1 \equiv u^{s_1}_{p_1} \) for short) and \( \bar{v}_2 \bar{p}_2 = -m_e \bar{v}_2 \). The \( k_\nu \) appears in

\[
k_\nu \bar{v}_2 \gamma^\nu u_1 = \bar{v}_2 (\gamma_1 + \bar{p}_2) u_1 = \bar{v}_2 \gamma_1 u_1 + \bar{v}_2 \bar{p}_2 u_1 = (m - m)\bar{v} u = 0.
\]

(The other factor is also zero, but one factor of zero is enough.) Therefore

\[
\mathcal{M} = \frac{e^2}{s} \bar{u}_3 \gamma_\mu v_4 \cdot \bar{v}_2 \gamma^\mu u_1
\]

where \( s \equiv k^2 = (p_1 + p_2)^2 = E_{\text{CoM}}^2 \) is the Mandelstam variable. And I am relying on you to remember which spinors refer to muons (3,4) and which to electrons (1,2).

**Squaring the amplitude.** We need to find \( \mathcal{M}^\dagger \) (the dagger here really just means complex conjugate, but let’s put dagger to remind ourselves to transpose and reverse the order of all the matrices). Recall the special role of \( \gamma^0 \) here:

\[
\gamma^\dagger_\mu \gamma_0 = \gamma_0 \gamma^\mu, \quad \gamma_0^\dagger = \gamma_0.
\]

This means that for any two Dirac spinors,

\[
(\bar{\Psi}_1 \gamma^\mu \Psi_2)^\dagger = \bar{\Psi}_2 \gamma^\mu \Psi_1.
\]

---

\(^{22}\)Who ordered that? (I. I. Rabi’s reaction to learning about the muon.) I hope you don’t find it too jarring that the number of ‘elementary’ particles in our discussion increased by three in the last two paragraphs. People used to get really disgruntled about this kind of thing. But here we have, at last, uncovered the true purpose of the muon, which is to halve the number of Feynman diagrams in this calculation (compare (3.28)).
(This is the same manipulation that showed that the Dirac Lagrangian was hermitian.) So
\[ \mathcal{M}^\dagger = \frac{e^2}{s} (\bar{v}_4 \gamma^\mu u_3) (\bar{u}_1 \gamma_\mu v_2). \]

and therefore
\[ |\mathcal{M}_{\mu^+\mu^-}\rangle|^2 = \frac{e^4}{s^2} (\bar{v}_4 \gamma^\mu u_3) (\bar{a}_3 \gamma^\nu v_4) \cdot (\bar{u}_1 \gamma_\mu v_2) (\bar{b}_2 \gamma_\mu u_1). \] (3.23)

These objects in parentheses are just c-numbers, so we can move them around, no problem. I’ve grouped them into a bit depending only on the initial state (the electron stuff 1, 2) and a bit depending only on the final state (the muon stuff 3, 4).

**Average over initial, sum over final.** In the amplitude above, we have fixed the spin states of all the particles. Only very sophisticated experiments are able to discern this information. So suppose we wish to predict the outcome of an experiment which does not measure the spins of the fermions involved. We must sum over the final-state spins using
\[ \sum_{s_4} v_{s_4}^a (p_4) \bar{v}_{s_4}^b (p_4) = (\tilde{p}_4 - m_\mu)_{ab} = \sum_{s_4} v_{s_4}^a (p_4) v_{s_4}^b (p_4) \]
(where I wrote the last expression to emphasize that these are just c-numbers) and
\[ \sum_{s_3} u_{s_3}^a (p_3) \bar{u}_{s_3}^b (p_3) = (\tilde{p}_3 + m_\mu)_{ab}. \]

Looking at just the ‘out’ factor of $|\mathcal{M}|^2$ in (3.23), we see that putting these together produces a spinor trace, as promised:
\[ \sum_{s_3, s_4} (\bar{u}(p_3) s_{\mu}^a \gamma^\mu (p_4) s_{\nu}^b) (\bar{v}(p_4) c_{\nu}^a \gamma^\nu (p_3) d) \]
\[ = (\tilde{p}_4 - m_\mu)_{bc} (\tilde{p}_3 + m_\mu)_{da} \]
\[ = \text{tr} (\gamma^\mu (\tilde{p}_4 - m_\mu) \gamma^\nu (\tilde{p}_3 + m_\mu)) \]
\[ = \text{tr} \gamma^\mu \gamma^\nu \gamma^\sigma - m_\mu^2 \text{tr} \gamma^\mu \gamma^\nu \]
\[ = 4 \left( p_{4\mu} p_{3\nu} + p_{3\mu} p_{4\nu} - p_{3\mu} p_{4\nu} \eta_{\mu\nu} - m_\mu^2 \eta_{\mu\nu} \right) \] (3.24)

If also we don’t know the initial (electron) spins, then the outcome of our experiment is the average over the initial spins, of which there are four possibilities. Therefore, the relevant probability for unpolarized scattering is
\[ \frac{1}{4} \sum_{s_{1,2,3,4}} |\mathcal{M}|^2 = \frac{e^4}{4s^2} \text{tr} (\gamma^\mu (\tilde{p}_4 - m_\mu) \gamma^\nu (\tilde{p}_3 + m_\mu)) \text{tr} (\gamma^\nu (\tilde{p}_2 - m_e) \gamma_\mu (\tilde{p}_1 + m_e)) \]

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\[
\begin{align*}
(3.24) \text{twice} & \quad \frac{8e^4}{s^2} (p_{13}p_{24} + p_{14}p_{23} + m_{\mu}^2 p_{12} + m_{e}^2 p_{34} + 2m_{e}^2 m_{\mu}^2) \\
\text{algebra} & \quad \frac{2e^4}{s^2} (t^2 + u^2 + 4s(m_{e}^2 + m_{\mu}^2) - 2(m_{e}^2 + m_{\mu}^2)^2)
\end{align*}
\] (3.25)

In the second step of (3.25) the \( p_{12}p_{34} \) terms cancel. In the last step of (3.25) we used all the Mandelstam variables:

\[
\begin{align*}
s & \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2 = \frac{E_{\text{CoM}}^2}{4} \\
t & \equiv (p_1 - p_3)^2 = (p_2 - p_4)^2 = m_e^2 + m_{\mu}^2 - 2E^2 + 2\vec{k} \cdot \vec{p} \\
u & \equiv (p_1 - p_4)^2 = (p_2 - p_3)^2 = m_e^2 + m_{\mu}^2 - 2E^2 - 2\vec{k} \cdot \vec{p}
\end{align*}
\]

where the particular kinematic variables (in the rightmost equalities) are special to this problem, in the center of mass frame (CoM), and are defined in the figure at right.

Really there are only two independent Lorentz-invariant kinematical variables, since \( s + t + u = \sum_i m_i^2 \).

Now we can use the formula for a differential cross section with a two-body final state, in the CoM frame (for the derivation, see these notes, §4.7):

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{CoM}} = \frac{1}{64\pi^2E_{\text{CoM}}^2} \frac{|\vec{p}|}{|\vec{k}|} \left( \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \right) \\
= \frac{\alpha^2}{16E^6} \frac{|\vec{p}|}{|\vec{k}|} \left( E^4 + |\vec{k}|^2 |\vec{p}|^2 \cos^2 \theta + E^2 (m_e^2 + m_{\mu}^2) \right)
\]

(3.26)

where \( \alpha \equiv \frac{e^2}{4\pi} \) is the fine structure constant. This can be boiled a bit with kinematical relations \( |\vec{k}| = \sqrt{E^2 - m_e^2}, |\vec{p}| = \sqrt{E^2 - m_{\mu}^2} \) to make manifest that it depends only on two independent kinematical variables, which we can take to be the CoM energy \( E \) and the scattering angle \( \theta \) in \( \vec{k} \cdot \vec{p} = |\vec{k}| |\vec{p}| \cos \theta \) (best understood from the figure). It simplifies a bit if we take \( E \gg m_e \), and more if we take \( E \gg m_{\mu} \sim 200m_e \), to

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{CoM}^2} \left( 1 + \cos^2 \theta \right).
\]

(3.27)

In fact, the two terms here come respectively from spins transverse to the scattering plane and in the scattering plane; see Schwartz §5.3 for an explanation.

There is a lot more to say about what happens when we scatter an electron and a positron! Another thing that can happen is that the final state could be an electron and positron again (Bhabha scattering\textsuperscript{23}).

\textsuperscript{23}See figure 3 here. Now remember that a person doesn’t have much control over their name. By the way, I totally believe the bit about non-perturbative strings = linst.
They are not necessarily the same $e^-$ and $e^+$, though (except in the sense that they are all the same), because another way to get there at tree level is the second, $t$-channel, diagram, at right. The intermediate photon in that diagram has $k_t = (p_1 - p_3)$, so that the denominator of the propagator is $t = k_t^2 = (p_1 - p_3)^2$ instead of $s$.

Squaring this amplitude gives

$$|M_s + M_t|^2 = |M_s|^2 + |M_t|^2 + 2\text{Re}(M_s M_t^*)$$

(3.28)

interference terms. Interference terms mean that you have to be careful about the overall sign or phase of the amplitudes.

You may be surprised that the cross section (3.27) decreases with energy. Mechanically this comes mainly from the $1/s^2$ from the photon propagator: as $s$ grows, the intermediate photon is more and more off-shell. But more deeply, it’s because above we’ve studied an exclusive cross-section, in the sense that we fixed the final state to be exactly a muon and an antimuon. At higher energies, nothing new happens, because the final state is fixed.

It has also been very valuable to think about inclusive cross-sections for $e^+e^-$ scattering, because in this way you can make anything that the $s$-channel photon couples to, if you put enough energy into it. The inclusive cross section for ($e^+e^-$ goes to anything) does grow with energy, and jumps at energies which are thresholds for new particles in the final state. In this way, for example, we can also make quarks (more specifically quark-antiquark pairs) since they also carry electric charge. See Peskin pp 139-140 (and our later discussion in §5.3) for a bit more about that, and in particular how this observable gives evidence that there are three colors of quarks.

What happens if instead we scatter two electrons (Möller scattering)? In that case, the leading order diagrams are the ones at right. Now the intermediate photons have $k_t = (p_1 - p_3)$ and $k_u = (p_1 - p_4)$ respectively, so that the denominator of the propagator is $t$ and $u$ in the two diagrams. The evaluation of these diagrams has a lot in common with the ones for $e^+e^- \rightarrow e^+e^-$, namely you just switch some of the legs between initial and final state.
The relation between such amplitudes is called *crossing symmetry*. Let’s illustrate it instead for $e^-\mu^- \leftrightarrow e^-\mu^-$, where again there is only one diagram, related by crossing to (3.30). The diagram is the one at right. (The muon is the thicker fermion line.)

\[ iM = \left( -ie\bar{u}_3\gamma^\mu u_1 \right)_{\text{electrons}} \frac{-i \left( \eta_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2} \right)}{k^2} \left( -ie\bar{u}_2\gamma^\nu u_4 \right)_{\text{muons}} \]  

(3.29)

with $k \equiv p_1 - p_3 = p_2 - p_4$. It differs from (3.30) by replacing the relevant \( v_s \) with \( u_s \) for the initial/final antiparticles that were moved into final/initial particles, and relabelling the momenta. After the spin sum,

\[ \frac{1}{4} \sum_{s_1, s_2, s_3, s_4} |M|^2 = \frac{e^4}{4t^2} \text{tr} \left( \gamma^\mu \left( p_4 + m_\mu \right) \gamma^\nu \left( p_2 + m_\mu \right) \right) \text{tr} \left( \gamma_\nu \left( p_3 + m_\epsilon \right) \gamma_\mu \left( p_1 + m_\epsilon \right) \right) \]

this amounts to the replacement \((p_1, p_2, p_3, p_4) \rightarrow (p_1, -p_3, p_4, -p_2)\); on the Mandelstam variables, this is just the permutation \((s, t, u) \rightarrow (t, u, s)\).

**Crossing symmetry more generally.** If you look at a Feynman diagram on its side (for example because someone else fails to use the convention that time goes to the left) it is still a valid amplitude for some process. Similarly, dragging particles between the initial and final state also produces a valid amplitude. Making this relation precise can save us some work. The precise relation for dragging an incoming particle into the final state, so that it is an outgoing antiparticle, is:

\[ iM_{f \leftarrow iA}(p_f; p_i, p_A) = iM_{A \leftarrow f}(p_f, k = -p_A; p_i) = f \]

(If you must, note that this is another sense in which an antiparticle is a particle going backwards in time.) If \( A \) is a spinor particle, the sum relations for particles and antiparticles are different:

\[ \sum_r u^r(p)\bar{u}^r(p) = \slashed{p} + m, \quad \sum_r v^r(k)\bar{v}^r(k) = \slashed{k} - m = -(\slashed{p} + m) \]

– after accounting for $k = -p_A$, they differ by an overall sign. Hence we must also append a fermion sign factor \((-1)^{\text{number of fermions shuffled between in and out}}\) in the unpolarized scattering probability. We’ll study a well-motivated example in more detail next.
**Mott formula.** By studying scattering of an electron from a heavy charged fermion (a muon is convenient) we can reconstruct the cross section for scattering off a Coulomb potential (named after Mott). This example will be important later in §4, where we’ll figure out how it is corrected by other QED processes.

\[
\mu^+\mu^- \leftrightarrow e^+e^-.
\]

Consider again the process \(\mu^+\mu^- \leftrightarrow e^+e^-\). To try to keep things straight, I’ll call the electron momenta \(p, p'\) and the muon momenta \(k, k'\), since that won’t change under crossing. We found the amplitude

\[
i \mathcal{M}_{\mu^+\mu^- \leftrightarrow e^+e^-} = \left( -ie\bar{u}(k)\gamma^\mu v^{\prime \mu}(k') \right)_{\text{muons}} \frac{-i}{q^2} \left( \eta_{\mu\nu} - \frac{(1-\xi)q_{\mu}q_{\nu}}{q^2} \right) \left( -ie\bar{v}^{\prime \nu}(p')\gamma^\nu u(p) \right)_{\text{electrons}} \tag{3.30}
\]

(with \(q \equiv p + p' = k + k'\)^24 and the (unpolarized) scattering probability density

\[
\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \text{ spinor traces} = \frac{1}{4} \frac{e^4}{s^2} E^{\mu\nu} M_{\mu\nu},
\]

where the tensor objects \(E^{\mu\nu}, M_{\mu\nu}\) come respectively from the electron and muon lines,

\[
\frac{1}{4} E_{\mu\nu} = p_\mu p'_\nu + p'_\mu p_\nu - \eta_{\mu\nu}(p \cdot p' + m_e^2)
\]

\[
\frac{1}{4} M_{\mu\nu} = k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k \cdot k' + m_{\mu}^2),
\]

and they are contracted by the photon line, with \(s = q^2 = (p + p')^2\).

\[
e^-\mu^- \leftrightarrow e^-\mu^-.
\]

To get from this the amplitude (tree level, so far) for the process \(e^-\mu^- \leftrightarrow e^-\mu^-\), we must move the incoming positron line to an outgoing electron line, and move the outgoing antimuon line to an incoming muon line (hence the sign in \(\sigma\) will be \((-1)^{\text{number of fermions shuffled between in and out}} = (-1)^2 = 1\)). Relative to the amplitude for \(\mu^+\mu^- \leftrightarrow e^+e^- \tag{3.30}\), we must replace the relevant \(v_s\) with \(u_s\) for the initial/final antiparticles that were moved into final/initial particles, and we must replace \(p' \rightarrow -p', k' \rightarrow -k'\):

\[
i \mathcal{M} = \left( -ie\bar{u}(p')\gamma^\mu u(p) \right)_{\text{electrons}} \frac{-i}{q_{\mu}^2} \left( \eta_{\mu\nu} - \frac{(1-\xi)q_{\mu}'q_{\nu}'}{q_{\mu}^2} \right) \left( -ie\bar{u}(k)\gamma^\nu u(k') \right)_{\text{muons}} \tag{3.31}
\]

---

^24Relative to the notation I used earlier, \(p_1 = p, p_2 = p', p_3 = k, p_4 = k'\).
with \( q_t \equiv p - p' = k - k' \). After the spin sum,

\[
\frac{1}{4} \sum_{s,s',r,r'} |M|^2 = \frac{e^4}{k^2} \left( -p_\mu p'_\nu - p'_\mu p_\nu - \eta_{\mu\nu}(-p \cdot p' + m_\epsilon^2) \right) \cdot \left( -k_\mu k'_\nu - k'_\mu k_\nu - \eta_{\mu\nu}(-k \cdot k' + m_\mu^2) \right)
\]

(3.32)

On the Mandelstam variables, this is just the permutation \((s, t, u) \rightarrow (t, u, s)\).

**Payoff: the Mott formula.** Recall other ways of figuring out the scattering cross section from a Coulomb potential from a point charge of charge \( ze \).

We think about scattering from a fixed electrostatic potential \( A_0 = \frac{ze}{r} \) and do classical mechanics. I can never remember how this goes. Instead, let’s just scatter an electron off a heavy charge, such as a muon. If the charge of the heavy object were \( z \) times that of the electron, we would multiply the amplitude by \( z \) and the cross section by \( z^2 \).

‘Heavy’ here means that we can approximate the CoM frame by its rest frame, and its initial and final energy as \( k'_0 = m_\mu, k_0 = \sqrt{m_\mu^2 + \vec{k}^2} = m_\mu + \frac{1}{2} \vec{k}^2/m_\mu + \cdots \approx m_\mu \). Also, this means the collision is approximately elastic, \( E' \approx E \). In the diagram of the kinematics at right, annoyingly, \( s \equiv \cos \theta, c \equiv \sin \theta \). (Sorry.)

This means that the muon-line tensor factor \( M_{\mu\nu} \) in (4.11) simplifies dramatically:

\[-\frac{1}{4} M_{\mu\nu} \simeq k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu} \left( \frac{k \cdot k' - m_\mu^2}{m_\mu^2 - m_\epsilon^2} \right) \simeq \delta_{\mu0} \delta_{\nu0} 2m_\mu^2. \]

In the electron line, we’ll need the ingredient

\[- p \cdot p' + m_\epsilon^2 = -E^2 + p^2 \cos \theta + m_\epsilon^2 = -\vec{p}^2(1 - \cos \theta). \quad (3.33)\]

So

\[
E^{\mu\nu} M_{\mu\nu} = 32m_\mu^2 E^{00} = 32m_\mu^2 (2E^2 + \eta^{00}(-p \cdot p' + m_\epsilon^2)) \overset{(3.33)}{=} 32m_\mu^2 (2E^2 - \vec{p}^2(1 - \cos \theta)) \overset{\text{trig}}{=} 32m_\mu^2 (E^2 - \vec{p}^2 \sin^2 \theta/2) \frac{\beta^2 = \vec{\beta}^2}{E^2} 64m_\mu^2 E^2 (1 - \beta^2 \sin^2 \theta/2). \]

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From the two-body phase space, the cross section is

\[ d\sigma = \frac{1}{v_{\text{rel}}} \frac{1}{2E} \frac{1}{2m_\mu} \frac{4z^2e^4}{t^2} \frac{m^2}{64m^2} E^2 (1 - \beta^2 \sin^2 \theta / 2) \frac{d\Omega}{16\pi^2} \frac{p}{E_{\text{total}}} \]

\[ E_{\text{total}} \sim \frac{4E}{\beta} z^2 e^4 \frac{(1 - \beta^2 \sin^2 \theta / 2)}{t^2} \frac{d\Omega}{16\pi^2}. \]

Noting that \( t = (p - p')^2 = -2p^2(1 - \cos \theta) \), we get

\[ \frac{d\sigma}{d\Omega_{\text{Mott}}} = z^2 \alpha^2 (1 - \beta^2 \sin^2 \theta / 2) \frac{4\beta^2 p^2 \sin^4 \theta / 2}{4\beta^2 p^2 \sin^4 \theta / 2}. \]

If we take \( \beta \ll 1 \) in this formula we get the Rutherford formula. Notice that it blows up at \( \theta \to 0 \). This is a symptom of the long-range nature of the Coulomb potential, i.e. the masslessness of the photon.

**Electron-proton scattering.** The answer is basically the same if we think of the heavy particle in (4.10) as a proton (we have to flip the sign of the charge but this gets squared away since there is no interference in this case). \( ep \to ep \) is called Rutherford scattering, for good reason\(^{25}\). More generally, the Mott formula applies to scattering electrons off of heavy pointlike charged particles. For \( ep \) collisions at high enough energies, this formula fails because the proton has structure. At even higher energies it works again because the electron scatters off pointlike, approximately free quarks.

**Electron-photon scattering.** In the case of the process \( e^-\gamma \leftarrow e^-\gamma \),\(^{26}\) we meet a new ingredient, namely external photons:

\[ iM = \begin{pmatrix} \gamma \mu \\ \bar{\Psi} / A \Psi \end{pmatrix}_{1} \begin{pmatrix} \bar{\Psi} / A \Psi \end{pmatrix}_{2} \equiv iM_s + iM_t \]

\[ = (-ie)^2 \epsilon_{\mu}^{\nu\sigma\tau} \bar{u}_3 \left( \gamma_\nu \frac{\gamma_\mu}{s - m^2} + \gamma_\mu \frac{i\gamma_\nu}{t - m^2} \gamma_\nu \right) u_2. \]

The two amplitudes have a relative plus since we only mucked with the photon contractions, they just differ by how the gamma matrices are attached. If you don’t believe me, draw the contractions on this:

\[ \langle \gamma e | (\bar{\Psi} A \Psi)_1 (\bar{\Psi} A \Psi)_2 | \gamma e \rangle \]

\(^{25}\)If you don’t know why, you should go read *Inward Bound*, by Abraham Pais, as soon as possible.\(^{26}\)which at high energy is called Compton scattering and at low energies is called Thomson scattering. Despite my previous curmudgeonly footnote chastising the innocent reader for an imagined incomplete knowledge of the history of science, I do have a hard time remembering which name goes where. Moreover, as much as I revere the contributions of many of these folks, I find that using their names makes me think about the people instead of the physics. No one owns the physics! It’s the same physics for lots of space aliens, too.
(I’m not going to TeX it, thank you).

Now, if we don’t measure the photon polarizations, we need

\[ P = \frac{1}{4} \sum_{\text{polarizations, spins}} |\mathcal{M}|^2. \]

The key ingredient is the completeness relation

\[ \sum_{i=1,2} \epsilon^i_{\mu}(k)\epsilon^i_{\nu}(k) = -\eta_{\mu\nu} + \text{something proportional to } k^{\mu}k^{\nu}. \]

We can do various incantations to find a definite coefficient of \( k^{\mu}k^{\nu} \), but it will not matter because of the Ward identity: anytime there is an external photon \( \epsilon(k)_{\mu} \), the amplitude is \( \mathcal{M} = \mathcal{M}_{\mu}\epsilon^{\mu}(k) \) and satisfies \( k^{\mu}\mathcal{M}_{\mu} = 0 \). Therefore, we can ignore the term about which I was vague and we have

\[ \sum_{\text{polarizations}} |\mathcal{M}|^2 = \sum_{i} \epsilon^{i*}_{\mu}\mathcal{M}^{\mu*}\mathcal{M}^{\nu}\epsilon^i_{\nu} = -\eta_{\mu\nu}\mathcal{M}_{\mu*}\mathcal{M}^{\nu} + (\text{terms with } \mathcal{M}_{\mu}k^{\mu}) \]

Don’t be scared of the minus sign, it’s because of the mostly minus signature, and makes the thing positive. But notice the opportunity to get negative probabilities if the gauge bosons don’t behave!

A dramatic process related by crossing to Compton scattering is pair annihilation, \( \mathcal{M}_{\gamma\gamma\rightarrow e^+e^-} \). See the end of Peskin §5, where he has a nice plot comparing to experimental data the result for \( \frac{d\sigma}{d\Omega} \) as a function of scattering angle.

[End of Lecture 8]
4 To infinity and beyond

At this point we are capable of successfully computing the amplitudes and cross-sections for many processes using QED. More precisely, we can do a good job of the leading-order-in-$\alpha$ amplitudes, using Feynman diagrams which are trees – no loops. The natural next step is to look at the next terms in the perturbation expansion in $\alpha$, which come from diagrams with one loop. When we do that we’re going to encounter some confusing stuff. A place we’ve already encountered this stuff is in the additive constant in the Hamiltonian; this has physical consequences as we’ll see in thinking about Casimir forces in §4.1.

We don’t encounter these short-distance issues in studying tree-level diagrams because in a tree-level diagram, the quantum numbers (and in particular the momenta) of the intermediate states are fixed by the external states. In contrast, once there is a loop, there are undetermined momenta which must be summed, and this sum includes, it seems, arbitrarily-high-momentum modes, about which surely we have no information yet.

In order to put ourselves in the right frame of mind to think about that stuff, we’ll make a brief retreat in §4.2 to systems with finitely many degrees of freedom.

Then we’ll apply some of these lessons to a toy field theory example (scalar field theory). Then we’ll come back to perturbation theory in QED. Reading assignment for this chapter: Zee §III.
4.1 Casimir effect: vacuum energy is real

[A. Zee, Quantum Field Theory in a Nutshell, §I.9] This subsection has two purposes. One is to show that the $\frac{1}{2}\hbar \omega$ energy of the vacuum of the quantum harmonic oscillator (which appeared in our discussion of quantum sound and light) is real. Sometimes we can get rid of it by choosing the zero of energy (which doesn’t matter unless we are studying dynamical gravity). But it is meaningful if we can vary $\omega$ (or the collection of $\omega$s in the case of many oscillators as for the radiation field) and compare the difference.

The other purpose is to give an object lesson in asking the right questions. In physics, the right question is often a question which can be answered by an experiment, at least in principle. The answers to such questions are less sensitive to our silly theoretical prejudices, e.g. about what happens to physics at very short distances.

In the context of the bunch of oscillators making up the radiation field, we can change the spectrum of frequencies of these oscillators $\{\omega_k\}$ by putting it in a box and varying the size of the box. In particular, two parallel conducting plates separated by some distance $d$ experience an attractive force from the change in the vacuum energy of the EM field resulting from their presence. The plates put boundary conditions on the field, and therefore on which normal modes are present.

To avoid some complications of E&M which are not essential for our point here, we’re going to make two simplifications:

- we’re going to solve the problem in 1+1 dimensions
- and we’re going to solve it for a scalar field.

To avoid the problem of changing the boundary conditions outside the plates we use the following device with three plates:

\[ | ← d → | ← L − d → | \]

(We will consider $L \gg d$, so we don’t really care about the far right plate.) The ‘perfectly conducting’ plates impose the boundary condition that our scalar field $q(x)$ vanishes there. The normal modes of the scalar field $q(x)$ in the left cavity are then

$$q_j = \sin \left( j\pi x/d \right), \quad j = 1, 2, ...$$

with frequencies $\omega_j = \frac{\pi j}{d}c$. There is a similar expression for the modes in the right cavity which we won’t need. We’re going to add up all the $\frac{1}{2}\hbar \omega$s for all the modes in both cavities to get the vacuum energy $E_0(d)$; the force on the middle plate is then $-\partial_d E_0$.  

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The vacuum energy in the whole region of interest between the outer plates is the sum of the vacuum energies of the two cavities

\[ E_0(d) = f(d) + f(L - d) \]

where

\[ f(d) = \frac{1}{2} \hbar \sum_{j=1}^{\infty} \omega_j = \hbar c \frac{\pi}{2d} \sum_{j=1}^{\infty} j \]

We have done something wrong. What?

Our crime is hubris: we assumed that we knew what the modes of arbitrarily large mode number \( k \) (arbitrarily short wavelength, arbitrarily high frequency) are doing, and in particular we assumed that they cared about our silly plates. In fact, no metal in existence can put boundary conditions on the modes of large enough frequency – those modes don’t care about \( d \). The reason a conductor puts boundary conditions on the EM field is that the electrons move around to compensate for an applied field, but there is a limit on how fast the electrons can move (e.g. the speed of light). The resulting cutoff frequency is called the plasma frequency but we don’t actually need to know about all these details. To parametrize our ignorance of what the high-frequency modes do, we must cut off (or regularize) the contribution of the high-frequency modes. Let’s call modes with \( \omega_j \gg \pi/a \) high frequency, where \( a \) is some short time\(^{27}\). Replace

\[ f(d) \sim f(a, d) = \hbar \frac{\pi}{2d} \sum_{j=1}^{\infty} e^{-\omega_j} \]

\[ = -\frac{\pi \hbar}{2} \partial_a \left( \sum_{j=1}^{\infty} e^{-aj} \right) \]

\[ = \frac{\pi \hbar}{2d} \frac{e^{aj}}{e^{aj} - 1} \]

\[ \sim \hbar \left( \frac{\pi d}{2a^2} - \frac{\pi}{24d} + \frac{\pi a^2}{480d^3} + \ldots \right) \] (4.1)

Answers which don’t depend on \( a \) have a chance of being meaningful. The thing we can measure is the force:

\[ F = -\partial_d E_0 = - (f'(d) - f'(L - d)) \]

\(^{27}\)You can think of \( a \) as the time it takes the waves to move by one lattice spacing. If we work in units where the velocity is \( c = 1 \), this is just the lattice spacing. I will do so for the rest of this discussion.
\[
\begin{align*}
\frac{a}{2a^2} + \frac{\pi}{24d^2} + O(a^2) - \left( \frac{\pi}{24 L - d^2} + O(a^2) \right)
&= -\hbar \left( \left( \frac{\pi}{2a^2} + \frac{\pi}{24d^2} + O(a^2) \right) - \left( \frac{\pi}{2a^2} + \frac{\pi}{24(L - d)^2} + O(a^2) \right) \right) \\
&\xrightarrow{a \to 0} -\frac{\pi \hbar}{24} \left( \frac{1}{d^2} - \frac{1}{(L - d)^2} \right) \\
&\equiv -\frac{\pi \hbar c}{24d^2} (1 + O(d/L)) . \tag{4.2}
\end{align*}
\]

This is an attractive force between the plates. (I put the \(c\) back in the last line.)

The analogous force between real conducting plates, caused by the change of boundary conditions on the electromagnetic field, has been measured.

The string theorists will tell you that \(\sum_{j=1}^{\infty} j = -\frac{1}{12}\), and our calculation above agrees with them in some sense. But what this foolishness means is that if we compute something which is not dependent on the cutoff we have to get the same answer no matter what cutoff we use. Notice that it is crucial to ask the right questions.

An important question is to what extent could we have picked a different cutoff function (instead of \(e^{-\pi \omega/a}\)) and gotten the same answer for the physics. This interesting question is answered affirmatively in Zee’s wonderful book, 2d edition, section I.9 (available electronically here!).

A comment about possible physical applications of the calculation we actually did: you could ask me whether there is such a thing as a Casimir force due to the vacuum fluctuations of phonons. Certainly it’s true that the boundary of a chunk of solid puts boundary conditions on the phonon modes, which change when we change the size of the solid. The problem with the idea that this might produce a measurable force (which would lead the solid to want to shrink) is that it is hard to distinguish the ‘phonon vacuum energy’ from the rest of the energy of formation of the solid, that is, the energy difference between the crystalline configuration of the atoms and the configuration when they are all infinitely separated. Certainly the latter is not well-described in the harmonic approximation (\(\lambda = 0\) in (1.1)).

**A few comments about the 3+1 dimensional case of E&M.** Assume the size of the plates is much larger than their separation \(L\). Dimensional analysis shows that the force per unit area from vacuum fluctuations must be of the form

\[
P = A \frac{\hbar c}{L^4}
\]

where \(A\) is a numerical number. \(A\) is not zero!

Use periodic boundary conditions in the \(xy\) planes (along the plates). The allowed
wave vectors are then
\[ \vec{k} = \left( \frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y} \right) \]
with \( n_x, n_y \) integers.

We have to do a bit of E&M here. Assume the plates are perfect conductors (this where the hubris about the high-frequency modes enters). This means that the transverse component of the electric field must vanish at the surface. Instead of plane waves in \( z \), we get standing waves: \( \phi(z) \propto \sin \left( n\pi z/L \right) \).

The frequencies of the associated standing waves are then
\[ \omega_n(\vec{k}) = c \sqrt{\frac{\pi^2 n^2}{L^2} + \vec{k}^2}, n = 0, 1, 2 \]
Also, there is only one polarization state for \( n = 0 \).

So the zero-point energy is
\[ E_0(L) = \frac{\hbar}{2} \left( 2 \sum_{n,\vec{k}}' \omega_n(\vec{k}) \right) \]
where it’s useful to define
\[ \sum_{n,\vec{k}}' \equiv \frac{1}{2} \sum_{n=0,\vec{k}} + \sum_{n \geq 1, \vec{k}} \]
Now you can imagine introducing a regulator like the one we used above, and replacing
\[ \sum_{n,\vec{k}}' \sim \sum_{n,\vec{k}} e^{-a\omega_n(\vec{k})/\pi} \]
and doing the sums and integrals and extracting the small-\( a \) behavior.

### 4.2 A parable from quantum mechanics on the breaking of scale invariance

Recall that the coupling constant in \( \phi^4 \) theory in \( D = 3 + 1 \) spacetime dimensions is dimensionless, and the same is true of the electromagnetic coupling \( e \) in QED in \( D = 3+1 \) spacetime dimensions. In fact, the mass parameters are the only dimensionful quantities in those theories, at least in their classical avatars. This means that if we ignore the masses, for example because we are interested in physics at much higher energies, then these models seem to possess scale invariance: the physics is unchanged under zooming in.
Here we will study a simple quantum mechanical example (that is: an example with a finite number of degrees of freedom) with such (classical) scale invariance. It exhibits many interesting features that can happen in strongly interacting quantum field theory – asymptotic freedom, dimensional transmutation. Because the model is simple, we can understand these phenomena without resort to perturbation theory. They will nevertheless illuminate some ways of thinking which we’ll need in examples where perturbing is our only option.

Consider the following (‘bare’) action:

\[ S[q] = \int dt \left( \frac{1}{2} \dot{\vec{q}}^2 + g_0 \delta^{(2)}(\vec{q}) \right) \equiv \int dt \left( \frac{1}{2} \dot{\vec{q}}^2 - V(\vec{q}) \right) \]

where \( \vec{q} = (x, y) \) are two coordinates of a quantum particle, and the potential involves \( \delta^{(2)}(\vec{q}) \equiv \delta(x)\delta(y) \), a Dirac delta function. I chose the sign so that \( g_0 > 0 \) is attractive.

(Notice that I have absorbed the inertial mass \( m \) in \( \frac{1}{2}mv^2 \) into a redefinition of the variable \( q, q \rightarrow \sqrt{m}q \).)

First, let’s do dimensional analysis (always a good idea). Since \( \hbar = c = 1 \), all dimensionful quantities are some power of a length. Let \( -[X] \) denote the number of powers of length in the units of the quantity \( X \); that is, if \( X \sim (\text{length})^\nu \) then we have \( -[X] = -\nu \). We have:

\[ [t] = [\text{length}/c] = -1 \implies [dt] = -1. \]

The action appears in the exponent in the path integrand, and is therefore dimensionless (it has units of \( \hbar \)), so we had better have:

\[ 0 = [S] = [\hbar] \]

and this applies to each term in the action. We begin with the kinetic term:

\[ 0 = \int dt \dot{\vec{q}}^2 \implies \]

\[ \dot{\vec{q}}^2 = +1 \implies [\vec{q}] = +\frac{1}{2} \implies [\vec{q}] = -\frac{1}{2}. \]

Since \( 1 = \int dq \delta(q) \), we have \( 0 = [dq] + [\delta(q)] \) and

\[ [\delta^D(\vec{q})] = -[\vec{q}]D = \frac{D}{2}, \quad \text{and in particular } [\delta^2(\vec{q})] = 1. \]

This implies that the naive (“engineering”) dimensions of the coupling constant \( g_0 \) are \( [g_0] = 0 \) – it is dimensionless. Classically, the theory does not have a special length scale; it is scale invariant.

\[ ^{28}\text{I learned this example from Marty Halpern.} \]
The Hamiltonian associated with the Lagrangian above is
\[ H = \frac{1}{2} \left( p_x^2 + p_y^2 \right) + V(\vec{q}). \]

Now we treat this as a quantum system. Acting in the position basis, the quantum Hamiltonian operator is
\[ H = -\frac{\hbar^2}{2} \left( \partial_x^2 + \partial_y^2 \right) - g_0 \delta^{(2)}(\vec{q}) \]

So in the Schrödinger equation \[ H \psi = \left( -\frac{\hbar^2}{2} \nabla^2 + V(\vec{q}) \right) \psi = E \psi, \] the second term on the LHS is
\[ V(\vec{q}) \psi(\vec{q}) = -g_0 \delta^{(2)}(\vec{q}) \psi(0). \]

To make it look more like we are doing QFT, let’s solve it in momentum space:
\[ \psi(\vec{q}) \equiv \int \frac{d^2p}{(2\pi \hbar)^2} e^{i\vec{p} \cdot \vec{q}/\hbar} \varphi(\vec{p}) \]

The delta function is
\[ \delta^{(2)}(q) = \int \frac{d^2p}{(2\pi \hbar)^2} e^{i\vec{p} \cdot \vec{q}/\hbar}. \]

So the Schrödinger equation says
\[ \left( -\frac{\hbar^2}{2} \nabla^2 - E \right) \psi(q) = -V(q) \psi(q) \]
\[ \int d^2p e^{ip \cdot q} \left( \frac{p^2}{2} - E \right) \varphi(p) = +g_0 \delta^{(2)}(q) \psi(0) \]
\[ = +g_0 \left( \int d^2p e^{ip \cdot q} \right) \psi(0) \quad (4.3) \]

which (integrating the both-hand side of (4.3) over \( q \): \( \int d^2q e^{-ip \cdot q} ((4.3)) \) ) says
\[ \left( \frac{p^2}{2} - E \right) \varphi(\vec{p}) = +g_0 \int \frac{d^2p'}{(2\pi \hbar)^2} \varphi(\vec{p}') \]
\[ = \psi(0) \]

There are two cases to consider:

- \( \psi(\vec{q} = 0) = \int d^2p \varphi(\vec{p}) = 0. \) Then this case is the same as a free theory, with the constraint that \( \psi(0) = 0, \)
\[ \left( \frac{p^2}{2} - E \right) \varphi(\vec{p}) = 0 \]

\( i.e. \) plane waves which vanish at the origin, \( e.g. \) \( \psi \propto \sin \frac{p_x x}{\hbar} e^{\pm ip_y y/\hbar}. \) These scattering solutions don’t see the delta-function potential at all.
• $\psi(0) \equiv \alpha \neq 0$, some constant to be determined. This means $\vec{p}^2/2 - E \neq 0$, so we can divide by it:

$$\varphi(\vec{p}) = \frac{g_0}{\frac{\vec{p}^2}{2} - E} \left( \int d^2p' \varphi(\vec{p}') \right) = \frac{g_0}{\frac{\vec{p}^2}{2} - E} \alpha.$$  

The integral of the RHS (for $\psi(0) = \alpha$) is a little problematic if $E > 0$, since then there is some value of $p$ where $p^2 = 2E$. Avoid this singularity by going to the boundstate region: consider $E = -\epsilon_B < 0$. So:

$$\varphi(\vec{p}) = \frac{g_0}{\frac{\vec{p}^2}{2} + \epsilon_B} \alpha.$$  

What happens if we integrate this $\int d^2p$ to check self-consistency – the LHS should give $\alpha$ again:

$$0 = \int d^2p \varphi(\vec{p}) \left( 1 - \int d^2p \frac{g_0}{\frac{\vec{p}^2}{2} + \epsilon_B} \right)$$  

$$\implies \int d^2p \frac{g_0}{\frac{\vec{p}^2}{2} + \epsilon_B} = 1$$

is a condition on the energy $\epsilon_B$ of possible boundstates.

But there’s a problem: the integral on the LHS behaves at large $p$ like

$$\int \frac{d^2p}{p^2} = \infty.$$  

At this point in an undergrad QM class, you would give up on this model. In QFT we don’t have that luxury, because this kind of thing happens all over the place. Here’s what we do instead.

We cut off the integral at some large $p = \Lambda$:

$$\int^\Lambda \frac{d^2p}{p^2} \sim \log \Lambda.$$  

This our first example of the general principle that a classically scale invariant system will exhibit logarithmic divergences (rather: logarithmic dependence on the cutoff). It’s the only kind allowed by dimensional analysis.

The introduction of the cutoff can be thought of in many ways: we could say there are no momentum states with $|p| > \Lambda$, or maybe we could say that the potential is not really a delta function if we look more closely. The choice of narrative here shouldn’t affect our answers to physics questions at energies far below the cutoff.
More precisely:

\[ \int_0^\Lambda \frac{d^2p}{p^2 + \epsilon_B} = 2\pi \int_0^\Lambda \frac{pdp}{p^2 + \epsilon_B} = 2\pi \log \left( 1 + \frac{\Lambda^2}{2\epsilon_B} \right). \]

So in our cutoff theory, the boundstate condition is:

\[ 1 = g_0 \int_0^\Lambda \frac{d^2p}{p^2 + \epsilon_B} = \frac{g_0}{2\pi \hbar^2} \log \left( 1 + \frac{\Lambda^2}{2\epsilon_B} \right). \]

A solution only exists for \( g_0 > 0 \). This makes sense since only then is the potential attractive (recall that \( V = -g_0\delta \)).

[End of Lecture 9]

Now here’s a trivial-seeming step that offers a dramatic new vista: solve for \( \epsilon_B \).

\[ \epsilon_B = \frac{\Lambda^2}{2} \frac{1}{e^{\frac{2\pi\hbar^2}{g_0}} - 1}. \] (4.4)

As we remove the cutoff (\( \Lambda \to \infty \)), we see that \( E = -\epsilon_B \to -\infty \), the boundstate becomes more and more bound – the potential is too attractive.

Suppose we insist that the boundstate energy \( \epsilon_B \) is a fixed thing – imagine we’ve measured it to be 200 MeV\(^{29}\). We should express everything in terms of the measured quantity. Then, given some cutoff \( \Lambda \), we should solve for \( g_0(\Lambda) \) to get the boundstate energy we have measured:

\[ g_0(\Lambda) = \frac{2\pi\hbar^2}{\log \left( 1 + \frac{\Lambda^2}{2\epsilon_B} \right)}. \]

This is the crucial step: this silly symbol \( g_0 \) which appeared in our action doesn’t mean anything to anyone (see Zee’s dialogue with the S.E. in section III). We are allowing \( g_0 \equiv \text{the bare coupling to be cutoff-dependent} \).

Instead of a dimensionless coupling \( g_0 \), the useful theory contains an arbitrary dimensional coupling constant (here \( \epsilon_B \)). This phenomenon is called dimensional transmutation (d.t.). The cutoff is supposed to go away in observables, which depend on \( \epsilon_B \) instead.

In QCD we expect that in an identical way, an arbitrary scale \( \Lambda_{QCD} \) will enter into physical quantities. (If QCD were the theory of the whole world, we would work in units where it was one.) This can be taken to be the rest mass of some mesons – boundstates of quarks. Unlike this example, in QCD there are many boundstates, but their energies are dimensionless multiplies of the one dimensionful scale, \( \Lambda_{QCD} \). Nature chooses \( \Lambda_{QCD} \approx 200 \text{ MeV} \).

\(^{29}\)Spoiler alert: I picked this value of energy to stress the analogy with QCD.
[This d.t. phenomenon was maybe first seen in a perturbative field theory in S. Coleman, E. Weinberg, *Phys Rev* **D7** (1973) 1898. We’ll come back to their example.]

There are more lessons in this example. Go back to (4.4):

\[ \epsilon_B = \frac{\Lambda^2}{2} \frac{1}{e^{2\pi\epsilon^2/\lambda_0}} - 1 \neq \sum_{n=0}^{\infty} g_0^n f_n(\Lambda) \]

it is not analytic (i.e. a power series) in \( g_0(\Lambda) \) near small \( g_0 \); rather, there is an essential singularity in \( g_0 \). (All derivatives of \( \epsilon_B \) with respect to \( g_0 \) vanish at \( g_0 = 0 \).) You can’t expand the dimensionful parameter in powers of the coupling. This means that you’ll never see it in perturbation theory in \( g_0 \). Dimensional transmutation is an inherently non-perturbative phenomenon.

Look at how the bare coupling depends on the cutoff in this example:

\[ g_0(\Lambda) = \frac{2\pi\hbar^2}{\log(1 + \Lambda^2/2\epsilon_B)} \xrightarrow{\Lambda^2 \gg 2\epsilon_B} \frac{2\pi\hbar^2}{\log(\frac{\Lambda^2}{2\epsilon_B})} \xrightarrow{\Lambda^2 \gg 2\epsilon_B} 0 \]

– the bare coupling vanishes in this limit, since we are insisting that the parameter \( \epsilon_B \) is fixed. This is called asymptotic freedom (AF): the bare coupling goes to zero (i.e. the theory becomes free) as the cutoff is removed. This also happens in QCD.

**RG flow equations.** Define the beta-function as the logarithmic derivative of the bare coupling with respect to the cutoff:

**Def:** \[ \beta(g_0) \equiv \Lambda \frac{\partial}{\partial \Lambda} g_0(\Lambda) \].

For this theory

\[ \beta(g_0) = \Lambda \frac{\partial}{\partial \Lambda} \left( \frac{2\pi\hbar^2}{\log(1 + \Lambda^2/2\epsilon_B)} \right) \xrightarrow{\text{calculate}} - \frac{g_0^2}{\pi\hbar^2} \left( 1 \xrightarrow{\text{perturbative}} - e^{-2\pi\hbar^2/g_0} \xrightarrow{\text{not perturbative}} \right) . \]

Notice that it’s a function only of \( g_0 \), and not explicitly of \( \Lambda \). Also, in this simple toy theory, the perturbation series for the beta function happens to stop at order \( g_0^2 \).

\( \beta \) measures the failure of the cutoff to disappear from our discussion – it signals a quantum mechanical violation of scale invariance. What’s \( \beta \) for? Flow equations:

\[ \dot{g}_0 = \beta(g_0) . \]
This is a tautology. The dot is
\[ \dot{A} = \partial_s A, \quad s \equiv \log \Lambda/\Lambda_0 \implies \partial_s = \Lambda \partial_\Lambda. \]
(\( \Lambda_0 \) is some reference scale.) But forget for the moment that this is just a definition:
\[ \dot{g}_0 = -\frac{g_0^2}{\pi \hbar^2} \left( 1 - e^{-2\pi \hbar^2/g_0} \right). \]
This equation tells you how \( g_0 \) changes as you change the cutoff. Think of it as a nonlinear dynamical system (fixed points, limit cycles...)

**Def:** A fixed point \( g_0^* \) of a flow is a point where the flow stops:
\[ 0 = \dot{g}_0|_{g_0^*} = \beta(g_0^*), \]
a zero of the beta function. (Note: if we have many couplings \( g_i \), then we have such an equation for each \( g_i: \dot{g}_i = \beta_i(g) \). So \( \beta_i \) is (locally) a vector field on the space of couplings.)

Where are the fixed points in our example?
\[ \beta(g_0) = -\frac{g_0^2}{\pi \hbar^2} \left( 1 - e^{-2\pi \hbar^2/g_0} \right). \]
There’s only one: \( g_0^* = 0 \), near which \( \beta(g_0) \sim -\frac{g_0^2}{\pi \hbar^2} \), the non-perturbative terms are small. What does the flow look like near this point? For \( g_0 > 0 \), \( \dot{g}_0 = \beta(g_0) < 0 \). With this (high-energy) definition of the direction of flow, \( g_0 = 0 \) is an attractive fixed point:
\[ *=<-<-<-<-<-<-<-<-<-<-<-<-<-<-<-<>-<>-<>-<>-<>-<>-<>-<>-<>-<>-<>-<>-<>-<>\]
\( g_0^* = 0 \).

We already knew this. It just says \( g_0(\Lambda) \sim \frac{1}{\log \Lambda^2} \to 0 \) at large \( \Lambda \). A lesson is that in the vicinity of such an AF fixed point, the non-perturbative stuff \( e^{-2\pi \hbar^2/g_0} \) is small. So we can get good results near the fixed point from the perturbative part of \( \beta \). That is: we can compute the behavior of the flow of couplings near an AF fixed point *perturbatively*, and be sure that it is an AF fixed point. This is the situation in QCD.

---

30 Warning: The sign in this definition carries a great deal of cultural baggage. With the definition given here, the flow (increasing \( s \)) is toward the UV, toward high energy. This is the high-energy particle physics perspective, where we learn more physics by going to higher energies. As we will see, there is a strong argument to be made for the other perspective, that the flow should be regarded as going from UV to IR, since we lose information as we move in that direction — in fact, the IR behavior does not determine the UV behavior in general, but UV does determine IR.
On the other hand, the d.t. phenomenon that we’ve shown here is something that we can’t prove in QCD. However, the circumstantial evidence is very strong!

Another example where this happens is quantum mechanics in any number of variables with a central potential \( V = -\frac{g_0^2}{r^2} \). It is also classically scale invariant:

\[
[r] = -\frac{1}{2}, \quad \left[ \frac{1}{r^2} \right] = +1 \quad \Rightarrow \quad [g_0] = 0.
\]

This model was studied in K.M. Case, *Phys Rev* 80 (1950) 797 and you will study it on the first homework. The resulting boundstates and d.t. phenomenon are called Efimov states; this model preserves a *discrete* scale invariance.

Here’s a quote from Marty Halpern from his lecture on this subject:

*I want you to study this set of examples very carefully, because it’s the only time in your career when you will understand what is going on.*

In my experience it’s been basically true. For real QFTs, you get distracted by Feynman diagrams, gauge invariance, regularization and renormalization schemes, and the fact that you can only do perturbation theory.
4.3 A simple example of perturbative renormalization in QFT

[Zee §III.1, Schwartz §15.4] Now let’s consider an actual field theory but a simple one, namely the theory of a real scalar field in four dimensions, with

$$\mathcal{L} = -\frac{1}{2} \phi \Box \phi - m^2 \phi^2 - \frac{g}{4!} \phi^4.$$  \hspace{1cm} (4.5)

Recall that $[\phi] = \frac{D-2}{2}$ so $[m] = 1$ and $[g] = \frac{4-D}{2}$, so $g$ is dimensionless in $D = 4$. As above, this will mean *logarithms*!

Let’s do $2 \leftrightarrow 2$ scattering of $\phi$ particles.

$$i\mathcal{M}_{2\leftrightarrow2} = \left( -ig + i\mathcal{M}_s + i\mathcal{M}_t + i\mathcal{M}_u \right) + \mathcal{O}(g^3)$$

where, in terms of $q_s \equiv k_1 + k_2$, the $s$-channel 1-loop amplitude is

$$i\mathcal{M}_s = \frac{1}{2}(-ig)^2 \int d^4k \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(q_s - k)^2 - m^2 + i\epsilon} \sim \int \Lambda d^4k \frac{1}{k^4}.$$  

**Parametrizing ignorance.** Recall our discovery of the scalar field at the beginning of the quarter by starting with a chain of springs, and looking at the long-wavelength (small-wavenumber) modes. In the sum, $\int d^4k$, the region of integration that’s causing the trouble is *not* the part where the system looks most like a field theory. That is: if we look closely enough (small enough $1/k$), we will see that the mattress is made of springs. In terms of the microscopic description with springs, there is a smallest wavelength, of order the inverse lattice spacing; the sum stops.

Field theories arise from many such models, which may differ dramatically in their short-distance physics. We’d like to not worry too much about which one, but rather say things which do not depend on this choice. Recall the discussion of the Casimir force from §1: in that calculation, many different choices of regulators for the mode sum corresponded to different material properties of the conducting plates. The leading Casimir force was independent of this choice; more generally, it is an important part of the physics problem to identify which quantities are UV sensitive and which are not.

Parametrizing ignorance is another way to say ‘doing science’. In the context of field theory, at least in the high-energy community, it is called ‘regularization’. 

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Now we need to talk about the integral a little more. The part which is causing the trouble is the bit with large $k$, which might as well be $|k| \sim \Lambda \gg m$, so let’s set $m = 0$ for simplicity.

We’ll spend lots of time learning to do integrals below. Here’s the answer:

$$i\mathcal{M} = -ig + iCg^2\left(\log \frac{\Lambda^2}{s} + \log \frac{\Lambda^2}{t} + \log \frac{\Lambda^2}{u}\right) + \mathcal{O}(g^3)$$

If you must know, $C = \frac{1}{16\pi^2}$.

**Observables can be predicted from other observables.** Again, the boldface statement might sound like some content-free tweet from some boring philosophy-of-science twitter feed, but actually it’s a very important thing to remember here.

What is $g$? As Zee’s Smart Experimentalist says, it is just a letter in some theorist’s lagrangian, and it doesn’t help anyone to write physical quantities in terms of it. Much more useful would be to say what is the scattering amplitude in terms of things that can be measured. So, suppose someone scatters $\phi$ particles at some given $(s, t, u) = (s_0, t_0, u_0)$, and finds for the amplitude $i\mathcal{M}(s_0, t_0, u_0) = -ig_P$ where $P$ is for ‘physical’.\(^{31}\) This we can relate to our theory letters:

$$-ig_P = i\mathcal{M}(s_0, t_0, u_0) = -ig + iCg^2L_0 + \mathcal{O}(g^3) \quad (4.6)$$

where $L_0 \equiv \log \frac{\Lambda^2}{s_0} + \log \frac{\Lambda^2}{t_0} + \log \frac{\Lambda^2}{u_0}$. (Note that quantities like $g_P$ are often called $g_R$ where ‘R’ is for ‘renormalized,’ whatever that is.)

**Renormalization.** Now here comes the big gestalt shift: Solve this equation $(4.6)$ for the stupid letter $g$

$$-ig = -ig_P - iCg^2L_0 + \mathcal{O}(g^3)$$

$$= -ig_P - iCg_P^2L_0 + \mathcal{O}(g_P^3). \quad (4.7)$$

and eliminate $g$ from the discussion:

$$i\mathcal{M}(s, t, u) = -ig + iCg^2L + \mathcal{O}(g^3)$$

$$\quad \overset{(4.7)}{=} -ig_P - iCg_P^2L_0 + iCg_P^2L + \mathcal{O}(g_P^3)$$

$$\quad = -ig_P + iCg_P^2\left(\log \frac{s_0}{s} + \log \frac{t_0}{t} + \log \frac{u_0}{u}\right) + \mathcal{O}(g_P^3). \quad (4.8)$$

\(^{31}\)You might hesitate here about my referring to the amplitude $\mathcal{M}$ as an ‘observable’. The difficult and interesting question of what can actually be measured in experiments can be decoupled a bit from this discussion. I’ll say more later, but if you are impatient see the beginning of Schwartz, chapter 18.

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This expresses the amplitude at any momenta (within the range of validity of the theory!) in terms of measured quantities, \(g_P, s_0, t_0, u_0\). The cutoff \(\Lambda\) is gone! Just like in our parable in §4.2, it was eliminated by letting the coupling vary with it, \(g = g(\Lambda)\), according to (4.7). We’ll say a lot more about how to think about that dependence.

**Renormalized perturbation theory.** To slick up this machinery, consider the following Lagrangian density (in fact the same as (4.5), with \(m = 0\) for simplicity):

\[
\mathcal{L} = -\frac{1}{2} \phi \Box \phi - \frac{g_P}{4!} \phi^4 - \frac{\delta g}{4!} \phi^4
\]

(4.9)

but written in terms of the measured coupling \(g_P\), and some as-yet-undetermined ‘counterterm’ \(\delta g\). Then

\[
\mathcal{M}(s, t, u) = -g_P - \delta g - Cg_P^2 \left( \log \frac{s}{\Lambda^2} + \log \frac{t}{\Lambda^2} + \log \frac{u}{\Lambda^2} \right) + \mathcal{O}(g_P^3).
\]

If, in order to enforce the renormalization condition \(\mathcal{M}(s_0, t_0, u_0) = -g_P\), we choose

\[
\delta g = -g_P^2 C \left( \log \frac{s_0}{\Lambda^2} + \log \frac{t_0}{\Lambda^2} + \log \frac{u_0}{\Lambda^2} \right)
\]

then we find

\[
\mathcal{M}(s, t, u) = -g_P - Cg_P^2 \left( \log \frac{s}{s_0} + \log \frac{t}{t_0} + \log \frac{u}{u_0} \right) + \mathcal{O}(g_P^3)
\]

– all the dependence on the unknown cutoff is gone, we satisfy the observational demand \(\mathcal{M}(s_0, t_0, u_0) = -g_P\), and we can predict the scattering amplitude (and others!) at any momenta.

The only price is that the ‘bare coupling’ \(g\) depends on the cutoff, and becomes infinite if we pretend that there is no cutoff. Happily, we didn’t care about \(g\) anyway. We can just let it go.

The step whereby we were able to absorb all the dependence on the cutoff into the bare coupling constant involved some apparent magic. It is not so clear that the same magic will happen if we study the next order \(\mathcal{O}(g_P^3)\) terms, or if we study other amplitudes. A QFT where all the cutoff dependence to all orders can be removed with a finite number of counterterms is called ‘renormalizable’. As we will see, such a field theory is less useful because it allows us to pretend that it is valid up to arbitrarily high energies. The alternative, where we must add more counterterms (such as something like \(\frac{\delta g}{\Lambda^2} \phi^6\) at each order in perturbation theory, is called an **effective field theory**, which is a field theory that has the decency to predict its regime of validity.

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4.4 Radiative corrections to the Mott formula

Recall that by studying scattering of an electron from a heavy charged fermion (a muon is convenient) we reconstructed the cross section for scattering off a Coulomb potential (named after Mott). Our next goal is to figure out how this cross section is corrected by other QED processes.

Recall that

$$iM = \left( -ie\bar{u}(p')\gamma^\mu u(p) \right)_{\text{electrons}} \frac{-i(\eta_{\mu\nu} - \frac{(1-\zeta)q'_\mu q'_\nu}{q'^2})}{q^2} \left( -ie\bar{u}(k)\gamma^\nu u(k') \right)_{\text{muons}}$$

with $q_t \equiv p - p' = k - k'$. After the spin sum,

$$\frac{1}{4} \sum_{s,s',r,r'} |M|^2 = 4e^4 \frac{1}{\ell^2} \left( -p_\mu p'_\nu - p'_\mu p_\nu - \eta_{\mu\nu}(-p \cdot p' + m_e^2) \right) \cdot \left( -k_\mu k'_\nu - k'_\mu k_\nu - \eta_{\mu\nu}(-k \cdot k' + m_\mu^2) \right)$$

Consider the limit where the target $\mu$ particle is much heavier than the electron. 'Heavy' here means that we can approximate the CoM frame by its rest frame, and its initial and final energy as $k'_0 = m_\mu, k_0 = \sqrt{m_\mu^2 + \vec{k}^2} = m_\mu + \frac{1}{2}\frac{k^2}{m_\mu} + \cdots \approx m_\mu$. Also, this means the collision is approximately elastic. In the diagram of the kinematics at right, $c \equiv \cos \theta, s \equiv \sin \theta$.

The answer we found after some boiling was:

$$\frac{d\sigma}{d\Omega_{\text{Mott}}} = \frac{\alpha^2(1 - \beta^2 \sin^2 \theta/2)}{4\beta^2 p^2 \sin^4 \theta/2}.$$  

If we take $\beta \ll 1$ in this formula we get the Rutherford formula.

**Radiative corrections.** Now it’s time to think about perturbative corrections to this cross section. Given that the leading-order calculation reproduced the classical physics of the Coulomb potential, you can think of what we are doing as effectively discovering (high-energy or short-distance) quantum corrections to the Coulomb law. The diagrams we must include are these (I made the muon lines thicker and also red):

$$iM_{e\mu-\mu e} =$$

\[\begin{pmatrix}
\end{pmatrix} + \begin{pmatrix}
\end{pmatrix} + \begin{pmatrix}
\end{pmatrix} \]
What do the one-loop diagrams in the second line have in common? They have an internal muon line. Why does this matter? When the energy going through the line is much smaller than the muon mass, then the propagator is \( \frac{i \not{k} + m_\mu}{k^2 - m_\mu^2} \sim \frac{1}{m_\mu} \) and its relative contribution is down by \( k/m_\mu \ll 1 \). So let’s neglect these for now.

Why don’t we include diagrams like ? The LSZ formula tells us that their effects on the \( S \)-matrix are accounted for by the wavefunction renormalization factors \( Z \)

\[
S_{e_\mu-e_\mu} = \sqrt{Z_e} \sqrt{Z_\mu} \left( \left( \begin{array}{c} \text{amputated, on-shell} \\ \end{array} \right) \right)
\]

and in determining the locations of the poles whose residues are the \( S \)-matrix elements.

Notice that the one-loop amplitudes are suppressed relative to the tree level amplitude by two factors of \( e \), hence one factor of the fine structure constant \( \alpha = \frac{e^2}{4\pi} \). Their leading effects on the cross section come from

\[
\sigma \sim \left| + \left( \begin{array}{c} \text{amputated, on-shell} \\ \end{array} \right) + \cdots \right|^2 \sim \sigma_{\text{tree}} + \mathcal{O}(\alpha^3)
\]

from the cross term between the tree and one-loop amplitudes.

In the above discussion, we encounter all three ‘primitive’ one-loop divergent amplitudes of QED, which we’ll study in turn:

- electron self-energy:

- vertex correction:

- vacuum polarization (photon self-energy):
4.5 Electron self-energy in QED

Let’s think about the electron two-point function in momentum space:

\[
\tilde{G}^{(2)}(p) = \sum + \quad + \quad + \quad + \quad \ldots
\]

As we did for the scalar field theory in §3 last quarter, we will denote the 1PI two-point function by

\[
-i\Sigma(p) \equiv \quad \text{a blob with nubbins; for fermions with conserved particle number, the nubbins carry arrows indicating the particle number flow. Let me call the tree level propagator}
\]

\[
iS(p) \equiv \frac{i(p + m_0)}{p^2 - m_0^2 + i\epsilon} = \frac{i}{p - m_0}
\]

– notice that I added a demeaning subscript to the notation for the mass appearing in the Lagrangian. Foreshadowing.

The full two point function is then:

\[
\tilde{G}^{(2)}(p) = iS + iS(-i\Sigma(p))iS + iS(-i\Sigma(p))iS(-i\Sigma(p))iS + \ldots
\]

\[
= iS (1 + \sum S + \Sigma S \Sigma S + \ldots) = iS \frac{1}{1 - \Sigma S}
\]

\[
= \frac{i}{p - m_0} \frac{1}{1 - \sum \frac{1}{p - m_0}} = \frac{i}{p - m_0 - \Sigma(p)}.
\]

Are you worried about these manipulations because \(\Sigma\) and \(S\) are matrices in the spinor indices? Don’t be: they are both made entirely from \(p\), and therefore they commute;
we could do these manipulations in the eigenbasis of $\not{p}$. This fully corrected propagator
has a pole at
\[ \not{p} = m \equiv m_0 + \Sigma(m) \]  
(4.14)
This means that the actual mass of the particle is this new quantity $m$. But what is
$m$ (it is called the ‘renormalized mass’)? To figure it out, we need to know about $\Sigma$.

In QED we must study $\Sigma$ in perturbation theory. As you can see from (4.12), the
leading (one-loop) contribution is
\[ -i \Sigma_2(p) = \left( -ie \right)^2 \int 4 d^4 k \ \gamma^\mu \frac{i(k + m_0)}{k^2 - m_0^2 + ie} \frac{-i \eta_{\mu\nu}}{(p-k)^2 - \mu^2 + ie}. \]
Notice that I am relying on the Ward identity to enforce the fact that only the traverse
bit of the photon propagator matters. Also, I added a mass $\mu$ for the photon as an
IR regulator. We must keep the external momentum $p$ arbitrary, since we don’t even
know where the mass-shell is!

Finally, I can’t put it off any longer: how are we going to do this loop-momentum
integral?

Step 1: Feynman parameter trick. It is a good idea to consider the integral
\[ \int_0^1 dx \frac{1}{(xA + (1-x)B)^2} = \int_0^1 dx \frac{1}{x(A-B) + B} = \frac{1}{A-B} \left( -\frac{1}{A} + \frac{1}{B} \right) = \frac{1}{AB}. \]
This allows us to combine the denominators into one:
\[ I = \frac{1}{k^2 - m_0^2 + ie (p-k)^2 - \mu^2 + ie} = \int_0^1 dx \frac{1}{x ((p^2 - 2pk + k^2) - \mu^2 + ie) + (1-x)(k^2 - m_0^2 + ie))^2} \]

Step 2: Now we can complete the square
\[ I = \int_0^1 dx \frac{1}{\left( (k-px)^2 - \Delta + ie \right)^2} \]
with
\[ \ell^\mu \equiv k^\mu - p^\mu x, \ \Delta \equiv +p^2 x^2 + x\mu^2 - xp^2 + (1-x)m_0^2 = x\mu^2 + (1-x)m_0^2 - x(1-x)p^2. \]
Step 3: Wick rotate. Because of the \( i \epsilon \) we’ve been dutifully carrying around, the poles of the \( p^0 \) integral don’t occur in the first and third octants of the complex \( p^0 \) plane. (And the integrand decays at large \( |p^0| \).) This means that we can rotate the contour to euclidean time for free: \( \ell^0 \equiv i \ell^4 \). Equivalently: the integral over the contour at right vanishes, so the real time contour gives the same answer as the (upward-directed) Euclidean contour.

Notice that \( \ell^2 = -\ell^2_E \). Altogether

\[-i \Sigma_2(p) = -e^2 \int d^4 \ell \int_0^1 dx \frac{N}{(\ell^2 - \Delta + i\epsilon)^2} = -e^2 \int_0^1 dx i \int d^4 \ell_E \frac{N}{(\ell^2_E + \Delta)^2} \]

where the numerator is

\[ N = \gamma^\mu (\ell + x\slashed{\Phi} + m_0) \gamma_\mu = -2 (\ell + x\slashed{\Phi}) + 4m_0. \]

Here I used two Clifford algebra facts: \( \gamma^\mu \gamma_\mu = 4 \) and \( \gamma^\mu \gamma_\mu = -2\slashed{\Phi} \). Think about the contribution from the term with \( \ell \) in the numerator: everything else is invariant under rotations of \( \ell \)

\[ d^4 \ell_E = \frac{1}{(2\pi)^4} d\Omega_3 \ell^3 d\ell = \frac{d\Omega_3}{(2\pi)^4} \ell^2 \frac{d\ell^2}{2}, \]

so this averages to zero. The rest is of the form (using \( \int_{S^3} d\Omega_3 = 2\pi^2 \))

\[ \Sigma_2(p) = e^2 \int_0^1 dx \int \frac{\ell^2 d\ell^2 (2\pi^2)}{2} \frac{2(2m_0 - x\Phi)}{(\ell^2 + \Delta)^2} \]

\[ = \frac{e^2}{8\pi^2} \int_0^1 dx (2m_0 - x\Phi) J \]

with

\[ J = \int_0^\infty d\ell^2 \frac{\ell^2}{(\ell^2 + \Delta)^2}. \]

In the large \( \ell \) part of the integrand this is

\[ \int^{\Lambda} d\ell^2 \frac{\ell^2}{\ell^2} \sim \log \Lambda. \]

You knew this UV divergence was coming. To be more precise, let’s add zero:

\[ J = \int d\ell^2 \left( \frac{\ell^2 + \Delta}{(\ell^2 + \Delta)^2} - \frac{\Delta}{(\ell^2 + \Delta)^2} \right) \]

\[ = \int_0^\infty d\ell^2 \left( \frac{1}{\ell^2 + \Delta} - \frac{\Delta}{(\ell^2 + \Delta)^2} \right) = \ln(\ell^2 + \Delta)|_{\ell^2=0}^\infty + \frac{\Delta}{\ell^2 + \Delta}|_{\ell^2=0}^\infty = \ln(\ell^2 + \Delta)|_{\ell^2=0}^\infty - 1. \]

Recall that

\[ \Delta = x\mu^2 + (1-x)m_0^2 - x(1-x)p^2 \equiv \Delta(\mu^2). \]
Pauli-Villars regularization. Here is a convenient fiction: when you exchange a photon, you also exchange a very heavy particle, with mass \( m^2 = \Lambda^2 \), with an extra \((-1)\) in its propagator. This means that (in this Pauli-Villars regulation scheme) the Feynman rule for the wiggly line is instead

\[
\begin{align*}
- i \eta_{\mu \nu} & \left( \frac{1}{k^2 - \mu^2 + i \epsilon} - \frac{1}{k^2 - \Lambda^2 + i \epsilon} \right) \\
& = - i \eta_{\mu \nu} \left( \frac{\mu^2 - \Lambda^2}{(k^2 - \mu^2 + i \epsilon)(k^2 - \Lambda^2 + i \epsilon)} \right)
\end{align*}
\]

This goes like \( \frac{1}{k^4} \) at large \( k \), so the integrals are more convergent. Yay.

Notice that the contribution from the Pauli-Villars photon to tree-level amplitudes goes like \( \frac{1}{k^2 - \Lambda^2} \approx \frac{1}{\Lambda^2} \) (where \( k \) is the momentum going through the photon line, determined by the external momenta), which innocuously vanishes as \( \Lambda \to \infty \).

Remembering that the residue of the pole in the propagator is the probability for the field operator to create a particle from the vacuum, you might worry that this is a negative probability, and unitarity isn’t manifest. This particle is a ghost. However, we will choose \( \Lambda \) so large that the pole in the propagator at \( k^2 = \Lambda^2 \) will never be accessed and we’ll never have external Pauli-Villars particles. We are using this as a device to define the theory in a regime of energies much less than \( \Lambda \). You shouldn’t take the regulated theory too seriously: for example, the wrong-sign propagator means wrong-sign kinetic terms for the PV fields. This means that very wiggly configurations will be energetically favored rather than suppressed by the Hamiltonian. It will not make much sense non-perturbatively.

I emphasize that this regulator is one possibility of many. They each have their drawbacks. They all break scale invariance. A nice thing about PV is that it is Lorentz invariant. A class of regulators which make perfect sense non-perturbatively is the lattice (as in the model with masses on springs). The price is that it really messes up the spacetime symmetries.

Applying this to the self-energy integral amounts to the replacement

\[
\mathcal{J} \sim \mathcal{J}_{\Delta(\mu^2)} - \mathcal{J}_{\Delta(\Lambda^2)} = \left[ \ln \left( \ell^2 + \Delta(\mu^2) \right) - 1 \right] - \left[ \ln \left( \ell^2 + \Delta(\Lambda^2) \right) - 1 \right] \\
= \ln \frac{\ell^2 + \Delta(\mu^2)}{\ell^2 + \Delta(\Lambda^2)} \\
= \ln \frac{1}{1 - \ln \frac{\Delta(\mu^2)}{\Delta(\Lambda^2)}} = \ln \frac{\Delta(\Lambda^2)}{\Delta(\mu^2)}.
\]

Notice that we can take advantage of our ignorance of the microphysics to make the
cutoff as big as we like and thereby simplify our lives:

$$\Delta(\Lambda^2) = x\Lambda^2 + (1 - x)m_0^2 - x(1 - x)p^2 \approx \Lambda \gg \text{everyone} \approx x\Lambda^2.$$ 

Finally then

$$\Sigma_2(p)_{PV} = \frac{\alpha}{2\pi} \int_0^1 dx (2m_0 - x\not{p}) \ln \frac{x\Lambda^2}{x\mu^2 + (1 - x)m_0^2 - x(1 - x)p^2}.$$

(4.16)

Having arrived at this regulated expression for the self-energy we need to “impose a renormalization condition,” i.e. introduce some observable physics in terms of which to parametrize our answers. We return to (4.14): the shift in the mass as a result of this one-loop self-energy is

$$\delta m \equiv m - m_0 = \Sigma_2(p = m) + \mathcal{O}(e^4) = \Sigma_2(p = m_0) + \mathcal{O}(e^4)$$

$$= \frac{\alpha}{2\pi} \int_0^1 dx (2 - x)m_0 \ln \left(\frac{x\Lambda^2}{x\mu^2 + (1 - x)m_0^2 + x(1 - x)m_0^2} = x\mu^2 + (1 - x)m_0^2 = f(x, m_0, \mu)\right)$$

$$= \frac{\alpha}{2\pi} \int_0^1 dx (2 - x)m_0 \left(\ln \frac{\Lambda^2}{m_0^2} + \ln \frac{xm_0^2}{f(x, m_0, \mu)}\right)$$

relatively small

$$= \frac{\alpha}{2\pi} \left(2 - \frac{1}{2}\right)m_0 \ln \frac{\Lambda^2}{m_0^2} = \frac{3\alpha}{4\pi}m_0 \ln \frac{\Lambda^2}{m_0^2}. \quad (4.17)$$

In the penultimate step (with the $\approx$), we’ve neglected the finite bit (labelled ‘relatively small’) compared to the logarithmically divergent bit: we’ve already assumed $\Lambda \gg$ all other scales in the problem.

**Mass renormalization.** Now the physics input: The mass of the electron is 511 keV (you can ask how we measure it and whether the answer we get depends on the resolution of the measurement, and indeed there is more to this story; this is a low-energy answer, for example we could make the electron go in a magnetic field and measure the radius of curvature of its orbit and set $m_e v^2/r = evB/c$), so

$$511 \text{ keV} \approx m_e = m_0 \left(1 + \frac{3\alpha}{4\pi} \ln \frac{\Lambda^2}{m_0^2}\right) + \mathcal{O}(\alpha^2).$$

In this equation, the LHS is a measured quantity. In the correction on the RHS $\alpha \approx \frac{1}{137}$ is small, but it is multiplied by $\ln \frac{\Lambda^2}{m_0^2}$ which is arbitrarily large. This means that the bare mass $m_0$, which is going to absorb the cutoff dependence here, must actually be really small. (Notice that actually I’ve lied a little here: the $\alpha$ we’ve been using is
still the bare charge; we will need to renormalize that one, too, before we are done.) I emphasize: \( m_0 \) and the other fake, bare parameters in \( \mathcal{L} \) depend on \( \Lambda \) and the order of perturbation theory to which we are working and other theorist bookkeeping garbage; \( m_e \) does not. At each order in perturbation theory, we eliminate \( m_0 \) and write our predictions in terms of \( m_e \). It is not too surprising that the mass of the electron includes such contributions: it must be difficult to travel through space if you are constantly emitting and re-absorbing photons.

**Wavefunction renormalization.** The actual propagator for the electron, near the electron pole is

\[
\tilde{G}^{(2)}(p) = \frac{i}{p - m_0 - \Sigma(p)} \overset{p \sim m}{\approx} \frac{iZ}{p - m} + \text{regular terms.} \tag{4.18}
\]

The residue of the pole at the electron mass is no longer equal to one, but rather \( Z \).

To see what \( Z \) actually is at this order in \( e^2 \), Taylor expand near the pole

\[
\Sigma(p) = \Sigma(p = m) + \frac{\partial \Sigma}{\partial \phi} |_{\phi = m}(p - m) + \cdots
\]

\[
= \Sigma(p = m_0) + \frac{\partial \Sigma}{\partial \phi} |_{\phi = m_0}(p - m_0) + \cdots + \mathcal{O}(e^4)
\]

So then (4.18) becomes

\[
\tilde{G}^{(2)}(p) p \sim m \overset{p \sim m}{\approx} \frac{i}{p - m - \frac{\partial \Sigma}{\partial \phi} |_{m_0}(p - m)} = \frac{i}{(p - m) \left(1 - \frac{\partial \Sigma}{\partial \phi} |_{m_0}\right)} \tag{4.19}
\]

So that

\[
Z = \frac{1}{1 - \frac{\partial \Sigma}{\partial \phi} |_{m_0}} \approx 1 + \frac{\partial \Sigma}{\partial \phi} |_{m_0} \equiv 1 + \delta Z
\]

and at leading nontrivial order

\[
\delta Z = \frac{\partial \Sigma_2}{\partial \phi} |_{m_0} = \frac{\alpha}{2\pi} \int_0^1 dx \left(-x \ln \frac{x\Lambda^2}{f(x, m_0, \mu)} + (2m_0 - x m_0 - 2x(1 - x)) \right)
\]

\[
= -\frac{\alpha}{4\pi} \left(\ln \frac{\Lambda^2}{m_0^2} + \text{finite}\right). \tag{4.20}
\]

Here \( f = f(x, m_0, \mu) \) is the same quantity defined in the second line of (4.17). We’ll see below that the cutoff-dependence in \( \delta Z \) plays a crucial role in making the \( S \) matrix (for example for the \( e\mu \rightarrow e\mu \) process we’ve been discussing) cutoff-independent and finite, when written in terms of physical variables.
4.6 Big picture interlude

OK, I am having a hard time just pounding away at one-loop QED. Let’s take a break and think about the self-energy corrections in scalar field theory. Then we will step back and think about the general structure of short-distance senstivity in (relativistic) QFT, before returning to the QED vertex correction and vacuum polarization.

4.6.1 Self-energy in $\phi^4$ theory

[Zee §III.3] Let’s return to the $\phi^4$ theory in $D = 3 + 1$ for a moment. The $\mathcal{M}_{\phi\phi\to\phi\phi}$ amplitude is not the only place where the cutoff appears.

Above we added a counterterm of the same form as the $\phi^4$ term in the Lagrangian. Now we will see that we need counterterms for everybody:

\[
\mathcal{L} = -\frac{1}{2} (\phi \Box \phi + m^2 \phi^2) - \frac{g_P}{4!} \phi^4 - \frac{\delta g}{4!} \phi^4 + \frac{1}{2} \delta Z \phi \Box \phi + \frac{1}{2} \delta m^2 \phi^2.
\]

Here is a way in which $\phi^4$ theory is weird: At one loop there is no wavefunction renormalization. That is,

\[
\delta \Sigma_1(k) = \int \mathcal{D}0(p) \mathcal{D}0(q) \mathcal{D}0(k-p-q) = -ig \int \mathcal{D}4(q) \frac{i}{q^2 - m^2 + i\epsilon} = \delta \Sigma_1(k = 0) \sim g\Lambda^2
\]

which is certainly quadratically divergent, but totally independent of the external momentum. This means that when we Taylor expand in $k$ (as we just did in (4.19)), this diagram only contributes to the mass renormalization.

So let’s see what happens if we keep going:

\[
\delta \Sigma_2(k) = (-ig)^2 \int \mathcal{D}4(p) \mathcal{D}4(q) \mathcal{D}4(k-p-q) \equiv I(k^2, m, \Lambda).
\]

Here $\mathcal{D}4(p) \equiv \frac{i}{p^2 - m^2 + i\epsilon}$ is the free propagator (the factor of $i$ is for later convenience), and we’ve defined $I$ by this expression. The fact that $I$ depends only on $k^2$ is a consequence of Lorentz invariance. Counting powers of the loop momenta, the short-distance bit of this integral is of the schematic form $\int^\Lambda \frac{d^4 p}{p^2} \sim \Lambda^2$, also quadratically divergent, but this time $k^2$-dependent, so there will be a nonzero $\delta Z \propto g^2$. As we just did for the electron self-energy, we should Taylor expand in $k$. (We’ll learn more about
why and when the answer is analytic in $k^2$ at $k = 0$ later.) The series expansion in $k^2$
(let’s do it about $k^2 = 0 \sim m^2$ to look at the UV behavior) is

$$\delta \Sigma_2(k^2) = A_0 + k^2 A_1 + k^4 A_2 + \cdots$$

where $A_0 = I(k^2 = 0) \sim \Lambda^2$. In contrast, dimensional analysis says $A_1 = \frac{\partial}{\partial k^2} I|_{k^2=0} \sim \int \frac{d^d P}{P^8} \sim \Lambda^{0+} \sim \ln \Lambda$ has two fewer powers of the cutoff. After that it’s clear sailing:

$A_2 = \left( \frac{\partial}{\partial k^2} \right)^2 I|_{k^2=0} \sim \int \Lambda \frac{d^d P}{P^{10}} \sim \Lambda^{-2}$ is finite as we remove the cutoff, and so are all the later coefficients.

Putting this together, the inverse propagator is

$$D^{-1}(k) = D_0^{-1}(k) - \Sigma(k) = k^2 - m^2 - \left( \delta \Sigma_1(0) + A_0 \right) - k^2 A_1 - k^4 A_2 + \cdots \equiv o - \Lambda^2$$

The $\cdots$ here includes both higher orders in $g$ ($O(g^3)$) and higher powers of $k^2$, i.e. higher derivative terms. If instead the physical pole were at a nonzero value of the mass, we should Taylor expand about $k^2 = m_P^2$ instead:

$$D^{-1}(k) = D_0^{-1}(k) - \Sigma(k) = k^2 - m_0^2 - \left( \delta \Sigma_1(0) + A_0 \right) - (k^2 - m_P^2) A_1 - (k^2 - m_P^2)^2 A_2 + \cdots$$

where now $A_n \equiv \frac{1}{n!} \left( \frac{\partial}{\partial k^2} \right)^n \Sigma_2(k^2)|_{k^2=m_P^2}$.

Therefore, the propagator is

$$D(k) = \frac{1}{(1 - A_1)(k^2 - m_P^2)} + \cdots = \frac{Z}{k^2 - m_P^2} + \cdots$$

with

$$Z = \frac{1}{1 - A_1}, \quad m_P^2 = m^2 + a.$$

Some points to notice: • $\delta Z = A_1$.

• The contributions $A_{n \geq 2}(k^2)^n$ can be reproduced by counterterms of the form $A_n \phi \Box^n \phi$. Had they been cutoff dependent we would have needed to add such (cutoff-dependent) counterterms.

• The mass-squared of the scalar field in $D = 3 + 1$ is quadratically divergent, while the mass of the spinor was only log divergent. This UV sensitivity of scalar fields is ubiquitous\(^3\) (see the homework) and is the source of many headaches.

• On the term ‘wavefunction renormalization’: who is $\phi$? Also just a theorist’s letter. Sometimes (in condensed matter) it is defined by some relation to observation (like

\(^3\)At least for most regulators. We’ll see that dim reg is special.
the height of a wave in the mattress), in high energy theory not so much. Classically, we fixed its (multiplicative) normalization by setting the coefficient of $\phi^2$ to one. If we want to restore that convention after renormalization, we can make a redefinition of the field $\phi_R \equiv Z^{-1/2} \phi$. This is the origin of the term ‘wavefunction renormalization’. A slightly better name would be ‘field renormalization’, but even better would be just ‘kinetic term renormalization’.

**Renormalized perturbation theory revisited.** The full story for the renormalized perturbation expansion in $\phi^4$ theory is

$$
\mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{g_N}{4!} \phi^4 + \mathcal{L}_{ct}
$$

with

$$
\mathcal{L}_{ct} = \frac{1}{2} \delta Z (\partial \phi)^2 + \frac{1}{2} \delta m^2 \phi^2 + \frac{\delta g}{4!} \phi^4.
$$

Here are the instructions for using it: The Feynman rules are as before: the coupling and propagator are

$$
\begin{align*}
\begin{array}{c}
\hline
\end{array} = -i g_P, & \hline \hline = \frac{i}{k^2 - m^2 + i\epsilon}.
\end{align*}
$$

(4.21)

but the terms in $\mathcal{L}_{ct}$ (the counterterms) are treated as new vertices, and treated perturbatively:

$$
\begin{align*}
\begin{array}{c}
\hline
\end{array} = i \delta g, & \hline \hline = i(\delta Z k^2 + \delta m^2).
\end{align*}
$$

All integrals are regulated, in the same way (whatever it is). The counterterm couplings $\delta g, \delta Z, \delta m^2$ are determined iteratively, as follows: given the $\delta_{N-1}$s up to $\mathcal{O}(g_N^N)$, we fix each one $\delta = \delta_{N-1} + g_N^N \Delta \delta_N + \mathcal{O}(g_N^{N+1})$ by demanding that (4.21) are actually true up to $\mathcal{O}(g_N^{N+1})$. This pushes the cutoff dependence back into the muck a bit further.

I say this is the full story, but wait: we didn’t try to compute amplitudes with more than four $\phi$s (such as $3 \leftarrow 3$ scattering of $\phi$ quanta). How do we know those don’t require new counterterms (like a $\phi^6$ term, for example)?

### 4.6.2 Where is the UV sensitivity?

[still Zee §III.3, Peskin ch. 10. We’ll follow Zee’s discussion pretty closely for a bit.] Given some process in a relativistic, perturbative QFT, how do we know if it will depend on the cutoff? We’d like to be able answer this question for a theory with scalars, spinors, vectors. Here’s how: First, look at each diagram $A$ (order by order in the loop expansion). Define the ‘superficial’ degree of divergence of $A$ to be $D_A$ if
\[ A \sim \Lambda^{D_A} \text{ (in the limit that } \Lambda \ll \text{ all other scales – this is an asymptotic statement).} \]

A log divergent amplitude has \( D_A = 0 \) (sometimes it’s called \( D_A = 0^+ \)).

Let’s start simple, and study the \( \phi^4 \) theory in \( D = 4 \). Consider a connected diagram \( A \) with \( B_E \) external scalar lines. I claim that \( D_A = 4 - B_E \). Why didn’t it depend on any other data of the diagram, such as

\[ B_I \equiv \# \text{ of internal scalar lines (i.e., propagators)} \]
\[ V \equiv \# \text{ of } \phi^4 \text{ vertices} \]
\[ L \equiv \# \text{ of loops} \]

? We can understand this better using two facts of graph theory and some power counting. I recommend checking my claims below with an example, such as the one at right.

**Graph theory fact #1:** These quantities are not all independent. For a connected graph,

\[ L = B_I - (V - 1). \tag{4.22} \]

Math proof\textsuperscript{33}: Imagine placing the vertices on the page and adding the propagators one at a time. You need \( V - 1 \) internal lines just to connect up all \( V \) vertices. After that, each *internal* line you add necessarily adds one more loop.

Another way to think about this fact makes clear that \( L = \# \text{ of loops} = \# \text{ of momentum integrals}. \) Before imposing momentum conservation at the vertices, each internal line has a momentum which we must integrate: \( \prod_{\alpha=1}^{B_I} \int d^D q_\alpha \). We then stick a \( \delta^{(D)}(\sum q) \) for each vertex, but one of these gives the overall momentum conservation \( \delta^{(D)}(k_T) \), so we have \( V - 1 \) fewer momentum integrals. For the example above, (4.22) says \( 4 = 8 - (5 - 1) \).

**Graph theory fact #2:** Each external line comes out of one vertex. Each internal line connects two vertices. Altogether, the number of ends of lines sticking out of vertices is

\[ B_E + 2B_I = 4V \]

where the RHS comes from noting that each vertex has four lines coming out of it (in \( \phi^4 \) theory). In the example, this is \( 4 + 2 \cdot 8 = 4 \cdot 5 \). So we can eliminate

\[ B_I = 2V - B_E/2. \tag{4.23} \]

\textsuperscript{33}I learned this one from my class-mate M.B. Schulz.
Now we count powers of momenta:

\[ A \sim \prod_{a=1}^{L} \int^{\Lambda} d^{D}k_{a} \prod_{a=1}^{B_{I}} \frac{1}{k_{a}^{2}}. \]

Since we are interested in the UV structure, I’ve set the mass to zero, as well as all the external momenta. The only scale left in the problem is the cutoff, so the dimensions of \( A \) must be made up by the cutoff:

\[ D_{A} = [A] = DL - 2B_{I} \]

\[ \overset{(4.22)}{=} B_{I}(D - 2) - D(V - 1) \]

\[ \overset{(4.23)}{=} D + \frac{2 - D}{2}B_{E} + V(D - 4). \]

If we set \( D = 3 + 1 = 4 \), we get \( D_{A} = 4 - B_{E} \) as claimed. Notice that with \( B_{E} = 2 \) we indeed reproduce \( D_{A} = 2 \), the quadratic divergence in the mass renormalization, and with \( B_{E} = 4 \) we get \( D_{A} = 0 \), the log divergence in the \( 2 \leftrightarrow 2 \) scattering. This pattern continues: with more than four external legs, \( D_{A} = 4 - B_{E} < 0 \), which means the cutoff dependence must go away when \( \Lambda \to 0 \). This is illustrated by the following diagram with \( B_{E} = 6 \):

\[ \sim \int^{\Lambda} d^{4}P \sim \Lambda^{-2}. \]

So indeed we don’t need more counterterms for higher-point interactions in this theory.

Why is the answer independent of \( V \) in \( D = 4 \)? This has the dramatic consequence that once we fix up the cutoff dependence in the one-loop diagrams, the higher orders have to work out, \( i.e. \) it strongly suggests that the theory is renormalizable. \(^{34}\)

Before we answer this, let’s explore the pattern a bit more. Suppose we include also a fermion field \( \psi \) in our field theory, and suppose we couple it to our scalar by a Yukawa interaction:

\[ S_{\text{bare}}[\phi, \psi] = - \int d^{D}x \left( \frac{1}{2} \phi (\Box + m_{\phi}^{2}) \phi + \bar{\psi} (-\partial + m_{\psi}) \psi + y \phi \bar{\psi} \psi + \frac{g}{4!} \phi^{4} \right). \]

\(^{34}\)Why isn’t it a proof of renormalizability? Consider the following integral:

\[ I = \int^{\Lambda} \frac{d^{4}p}{(p^{2} + m^{2})^{\delta}} \int^{\Lambda} d^{4}k. \]

According to our method of counting, we would say \( D_{I} = 4 + 4 - 10 = -2 \) and declare this finite and cutoff-independent. On the other hand, it certainly does depend on the physics at the cutoff. (I bet it is possible to come up with more pathological examples.) The rest of the work involving ‘nested divergences’ and forests is in showing that the extra structure in the problem prevents things like \( I \) from being Feynman amplitudes.
To find the degree of divergence in an amplitude in this model, we have to independently keep track of the number fermion lines $F_E, F_I$, since a fermion propagator has dimension $\left[ \frac{1}{p} \right] = -1$, so that $D_A = [A] = DL - 2B_I - F_I$. The number of ends-of-fermion-lines is $2V_g = 2F_E + F_I$ and the number of ends-of-boson-lines is $V_y + 4V_g = B_E + 2B_I$. The number of loops is $L = B_I + F_I - (V_y + V_g - 1)$. Putting these together (I used Mathematica) we get

$$D_A = D + (D - 4) \left( V_g + \frac{1}{2}V_y \right) + B_E \left( \frac{2 - D}{2} \right) + F_E \left( \frac{1 - D}{2} \right).$$

Again in $D = 4$ the answer is independent of the number of vertices! Is there something special about four spacetime dimensions?

To temper your enthusiasm, consider adding a four-fermion interaction: $G(\bar{\psi}\psi)(\bar{\psi}\psi)$ (or maybe $G_V(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi)$ or $G_A(\bar{\psi}\gamma^\mu\gamma^5\psi)(\bar{\psi}\gamma^\mu\gamma^5\psi)$ or any other pile of gamma matrices in between, with the indices contracted). When you redo this calculation on the homework, you’ll find that in $D = 4$ a diagram (for simplicity, one with no $\phi^4$ or Yukawa interactions) has

$$D_A = 4 - (1)B_E - \left( \frac{3}{2} \right) F_E + 2V_G.$$

This dependence on the number of four-fermi vertices means that there are worse and worse divergences as we look at higher-order corrections to a given process. Even worse, it means that for any number of external lines $F_E$ no matter how big, there is a large enough order in perturbation theory in $G$ where the cutoff will appear! This means we need $\delta_n(\bar{\psi}\psi)^n$ counterterms for every $n$, which we’ll need to fix with physical input. This is a bit unappetizing, and such an interaction is called “non-renormalizable”. However, when we remember that we only need to make predictions to a given precision (so that we only need to go to a finite order in this process) we will see that such theories are nevertheless quite useful.

So why were those other examples independent of $V$? It’s because the couplings were dimensionless. Those theories were classically scale invariant (except for the mass terms).

### 4.6.3 Naive scale invariance in field theory

[Halpern] Consider a field theory of a scalar field $\phi$ in $D$ spacetime dimensions, with an action of the form

$$S[\phi] = \int d^Dx \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - g \phi^p \right)$$

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for some constants $p, g$. Which value of $p$ makes this scale invariant? (That is: when is $g$ dimensionless, and hence possibly the coupling for a renormalizable interaction.)

Naive dimensions:

$$ [S] = [h] = 0, \quad [x] \equiv -1, \quad [d^D x] = -D, \quad [\partial] = 1 $$

The kinetic term tells us the engineering dimensions of $\phi$:

$$ 0 = [S_{\text{kinetic}}] = -D + 2 + 2[\phi] \implies [\phi] = \frac{D - 2}{2}. $$

Notice that the $D = 1$ case agrees with our quantum mechanics counting from §4.2. Quantum field theory in $D = 1$ spacetime dimensions is quantum mechanics.

Then the self-interaction term has dimensions

$$ 0 = [S_{\text{interaction}}] = -D + [g] + p[\phi] \implies [g] = D - p[\phi] = D + p \frac{2 - D}{2} $$

We expect scale invariance when $[g] = 0$ which happens when

$$ p = p_D \equiv \frac{2D}{D - 2} $$

\text{i.e.} the scale invariant scalar-field self-interaction in $D$ spacetime dimensions is $\phi^{\frac{2D}{D - 2}}$.

<table>
<thead>
<tr>
<th>$D$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>$\ldots$</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\phi]$</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>2</td>
<td>$\ldots$</td>
<td>$\frac{D}{2}$</td>
</tr>
<tr>
<td>scale-inv’t $p \equiv p_D$</td>
<td>$-2$</td>
<td>$\infty^*$</td>
<td>6</td>
<td>4</td>
<td>$\frac{10}{3}$</td>
<td>3</td>
<td>$\ldots$</td>
<td>2</td>
</tr>
</tbody>
</table>

$^*$ What is happening in $D = 2$? The field is dimensionless, and so any power of $\phi$ is naively scale invariant, as are more complicated interactions like $g_{ij}(\phi)\partial^\mu \phi^i \partial^\nu \phi^j$, where the coupling $g(\phi)$ is a function of $\phi$. This allows for scale-invariant \textit{non-linear sigma models}, where the fields are coordinates on a curved manifold with metric $ds^2 = g_{ij} d\phi^i d\phi^j$.

In dimensions where we get fractional powers, this isn’t so nice.

Notice that the mass term $\Delta S = \int d^D x \frac{m^2}{2} \phi^2$ gives

$$ 0 = -D + 2[m] + 2[\phi] \implies [m] = 1 \quad \forall D < \infty $$

– it’s a mass, yay.
What are the consequences of this engineering dimensions calculation in QFT? For $D > 2$, an interaction of the form $g\phi^p$ has

$$[g] = D \cdot \frac{p_D - p}{p_D} \begin{cases} < 0 & \text{when } p > p_D, \text{ non-renormalizable or irrelevant} \\ = 0 & \text{when } p = p_D, \text{ renormalizable or marginal} \\ > 0 & \text{when } p < p_D, \text{ super-renormalizable or relevant.} \end{cases} \quad (4.25)$$

Consider the ‘non-renormalizable’ case. Suppose we calculate in QFT some quantity $f$ with $[f]$ as its naive dimension, in perturbation theory in $g$, e.g. by Feynman diagrams. We’ll get:

$$f = \sum_{n=0}^{\infty} g^n c_n$$

with $c_n$ independent of $g$. So

$$[f] = n[g] + [c_n] \implies [c_n] = [f] - n[g]$$

So if $[g] < 0$, $c_n$ must have more and more powers of some mass (inverse length) as $n$ increases. What dimensionful quantity makes up the difference? Sometimes it is masses or external momenta. But generically, it gets made up by UV divergences (if everything is infinite, dimensional analysis can fail, nothing is real, I am the walrus). More usefully, in a meaningful theory with a UV cutoff, $\Lambda_{UV}$, the dimensions get made up by the UV cutoff, which has $[\Lambda_{UV}] = 1$. Generically: $c_n = \tilde{c}_n (\Lambda_{UV})^{-n[g]}$, where $\tilde{c}_n$ is dimensionless, and $n[g] < 0$ – it’s higher and higher powers of the cutoff.

Consider the renormalizable (classically scale invariant) case: $[c_n] = [f]$, since $[g] = 0$. But in fact, what you’ll get is something like

$$c_n = \tilde{c}_n \log^{\nu(n)} \left( \frac{\Lambda_{UV}}{\Lambda_{IR}} \right),$$

where $\Lambda_{IR}$ is an infrared cutoff or a mass, $[\Lambda_{IR}] = 1$.

Some classically scale invariant examples (so that $m = 0$ and the bare propagator is $1/k^2$) where you can see that we get logs from loop amplitudes:

$\phi^4$ in $D = 4$: \[ \sim \int d^4k \left( \frac{1}{k^2} \right)^2 \]

$\phi^6$ in $D = 3$: \[ \sim \int d^k \kappa^3 \kappa \left( \frac{1}{\kappa^2} \right)^2 \]

$\phi^3$ in $D = 6$: \[ \sim \int d^3k \left( \frac{1}{\kappa^2} \right)^3 \]

In $D = 2$, even the propagator for a massless scalar field has logs:

$$\langle \phi(x)\phi(0) \rangle = \int d^2k \frac{e^{-ikx}}{k^2} \sim \log \frac{|x|}{\Lambda_{UV}}.$$
The terms involving ‘renormalizable’ in (4.25) are somewhat old-fashioned and come from a high-energy physics point of view where the short-distance physics is unknown, and we want to get as far as we can in that direction with our limited knowledge (in which case the condition ‘renormalizability’ lets us get away with this indefinitely – it lets us imagine we know everything). The latter terms are natural in the opposite situation (like condensed matter physics) where we know some basically correct microscopic description but want to know what happens at low energies. Then an operator like $\frac{1}{M^2} \phi^{28}$ whose coefficient is suppressed by some large mass scale $M$ is irrelevant for physics at energies far below that scale. Inversely, an operator like $m^2 \phi^2$ gives a mass to the $\phi$ particles, and matters very much (is relevant) at energies $E < m$. In the marginal case, the quantum corrections have a chance to make a big difference.

4.7 Vertex correction in QED

[Peskin chapter 6, Schwartz chapter 17, Zee chapter III.6] Back to work on QED. The vertex correction has some great physics payoffs:

- We’ll cancel the cutoff dependence we found in the $S$ matrix from $\delta Z$.

- We’ll compute $g - 2$ (the anomalous magnetic moment) of the electron, the locus of some of the most precise agreement between theory and experiment. (Actually the agreement is so good that it’s used as the definition of the fine structure constant. But a similar calculation gives the leading anomalous magnetic moment of the muon.)

- We’ll see that the exclusive differential cross section $\left( \frac{d\sigma}{d\Omega} \right)_{e\mu \rightarrow e\mu}$ that we’ve been considering is not really an observable. Actually it is infinity! The key word here is ‘exclusive,’ which means that we demand that the final state is exactly one electron and one muon and absolutely nothing else. Think for a moment about how you might do that measurement.

This is an example of an IR divergence. While UV divergences mean you’re overstepping your bounds (by taking too seriously your Lagrangian parameters or your knowledge of short distances), IR divergences mean you are asking the wrong question.

35More accurately, the exclusive cross section is zero; the one-loop correction is minus infinity, which is perturbation theory’s clumsy attempt to correct the finite tree level answer to make it zero.
To get started, consider the following class of diagrams.

$$\equiv iM = ie^2 (\bar{u}(p') \Gamma^\mu(p,p') u(p)) \frac{1}{q^2} \bar{u}(K') \gamma_\mu u(K)$$ (4.26)

The shaded blob is the vertex function $\Gamma$. The role of the light blue factors is just to make and propagate the photon which hits our electron; let’s forget about them. Denote the photon momentum by $q = p' - p$. We’ll assume that the electron momenta $p, p'$ are on-shell, but $q^\mu$ is not, as in the $e\mu$ scattering process. Then $q^2 = 2m^2 - 2p' \cdot p$.

Before calculating the leading correction to the vertex $\Gamma^\mu = \gamma^\mu + O(e^2)$, let’s think about what the answer can be. It is a vector made from $p, p', \gamma^\mu$ and $m, e$ and numbers. It can’t have any $\gamma_5$ or $\epsilon_{\mu\nu\rho\sigma}$ by parity symmetry of QED. So on general grounds we can organize it as

$$\Gamma^\mu(p,p') = A\gamma^\mu + B(p + p')^\mu + C(p - p')^\mu$$ (4.27)

where $A, B, C$ are Lorentz-invariant functions of $p^2 = (p')^2 = m^2$, $p \cdot p'$, $\gamma, \gamma'$. But, for example, $\bar{\psi}\gamma^\mu u(p) = (m\gamma^\mu - p^\mu)u(p)$ which just mixes up the terms; really $A, B, C$ are just functions of the momentum transfer $q^2$. Gauge invariance, in the form of the Ward identity, says that contracting the photon line with the photon momentum should give zero:

$$0 \overset{\text{Ward}}{=} q_\mu \bar{u}(p') \Gamma^\mu u(p) \overset{(4.27)}{=} \bar{u}(p') \\
\left. Aq + B(p + p') \cdot (p - p') + C q^2 \right|_{=m^2-m^2=0} u(p)
$$

Therefore $0 = Cq^2 \bar{u}(p') u(p)$ for general $q^2$ and general spinors, so $C = 0$. This is the moment for the Gordon identity to shine:

$$\bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') \left( \frac{p^\mu + p'^\mu}{2m} + \frac{ie^{\mu\nu} q_\nu}{2m} \right) u(p)$$

(where $\sigma^{\mu\nu} \equiv \frac{1}{2} [\gamma^\mu, \gamma^\nu]$) can be used to eliminate the $p + p'$ term\(^{36}\). The Gordon identity

\(^{36}\)Actually this is why we didn’t include a $\sigma^{\mu\nu}$ term. You could ask: what about a term like $\sigma^{\mu\nu}(p + p')^\nu$? Well, there’s another Gordon identity that relates to things we’ve already included:

$$\bar{u}_2 \sigma^{\mu\nu} (p_1 + p_2)^\nu u_1 = i\bar{u}_2 (q_\mu - (m_1 - m_2)\gamma_\mu) u_1.$$

It is proved the same way: just use the Dirac equation $\gamma^\mu p_1 u_1 = m_1 u_1, \bar{u}_2 p_2 = \bar{u}_2 m_2$ and the Clifford algebra. We are interested here in the case where $m_1 = m_2$. 
shows that the QED interaction vertex \( \bar{u}(p')\gamma^\mu u(p)A_\mu \) contains a magnetic moment bit in addition to the \( p + p' \) term (which is there for a charged scalar field).

It is then convenient (and conventional) to parametrize the vertex in terms of the two form factors \( F_{1,2} \):

\[
\Gamma^\mu(p,p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^\mu\nu}{2m} F_2(q^2).
\]

(4.28)

This little monstrosity has the complete information about the coupling of the electron to the electromagnetic field, such as for example a background electromagnetic field. It is a parametrization of the matrix elements of the current between two one-electron states, incorporating the fact of gauge invariance.

The first term at zero momentum \( e F_1(q^2 = 0) \) is the electric charge of the electron (if you don’t believe it, use the vertex (4.28) to calculate the Coulomb field of the electron; there are some details on page 186 of Peskin). Since the tree-level bit of \( F_1 \) is 1, if by the letter \( e \) here we mean the actual charge, then we’d better include counterterms (\( \mathcal{L}_{ct} \ni \bar{\psi}\delta e\gamma^\mu A_\mu \psi \)) to make sure it isn’t corrected: \( F_1(0) = 1 \).

On the homework last quarter you showed (or see Peskin p. 187) that the magnetic moment of the electron is

\[
\vec{\mu} = g e \frac{\vec{S}}{2m},
\]

where \( \vec{S} \equiv \xi\dagger \vec{\sigma} \xi \) is the electron spin. Comparing with the vertex function, this says that the \( g \) factor is

\[
g = 2(F_1(0) + F_2(0)) = 2 + 2F_2(0) = 2 + \mathcal{O}(\alpha).
\]

We see that the anomalous magnetic moment of the electron is \( 2F_2(q^2 = 0) \).

Now that we have some expectation about the form of the answer, and some ideas about what it’s for, we sketch the evaluation of the one-loop QED vertex correction:

\[
= -ie^3 \int d^4k \bar{u}(p')\gamma^\nu \frac{k'}{k' - m_e^2} \frac{k + m_e}{k^2 - m_e^2} \gamma^\rho u(p) \cdot \frac{\eta_{\nu\rho}}{(p - k)^2 - m_e^2}
\]

with \( k' \equiv k + q \).

(1) Feynman parameters again. The one we showed before can be rewritten more
symmetrically as:

\[
\frac{1}{AB} = \int_0^1 dx \int_0^1 dy \frac{\delta(x + y - 1)}{(xA + yB)^2} \frac{1}{(xA + yB)^2}
\]

Now how can you resist the generalization\(^\text{37}\):

\[
\frac{1}{ABC} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\delta(x + y + z - 1)}{(xA + yB + zC)^3} \]

So, set \(A = (k')^2 - m_e^2, B = k^2 - m_e^2, C = (p - k)^2 - m_\gamma^2\) (with the appropriate \(i\)cs), so that the integral we have to do is

\[
\int \frac{d^4kN^\mu}{(k^2 + k \cdot (\cdots) + \cdots)^3}
\]

(2) Complete the square, \(\ell = k - zp + xq\) to get \(\int \frac{d^4\ell N^\mu}{(\ell^2 - \Delta)^3}\) where \(\Delta = -xyq^2 + (1 - z)^2m^2 + zm_\gamma^2\).

The \(\ell\)-dependence in the numerator is either 1 or \(\ell^\mu\) or \(\ell^\nu\ell^\mu\). In the integral over \(\ell\), the second averages to zero, and the third averages to \(\eta^\mu\ell^2\frac{1}{4}\). As a result, the momentum

\(^{37}\)Peskin outlines a proof by induction of the whole family of such identities on page 190. But here’s a simpler proof using Schwinger parameters. You’ll agree that

\[
\frac{1}{A} = \int_0^\infty ds \ e^{-sA}.
\]

Applying this identity to each factor gives

\[
\frac{1}{A_1A_2 \cdots A_n} = \int_0^\infty ds_1 \cdots \int_0^\infty ds_n \ e^{-\sum_{i=1}^n s_i A_i}.
\]

Now use scaling to set \(\tau = \sum_{i=1}^n s_i\), and \(x_i \equiv s_i/\tau\). Then

\[
\frac{1}{A_1A_2 \cdots A_n} = \int_0^\infty d\tau \tau^{n-1} \prod_{i=1}^n \int_0^1 dx_i \delta \left( \sum_{i=1}^n x_i - 1 \right) e^{-\tau \sum_{i=1}^n x_i A_i}.
\]

Now do the integral over \(\tau\), using \(\int_0^\infty d\tau \tau^{n-1} e^{-\tau X} = \frac{(n-1)!}{X^n}\) (differentiate (4.29) wrt \(A\)), to arrive at

\[
\frac{1}{A_1A_2 \cdots A_n} = \prod_{i=1}^n \int_0^1 dx_i \delta \left( \sum_{i=1}^n x_i - 1 \right) \frac{(n-1)!}{(\sum_{i=1}^n x_i A_i)^n}.
\]
integrals we need are just
\[ \int \frac{d^D \ell}{(\ell^2 - \Delta)^m} \quad \text{and} \quad \int \frac{d^D \ell \ell^2}{(\ell^2 - \Delta)^m}. \]

Right now we only need \( D = 4 \) and \( m = 3 \), but it turns out to be quite useful to think about them all at once. Like in our discussion of the electron self-energy diagram, we can evaluate them by Wick rotating (which changes the denominator to \( \ell^2_E + \Delta \)) and going to polar coordinates. This gives:

\[ \int \frac{d^D \ell}{(\ell^2 - \Delta)^m} = (-1)^m \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(m - \frac{D}{2})}{\Gamma(m)} \left( \frac{1}{\Delta} \right)^{m - \frac{D}{2}}. \quad (4.30) \]

\[ \int \frac{d^D \ell \ell^2}{(\ell^2 - \Delta)^m} = (-1)^{m-1} \frac{D}{2} \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(m - \frac{D}{2} - 1)}{\Gamma(m)} \left( \frac{1}{\Delta} \right)^{m - \frac{D}{2} - 1}. \quad (4.31) \]

Notice that these integrals are not equal to infinity when the parameter \( D \) is not an integer. This is the idea behind dimensional regularization.

(0) But for now let’s persist in using the Pauli Villars regulator. (I call this step (0) instead of (3) because it should have been there all along.) Here this means we subtract from the amplitude the same quantity with \( m \gamma \) replaced by \( \Lambda \). The dangerous bit comes from the \( \ell^2 \) term we just mentioned, since \( m - D/2 - 1 = 3 - 4/2 - 1 = 0 \) means logs.

The numerator is

\[ N^\mu = \bar{u}(p')\gamma^\mu (\not{k} + \not{q} + m_e) \gamma^\mu (\not{k} + m_e) \gamma_\nu u(p) = -2 \left( A\bar{u}(p')\gamma^\mu u(p) + B\bar{u}(p')\sigma^{\mu\nu} q_\nu u(p) + C\bar{u}(p')q^\mu u(p) \right) \quad (4.32) \]

where

\[ A = -\frac{1}{2} \ell^2 + (1 - x)(1 - y)q^2 + (1 - 4z + z^2)m^2 \]
\[ B = imz(1 - z) \]
\[ C = m(z - 2)(y - x). \quad (4.33) \]

The blood of many men was spilled to arrive at these simple expressions (actually most of the algebra is done explicitly on page 319 of Schwartz). Now you say: but you promised there would be no term like \( C \) because of the Ward identity. Indeed I did and indeed there isn’t because \( C \) is odd in \( x \leftrightarrow y \) while everything else is even, so this term integrates to zero.

The first term (with \( A \)) is a correction to the charge of the electron and will be UV divergent. More explicitly, we get, using Pauli-Villars,

\[ \int \frac{d^4 \ell}{(\ell^2 - \Delta_{m\gamma})^3} - \frac{\ell^2}{(\ell^2 - \Delta)^3} = \frac{i}{(4\pi)^2} \ln \frac{\Delta^2}{\Delta_{m\gamma}}. \]

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The other bits are finite, and we ignore the terms that go like negative powers of \( \Lambda \). More on this cutoff dependence soon. But first something wonderful:

### 4.7.1 Anomalous magnetic moment

The second term \( B \) contains the anomalous magnetic moment:

\[
F_2(q^2) = \frac{2m}{e} \cdot \text{(the term with } \mathcal{B} \text{)}
\]

\[
= \frac{2m}{e} 4e^3m \int dx dy dz \delta(x + y + z - 1)z(1 - z) \int \frac{d^4\ell}{(\ell^2 - \Delta)^3}
\]

\[
= \frac{\alpha}{\pi} m^2 \int dx dy dz \delta(x + y + z - 1) \frac{z(1 - z)}{(1 - z)^2m^2 - xyq^2}.
\]  

The magnetic moment is the long-wavelength bit of this:

\[
F_2(q^2 = 0) = \frac{\alpha}{\pi} m^2 \int_0^1 dz \int_0^{1-z} dy \frac{z}{(1 - z)m^2} = \frac{\alpha}{2\pi}.
\]

\[
g = 2 + \frac{\alpha}{\pi} + \mathcal{O}(\alpha^2).
\]

A rare opportunity for me to plug in numbers: \( g = 2.00232 \).

### 4.7.2 IR divergences mean wrong questions.

There is a term in the numerator from the \( \mathcal{A}^\mu \) bit

\[
\int \frac{d^4\ell}{(\ell^2 - \Delta)^3} = c \frac{1}{\Delta}
\]

(with \( c = -\frac{1}{32\pi^2} \) again), but without the factor of \( z(1 - z) \) we had in the magnetic moment calculation. It looks like we’ve gotten away without having to introduce a UV regulator here, too (so far). But now look at what happens when we try to do the Feynman parameter integrals. For example, at \( q^2 = 0 \), we get (if we had set \( m_\gamma = 0 \))

\[
\int dx dy dz \delta(x + y + z - 1) \frac{m^2(1 - 4z + z^2)}{\Delta} = m^2 \int_0^1 dz \int_0^{1-z} dy \frac{-2 + 2(1 - z) + (1 - z)^2}{(1 - z)^2m^2}
\]

\[
= \int_0^1 dz \frac{-2}{(1 - z)} + \text{finite},
\]

which diverges at the upper limit of integration. In fact it’s divergent even when \( q^2 \neq 0 \). This is a place where we actually need to include the photon mass, \( m_\gamma \), for our own safety.
The (IR singular bit of the) vertex (to $O(\alpha)$) is of the form
\[
\Gamma^\mu = \gamma^\mu \left( 1 - \frac{\alpha}{2\pi} f_{IR}(q^2) \ln \left( \frac{-q^2}{m_\gamma^2} \right) \right) + \text{stuff which is finite as } m_\gamma \to 0. \tag{4.36}
\]

Notice that the IR divergent stuff depends on the electron momenta $p, p'$ only through $q$, the momentum of the photon. So it looks like we are led to conclude
\[
\frac{d\sigma}{d\Omega}_{\mu e^-\mu e^+} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} \left( 1 - \frac{\alpha}{\pi} f_{IR}(q^2) \ln \left( \frac{-q^2}{m_\gamma^2} \right) \right) + O(\alpha^2)
\]
which blows up when we remove the fake photon mass $m_\gamma \to 0$.

[Schwartz §20.1] I wanted to just quote the above result for (4.36) but I lost my nerve, so here is a bit more detail leading to it. The IR dangerous bit comes from the second term in $A$ above. That is,
\[
F_1(q^2) = 1 + f(q^2) + \delta_1 + O(\alpha^2)
\]
with
\[
f(q^2) = \frac{e^2}{8\pi^2} \int_0^1 dx dy dz \delta(x + y + z - 1) \left( \ln \frac{z\Lambda^2}{\Delta} + \frac{q^2(1-x)(1-y) + m_\gamma^2(1-4z + z^2)}{\Delta} \right).
\]

$\delta_1$ here is a counterterm for the $\Psi\gamma^\mu A_\mu \Psi$ vertex.

We can be more explicit if we consider $-q^2 \gg m_e^2$ so that we can ignore the electron mass everywhere. Then we would choose the counterterm $\delta_1$ so that
\[
1 = F_1(0) \implies \delta_1 = -f(0) \xrightarrow{m_e \to 0} -\frac{e^2}{8\pi^2} \frac{1}{2} \ln \frac{\Lambda^2}{m_\gamma^2}.
\]

And the form of $f(q^2)$ is
\[
f(q^2)\big|_{m_e=0} = \frac{e^2}{8\pi^2} \int dx dy dz \delta(x + y + z - 1) \left( \ln \frac{(1-x-y)\Lambda^2}{\Delta} \right)_{\text{IR finite}} + \frac{q^2(1-x)(1-y)}{-xyq^2 + (1-x-y)m_\gamma^2} + \text{finite}.
\]

In doing the integrals, we had to remember the $i\epsilon$ in the propagators, which can be reproduced by the replacement $q^2 \to q^2 + i\epsilon$. This $\ln^2(q^2/m_\gamma^2)$ is called a Sudakov double logarithm. Notice that taking differences of these at different $q^2$ will not make it finite.
Diversity and inclusion to the rescue. Before you throw up your hands in despair, I would like to bring to your attention another consequence of the masslessness of the photon: It means real (as opposed to virtual) photons can be made with arbitrarily low energy. But a detector has a minimum triggering energy: the detector works by particles doing some physical something to stuff in the detector, and it has a finite energy resolution – it takes a finite amount of energy for those particles to do stuff. This means that a process with exactly one $e$ and one $\mu$ in the final state cannot be distinguished from a process ending in $e\mu$ plus a photon of arbitrarily small energy, such as would result from (final-state radiation) or (initial-state radiation). This ambiguity is present for any process with external charged particles.

Being more inclusive, then, we cannot distinguish amplitudes of the form

$$\bar{u}(p')\mathcal{M}_0(p', p)u(p) \equiv -\frac{1}{\Lambda} \left( \begin{array}{c} 1 \\ \mathcal{M}_0(p', p)u(p) \\ \bar{u}(p') \end{array} \right)$$

from more inclusive amplitudes like

$$\bar{u}(p')\mathcal{M}_0(p', p)u(p) = -\frac{1}{\Lambda} \left( \begin{array}{c} 1 \\ \mathcal{M}_0(p', p)u(p) \\ \bar{u}(p') \end{array} \right)$$

$$= \bar{u}(p')\gamma^\mu \left( \frac{e}{p' + \gamma - m_e} \mathcal{M}_0(p', p)u(p)\gamma^\mu(k) + \bar{u}(p')\mathcal{M}_0(p', p)\frac{e}{p' - \gamma - m_e} \gamma^\mu(p)\gamma^\mu(k) \right).$$

Now, by assumption the photon is real ($k^2=0$) and it is soft, in the sense that $k^0 < E_c$, the detector cutoff. So we can approximate the numerator of the second term as

$$(\not{p} - \not{k} - m_e) \gamma^\mu u(p) \simeq (\not{p} + m_e) \gamma^\mu u(p) = (2p^\mu + \gamma^\mu (-\not{p} + m_e))u(p) = 2p^\mu u(p).$$

In the denominator we have e.g. $(p - k)^2 = m_e^2 - 2p \cdot k + k^2 \sim -2p \cdot k$ since the electron is on shell and $k \ll p$. Therefore

$$\mathcal{M}(e\mu + \text{one soft } \gamma \leftrightarrow e\mu) = e\bar{u}(p')\mathcal{M}_0(p', p)u(p) \left( \frac{p' \cdot e^*}{p' \cdot k + i\epsilon} - \frac{p \cdot e^*}{p \cdot k - i\epsilon} \right)$$  \hspace{1cm} (4.37)
**This is bremsstrahlung.** Before we continue this calculation to find the inclusive amplitude which a real detector actually measures, let’s pause to relate the previous expression to some physics we know. Where have we seen this kind of expression

\[
\frac{p'\mu}{p' \cdot k + i\epsilon} - \frac{p\mu}{p \cdot k - i\epsilon} \equiv \frac{1}{i\epsilon} j^\mu(k)
\]

before? Notice that the \(i\epsilon\) are different because one comes from final state and one from initial. Well, this object is the Fourier transform \(\tilde{j}^\mu(k) = \int d^4 x \ e^{i k x} j^\mu(x)\) of the current

\[
j^\mu(x) = e \int d\tau \frac{dy^\mu}{d\tau} \delta^{(4)}(x - y(\tau))
\]

associated with a particle which executes a piecewise linear motion

\[
y(\tau) = \begin{cases} 
\frac{p^\mu}{m} \tau, & \tau < 0 \\
\frac{p'^\mu}{m} \tau, & \tau > 0 
\end{cases}
\]

This is a good approximation to the motion a free particle which experiences a sudden acceleration; sudden means that the duration of the pulse is short compared to \(\omega^{-1}\) for any frequency we’re going to measure. The electromagnetic radiation that such an accelerating charge produces is given classically by Maxwell’s equation: \(\tilde{A}^\mu(k) = -\frac{1}{k^2} \tilde{j}^\mu(k)\).

I claim further that the factor \(f_{IR}(q^2) = \frac{a}{\pi} \ln \left( \frac{-q^2}{m^2} \right)\) (which entered our lives in (4.36)) arises classically as the number of soft photons produced by such a process in each decade of wavenumber. You can figure this out by plugging \(\tilde{A}^\mu(k) = -\frac{1}{k^2} \tilde{j}^\mu(k)\) into the electromagnetic energy \(\frac{1}{2} \int d^3 x \ (E^2 + B^2)\). See Peskin §6.1 for help.

\[
\int d^4x j^\mu(x) e^{i k x} = e \int d\tau \frac{dy^\mu}{d\tau} e^{i k y(\tau)} = e \int_0^\infty d\tau \frac{p^\mu}{m} e^{i \left( \frac{k^\mu}{m} + i\epsilon \right) \tau} + e \int_0^\infty d\tau \frac{p'^\mu}{m} e^{i \left( \frac{k'^\mu}{m} - i\epsilon \right) \tau} = j^\mu(k).
\]

Notice that the \(i\epsilon\) are convergence factors in the Fourier transforms.
which means that the lower limit of the \( k \) integral gets cut off at \( m_\gamma \):

\[
\int_0^{E_c} \frac{dk}{E_k} = \left( \int_0^{m_\gamma} + \int_{m_\gamma}^{E_c} \right) \frac{dk}{\sqrt{k^2 + m_\gamma^2}} \sim \int_0^{m_\gamma} \frac{dk}{m_\gamma} + \int_{m_\gamma}^{E_c} \frac{dk}{k} \ln \frac{E_c}{m_\gamma}.
\]

Being careful about the factors, the actual cross section measured by a detector with energy resolution \( E_c \) is\(^{39}\)

\[
\frac{d\sigma}{d\Omega}^{\text{observed}} = \frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_{e\mu\leftrightarrow\mu e} + \left( \frac{d\sigma}{d\Omega} \right)_{\mu e\gamma\leftrightarrow\mu e} + \mathcal{O}(\alpha^3)
\]

\[
= \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} \left( 1 - \frac{\alpha}{\pi} f_{IR}(q^2) \ln \left( \frac{-q^2}{m_\gamma^2} \right) + \frac{\alpha}{\pi} f_{IR}(q^2) \ln \left( \frac{E_c^2}{m_\gamma^2} \right) \right)_{\text{vertex correction}} + \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} \left( 1 - \frac{\alpha}{\pi} f_{IR}(q^2) \ln \left( \frac{-q^2}{E_c^2} \right) \right)_{\text{soft photons}}
\]

The thing we can actually measure is independent of the IR regulator photon mass \( m_\gamma \), and finite when we remove it. On the other hand, it depends on the detector resolution. Like in the plot of some kind of Disney movie, an apparently minor character whom you may have been tempted to regard as an ugly detail has saved the day.

I didn’t show explicitly that the coefficient of the log is the same function \( f_{IR}(q^2) \). In fact this function is \( f_{IR}(q^2) = \frac{1}{2} \log(-q^2/m^2) \), so the product \( f_{IR} \ln q^2 \sim \ln^2 q^2 \) is the Sudakov double logarithm. A benefit of the calculation which shows that the same \( f_{IR} \) appears in both places (Peskin chapter 6.5) is that it also shows that this pattern persists at higher order in \( \alpha \): there is a \( \ln^2(q^2/m_\gamma^2) \) dependence in the two-loop vertex correction, and a matching \( -\ln^2(E_c^2/m_\gamma^2) \) term in the amplitude to emit two soft photons. There is a \( \frac{1}{2!} \) from Bose statistics of these photons. The result exponentiates, and we get

\[
e^{-\frac{\alpha}{\pi} f \ln(-q^2/m_\gamma^2)} e^{-\frac{\alpha}{\pi} f(E_c^2/m_\gamma^2)} = e^{-\frac{\alpha}{\pi} f \ln(-q^2/E_c^2)}.
\]

You may be bothered that I’ve made all this discussion about the corrections from the electron line, but said nothing about the muon line. But the theory should make

\(^{39}\)Notice that we add the cross-sections, not the amplitudes, for these processes with different final states. Here’s why: even though we don’t measure the existence of the photon, something does: it gets absorbed by some part of the apparatus or the rest of the world and therefore becomes entangled some of its degrees of freedom; when we fail to distinguish between those states, we trace over them, and this erases the interference terms we would get if we summed the amplitudes.
sense even if the electron and muon charges $Q_e, Q_m$ were different, so the calculation should make sense term-by-term in an expansion in $Q_m$.

Some relevant names for future reference: The name for the guarantee that this always works in QED is the *Bloch-Nordsieck theorem*. Closely-related but more serious issues arise in QCD, the theory of quarks and gluons; this is the beginning of the story of *jets* (a jet is some IR-cutoff dependent notion of a QCD-charged particle plus the cloud of stuff it carries with it) and *parton distribution functions*.

---

**Sketch of exponentiation of soft photons.** [Peskin §6.5] Consider a diagram with $n$ soft external photons, summed over ways of distributing them on an initial and final electron line:

$$
\sum_{n_f=1}^n \sum_{n_i=1}^n u(p') i \mathcal{M}_0 u(p) e^n \prod_{\alpha=1}^{n_f} \left( \frac{p'_{\mu_\alpha}}{p' \cdot k_\alpha} - \frac{p_{\mu_\alpha}}{p \cdot k_\alpha} \right) \equiv \mathcal{A}_n.
$$

Here the difference in each factor is just as in (4.37), one term from initial and one from final-state emission; expanding the product gives the sum over $n_f = 1 - n_i$, the number coming from the final-state line. From this expression, we can make a diagram with a soft-photon loop by picking an initial line $\alpha$ and a final line $\beta$ setting $k_\alpha = -k_\beta \equiv k$ and tying them together with a propagator and summing over $k$:

$$\mathcal{A}_n = \mathcal{A}_{n-2} e^2 \int d^4k \frac{-i\eta_{\mu\sigma}}{k^2} \left( \frac{p'}{p' \cdot k} - \frac{p}{p \cdot k} \right)^{\alpha} \left( \frac{p'}{-p' \cdot k} - \frac{p}{-p \cdot k} \right)^{\sigma}$$

The factor of $\frac{1}{2}$ accounts for the symmetry under exchange of $\alpha \leftrightarrow \beta$. For the case of $n = 2$, this is the whole story, and this is

$$\bar{u} i \mathcal{M}_0 u \cdot X = \left( \begin{array}{c}
\end{array} \right) \cdot \left( \begin{array}{c}
\end{array} \right)_{\text{soft part}}$$

(where here ‘soft part’ means the part which is singular in $m_\gamma$) from which we conclude that

$$X = -\frac{\alpha}{2\pi} f_{IR}(q^2) \ln \left( \frac{-q^2}{m_\gamma^2} \right) + \text{finite}.$$
gives

\[ \mathcal{M}_\text{virtual soft} = \sum_{m=0}^{\infty} \left( \begin{array}{c} \text{virtual soft} \\ \infty \end{array} \right) \sum_{m} \frac{1}{m!} X^m \]

where the \(1/m!\) is a symmetry factor from interchanging the virtual soft photons.

Now consider the case of one real external soft (\(E \in [m_\gamma, E_c]\)) photon in the final state. The cross section is

\[ d\sigma_{1\gamma} = \int d\Pi \sum_{\text{pol}} \epsilon^\mu \epsilon'^\nu \mathcal{M}_\mu \mathcal{M}_\nu^* \]

\[ = |\bar{u}(p')M_0 u(p)|^2 \int \frac{d^3k}{2E_k} (-\eta_{\mu\nu}) \epsilon^2 \left( \frac{p'}{p'} \cdot k - \frac{p}{p} \cdot k \right) \epsilon^\mu \left( \frac{p'}{-p'} \cdot k - \frac{p}{-p} \cdot k \right) \epsilon^\nu \]

\[ \equiv d\sigma_0 Y, \]

\[ Y = \frac{\alpha}{\pi} f_{IR}(q^2) \ln \left( \frac{E_c^2}{m_\gamma^2} \right). \]

(The integral is done in Peskin, page 201.) Therefore, the exclusive cross section, including contributions of soft real photons gives

\[ \sum_{n=0}^{\infty} d\sigma_{n\gamma} = d\sigma_0 \sum_{n} \frac{1}{n!} Y^n = d\sigma_0 e^Y. \]

Here the \(n!\) is because the final state contains \(n\) identical bosons.

Putting the two effects together gives the promised cancellation of \(m_\gamma\) dependence to all orders in \(\alpha\):

\[ d\sigma = d\sigma_0 e^{2X} e^Y \]

\[ = d\sigma_0 \exp \left( -\frac{\alpha}{\pi} f_{IR}(q^2) \ln \frac{q^2}{m_\gamma^2} + \frac{\alpha}{\pi} f_{IR}(q^2) \ln \frac{E_c^2}{m_\gamma^2} \right) \]

\[ = d\sigma_0 \exp \left( -\frac{\alpha}{\pi} f_{IR}(q^2) \ln \frac{q^2}{E_c^2} \right) \]

This might seem pretty fancy, but unpacking the sum we did, the basic statement is that the probability of finding \(n\) photons with energy in a given (low-energy) range \([E_-, E_+]\) is

\[ P_{[E_-, E_+]} = \frac{1}{n!} \lambda^n e^{-\lambda}, \quad \lambda = \frac{\alpha}{\pi} f_{IR}(q^2) \ln \frac{E_+}{E_-} = \langle n \rangle = \langle n^2 \rangle - \langle n \rangle^2 \]

a Poisson distribution. This is just what one finds in a coherent state of the radiation field.
4.7.3 Some magic from gauge invariance of QED

We found that the self-energy of the electron gave a wavefunction renormalization factor

\[ Z_2 = 1 + \frac{\partial \Sigma}{\partial \not{p}} \big|_{\not{p}=0} + \mathcal{O}(e^4) = 1 - \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \text{finite} + \mathcal{O}(\alpha^2). \]

We care about this because there is a factor of \( Z_2 \) in the LSZ formula for an \( S \)-matrix element with two external electrons. On the other hand, we found a cutoff-dependent correction to the vertex \( e\gamma^\mu F_1(q^2) \) of the form

\[ F_1(q^2) = 1 + \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \text{finite} + \mathcal{O}(\alpha^2). \]

Combining these together

\[ S_{e\mu\leftrightarrow e\mu} = \left( \sqrt{Z_2(e)} \right)^2 \left( \gamma^\mu + \left( \gamma^\nu \gamma^\mu \gamma^\nu \right) + \cdots \right) \]

\[ = \left( 1 - \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \cdots \right) e^2 \bar{u}(p') \left( \gamma^\mu \left( 1 + \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \cdots \right) + \alpha \frac{i\sigma^{\mu\nu}q_\nu}{2m} \right) u(p) \]

the UV divergence from the vertex cancels the one in the self-energy. Why did this have to happen? During our discussion of the IR divergences, I mentioned a counterterm \( \delta_1 \) for the vertex. But how many counterterms do we get here? Is there a point of view which makes this cancellation obvious? Notice that the \( \cdots \) multiplying the \( \gamma^\mu \) term still contain the vacuum polarization diagram, which is our next subject, and which may be (is) cutoff dependent. Read on.

4.8 Vacuum polarization

[Zee, III.7] We’ve been writing the QED lagrangian as

\[ \mathcal{L} = \bar{\psi} \left( \not{\partial} + ie\not{A} - m \right) \psi - \frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}. \]

I’ve put tildes on the photon field because of what’s about to happen: Suppose we rescale the definition of the photon field \( e\tilde{A}_\mu \equiv A_\mu, e\tilde{F}_{\mu\nu} \equiv F_{\mu\nu} \). Then the coupling \( e \) moves to the photon kinetic term:

\[ \mathcal{L} = \bar{\psi} \left( \not{\partial} + i\tilde{A} - m \right) \psi - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}. \]

With this normalization, instead of measuring the coupling between electrons and photons, the coupling constant \( e \) measures the difficulty a photon has propagating through space:

\[ \langle A_\mu A_\nu \rangle \sim \frac{-i\eta_{\mu\nu}e^2}{q^2}. \]
None of the physics is different, since each internal photon line still has two ends on a $\bar{\psi}A\psi$ vertex.

But from this point of view it is clear that the magic of the previous subsection is a consequence of gauge invariance, here’s why: the demand of gauge invariance relates the coefficients of the $\bar{\psi}\partial\psi$ and $\bar{\psi}A\psi$ terms$^{40}$. Therefore, any counterterm we need for the $\bar{\psi}\partial\psi$ term (which comes from the electron self-energy correction and is traditionally called $\delta Z_2$) must be the same as the counterterm for the $\bar{\psi}A\psi$ term (which comes from the vertex correction and is called $\delta Z_1$). No magic, just gauge invariance.

A further virtue of this reshuffling of the factors of $e$ (emphasized by Zee on page 205) arises when we couple more than one species of charged particle to the electromagnetic field, e.g. electrons and muons or, more numerously, protons: once we recognize that charge renormalization is a property of the photon itself, it makes clear that quantum corrections cannot mess with the ratio of the charges. A deviation from $−1$ of the ratio of the charges of electron and proton as a result of interactions might seem plausible given what a mess the proton is, and would be a big deal for atoms. Gauge invariance forbids it.

Just as we defined the electron self-energy (amputated 2-point function) as $−i\Sigma(p)$ (with two spinor indices implied), we define the photon self-energy as

$$i\Pi_{\mu\nu}(q^2) = \frac{1}{e^2}$$

(the diagrams on the RHS are amputated). It is a function of $q^2$ by Lorentz symmetry. (The reason for the difference in sign is that the electron propagator is $+i\frac{1}{p−m}$ while the photon propagator is $−i\frac{\eta_{\mu\nu}}{q^2}$.) We can parametrize the answer as

$$\Pi_{\mu\nu}(q^2) = A(q^2)\eta_{\mu\nu} + B(q^2)q^\mu q^\nu.$$  

The Ward identity says

$$0 = q_\mu \Pi_{\mu\nu}(q^2) \implies 0 = Aq^\nu + Bq^2 q^\nu \implies B = −A/q^2.$$  

Let $A \equiv q^2$ so that

$$\Pi_{\mu\nu}(q^2) = q^2 \left( \eta_{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right).$$

$^{40}$Notice that the gauge transformation of the rescaled $A_\mu$ is $A_\mu \rightarrow A_\mu + \partial_\mu \lambda(x)$, $\psi(x) \rightarrow e^{iq\lambda(x)}\psi(x)$ so that $D_\mu \psi \equiv (\partial + qA_\mu)\psi \rightarrow e^{iq\lambda}D_\mu \psi$ where $q$ is the charge of the field ($q = −1$ for the electron). This is to be contrasted with the transformation of $\tilde{A}_\mu \rightarrow \tilde{A}_\mu - \partial_\mu \lambda(x)/e$.  

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This object $\Delta_{T}^{\mu\nu}$ is a projector

$$\Delta_{T}^{\mu\rho}\Delta_{T\nu}^{\rho} = \Delta_{T}^{\mu\nu} \quad (4.38)$$

onto modes transverse to $q^\mu$. Recall that we can take the bare propagator to be

$$\tilde{G}(2)(q) = \frac{-i\Delta_{T}}{q^2}$$

without changing any gauge-invariant physics. This is useful because then

$$\tilde{G}(2)(q) = \frac{-i\Delta_{T}}{q^2} \left(1 + i\Pi q^2 \Delta_{T} \left(\frac{-i\Delta_{T}}{q^2}\right) + i\Pi q^2 \Delta_{T} \left(\frac{-i\Delta_{T}}{q^2}\right) + \cdots\right)$$

$$\Delta_{T}^{2} = \frac{-i\Delta_{T}}{q^2} \left(1 + \Pi \Delta_{T} + \Pi^2 \Delta_{T} + \cdots\right) = \frac{-i\Delta_{T}}{q^2} \frac{1}{1 - \Pi(q^2)}. \quad (4.39)$$

Does the photon get a mass? If the thing I called $A$ above $q^2 \Pi(q^2) \to 0$ $A_0 \neq 0$ (that is, if $\Pi(q^2) \to A_0$ or worse), then $\tilde{G} q^2 \to 0 \frac{1}{q^2 - A_0}$ does not have a pole at $q^2 = 0$. If $\Pi(q^2)$ is regular at $q^2 = 0$, then the photon remains massless. In order to get such a singularity in the photon self energy $\Pi(q^2) \sim \frac{A_0}{q^2}$, we need a process like $\delta \Pi \sim \cdots$', where the intermediate state is a massless boson with propagator $\sim \frac{A_0}{q^2}$. As I will explain below, this is the Higgs mechanism (not the easiest way to understand it).

The Ward identity played an important role here. Why does it work for the vacuum polarization?

$$q^\mu \Pi_2^{\mu\nu}(q^2) = q^\mu \frac{1}{p + q - m} \frac{1}{p - m} \gamma^\nu. \quad (4.40)$$

But here is an identity:

$$\frac{1}{p + q - m} \frac{1}{p - m} = \frac{1}{p - m} - \frac{1}{p + q - m}. \quad (4.40)$$

Now, if we shift the integration variable $p \to p + q$ in the second term, the two terms cancel.

Why do I say ‘if’? If the integral depends on the UV limit, this shift is not innocuous. So we have to address the cutoff dependence.

In addition to the (lack of) mass renormalization, we’ve figured out that the electromagnetic field strength renormalization is

$$Z_{\gamma} \equiv Z_{3} = \frac{1}{1 - \Pi(0)} \sim 1 + \Pi(0) + \mathcal{O}(e^4).$$

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We need $Z_\gamma$ for example for the $S$-matrix for processes with external photons, like Compton scattering.

Claim: If we do it right\textsuperscript{41}, the cutoff dependence looks like\textsuperscript{42}:

$$\Pi_2(q^2) = \frac{\alpha_0}{4\pi} \left( - \frac{2}{3} \ln \Lambda^2 + \frac{2D(q^2)}{\text{finite}} \right)$$

where $\Lambda$ is the UV scale of ignorance. The photon propagator gets corrected to

$$\frac{e_0^2 \Delta_T}{q^2} \rightarrow \frac{Z_3e_0^2 \Delta_T}{q^2},$$

and $Z_3 = \frac{1}{1-\Pi(0)}$ blows up logarithmically if we try to remove the cutoff. You see that the fine structure constant $\alpha_0 = \frac{e_0^2}{4\pi}$ has acquired the subscript of deprecation: we can make the photon propagator sensible while removing the cutoff if we are willing to recognize that the letter $e_0$ we’ve been carrying around is a fiction, and write everything in terms of $e \equiv \sqrt{Z_3}e_0$ where $\frac{e^2}{4\pi} = \frac{1}{137}$ is the measured fine structure constant. To this order, then, we write

$$e_0^2 = e^2 \left( 1 + \frac{\alpha_0}{4\pi} \frac{2}{3} \ln \Lambda^2 \right) + O(\alpha^2).$$

$m_0 = m + O(\alpha_0) = m + O(\alpha)$. \hfill (4.42)

Since the difference between $\alpha_0$ and $\alpha$ is higher order (in either), our book-keeping is unchanged. Inverting the relationship perturbatively, the renormalized charge is

$$e^2 = e_0^2 \left( 1 - \frac{\alpha_0}{4\pi} \frac{2}{3} \ln \Lambda^2 + O(\alpha^2) \right)$$

– in QED, the quantum fluctuations reduce the charge, as you might expect from the interpretation of this phenomenon as dielectric screening.

In the example case of $e\mu \leftrightarrow e\mu$ scattering, the UV cutoff dependence looks like

$$S_{e\mu\leftrightarrow e\mu} = \sqrt{Z_\mu^2} \left( 1 - \frac{\alpha_0}{4\pi} \ln \Lambda^2 + \frac{\alpha_0}{2\pi} A(m_0) \right) e_0^2$$

\textsuperscript{41}What I mean here is: if we do it in a way which respects the gauge invariance and hence the Ward identity. The simple PV regulator we’ve been using does not quite do that. However, an only slightly more involved implementation, explained in Zee page 202-204, does. Alternatively, we could use dimensional regularization everywhere.

\textsuperscript{42}The factor in front of the $\ln \Lambda$ can be made to look like it does in other textbooks using $\alpha = \frac{e^2}{4\pi}$, so that

$$\frac{\alpha_0}{4\pi} \left( \frac{2}{3} \ln \Lambda^2 \right) = \frac{e_0^2}{12\pi^2} \ln \Lambda.$$
\[
L_\mu \bar{u}(p') \left[ \gamma^\mu \left( 1 + \frac{\alpha_0}{4\pi} \ln \Lambda^2 + \frac{\alpha_0}{2\pi} (B + D) + \frac{\alpha_0}{4\pi} \left( -\frac{2}{3} \ln \Lambda^2 \right) \right) + \frac{i\sigma^{\mu\nu} q_\nu \alpha_0}{2\pi} C(q^2, m_0) \right] u(p)
= e^2 L_\mu \bar{u}(p') \left[ \gamma^\mu \left( 1 + \frac{\alpha}{2\pi} (A + B + D) \right) + \frac{i\sigma^{\mu\nu} q_\nu \alpha_2}{2m} C \right] u(p) + \mathcal{O}(\alpha^2) \quad (4.43)
\]

where \( L_\mu \) is the stuff from the muon line, and \( A, B, C, D \) are finite functions of \( m, q \).

In the second step, two things happened: (1) we cancelled the UV divergences from the \( Z \)-factor and from the vertex correction: this had to happen because there was no possible counterterm. (2) we used (4.41) and (4.42) to write everything in terms of the measured \( e, m \).

Claim: this works for all processes to order \( \alpha^2 \). For example, Bhabha scattering gets a contribution of the form

\[
\propto e_0 \frac{1}{1 - \Pi(0)} e_0 = e^2.
\]

In order to say what are \( A, B, D \) we need to specify more carefully a renormalization scheme. To do that, I need to give a bit more detail about the integral.

### 4.8.1 Under the hood

The vacuum-polarization contribution of a fermion of mass \( m \) and charge \( e \) at one loop is

\[
q,\mu \quad \text{---} \quad q,\nu = - \int d^Dk \text{tr} \left( ie\gamma^\mu \right) \frac{i k^2}{k^2 - m^2} \left( ie\gamma^\nu \right) \frac{i (q + \bar{k} + m)}{(q + k)^2 - m^2}
\]

The minus sign out front is from the fermion loop. Some boiling, which you can find in Peskin (page 247) or Zee (§III.7), reduces this to something manageable. The steps involved are: (1) a trick to combine the denominators, like the Feynman trick \( \frac{1}{AB} = \int_0^1 dx \left( \frac{1}{(1-x)A + xB} \right)^2 \). (2) some Dirac algebra, to turn the numerator into a polynomial in \( k, q \). As Zee says, our job in this course is not to train to be professional integrators. The result of this boiling can be written

\[
i\Pi_2^{\mu\nu}(q) = -e^2 \int d^D\ell \int_0^1 dx \frac{N^{\mu\nu}}{(\ell^2 - \Delta)^2}
\]

with \( \ell = k + xq \) is a new integration variable, \( \Delta \equiv m^2 - x(1-x)q^2 \), and the numerator is

\[
N^{\mu\nu} = 2\ell^\mu \ell^\nu - \eta^{\mu\nu} \ell^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu} (m^2 + x(1-x)q^2) + \text{terms linear in } \ell^\mu.
\]

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At this point I have to point out a problem with applying the regulator we’ve been using (I emphasize that this is a distinct issue from the choice of RG scheme). With a euclidean momentum cutoff, the diagram \( \sim \gamma \) gives something of the form
\[
\begin{align*}
\mathcal{I}_2^{\mu \nu} &\propto e^2 \int_0^{\Lambda} d^4 \ell_E \frac{\ell_E^{\mu \nu}}{(\ell_E^2 - \Delta)^2} + \cdots \propto e^2 \Lambda^2 \eta^{\mu \nu}
\end{align*}
\]
This is NOT of the form \( \Pi^{\mu \nu} = \Delta_T^{\mu \nu} \Pi(p^2) \); rather it produces a correction to the photon mass proportional to the cutoff. What happened? Our cutoff was not gauge invariant. Oops.

**Fancier PV regularization.** [Zee page 202] We can fix the problem by adding also heavy Pauli-Villars electron ghosts. Suppose we add a bunch of them with masses \( m_a \) and couplings \( \sqrt{c_a} e \) to the photon. Then the vacuum polarization is that of the electron itself plus
\[
- \sum_a c_a \int \mathcal{D} k \text{tr} \left( (ie \gamma^\mu) \frac{i}{\not{q} + \not{k} - m_a} (ie \gamma^\nu) \frac{i}{\not{q} - m_a} \right) \sim \int \mathcal{D} k \left( \sum_a \frac{c_a}{k^2} + \sum_a \frac{c_a m_a^2}{p^4} + \cdots \right).
\]
So, if we take \( \sum_a c_a = -1 \) we cancel the \( \Lambda^2 \) term, and if we take \( \sum_a c_a m_a^2 = -m^2 \), we also cancel the \( \ln \Lambda \) term. This requires at least two PV electron fields, but so what? Once we do this, the momentum integral converges, and the Ward identity applies, so the answer will be of the promised form \( \Pi^{\mu \nu} = q^2 \Pi \Delta_T^{\mu \nu} \). After some more boiling, the answer is
\[
\Pi_2(q^2) = \frac{1}{2 \pi^2} \int dx x (1 - x) \ln \frac{M^2}{m^2 - x(1 - x)q^2}
\]
where \( \ln M^2 \equiv - \sum_a c_a \ln m_a^2 \). This \( M \) plays the role of the UV scale of ignorance thenceforth.

Notice that this is perfectly consistent with our other two one-loop PV calculations: in those, the extra PV electrons never get a chance to run. At higher loops, we would have to make sure to be consistent.

**Dimensional regularization.** A regulator which is more automatically gauge invariant is dimensional regularization (dim reg). I have already been writing many of the integrals in \( D \) dimensions. One small difference when we are considering this as a

\[43 \] Two points: How could we have predicted that the cutoff on euclidean momentum \( \ell_E^2 < \Lambda^2 \) would break gauge invariance? Its violation of the Ward identity here is a proof, but involved some work. The idea is that the momentum of a charged field shifts under a gauge transformation. Second: it is possible to construct a gauge invariant regulator with an explicit UV cutoff, using a lattice. The price, however, is that the gauge field enters only via the link variables \( U(x, \hat{e}) = e^{i \sum_{x \neq x'} A} \) where \( x \) is a site in the lattice and \( \hat{e} \) is the direction to a neighboring site in the lattice. For more, look up ‘lattice gauge theory’ in Zee’s index. More on this later.
regulator for an integral of fixed dimension is that we don’t want to violate dimensional analysis, so we should really replace

\[ \int d^4 \ell \rightarrow \int \frac{d^{4-\epsilon} \ell}{\bar{\mu}^{-\epsilon}} \]

where \( D = 4 - \epsilon \) and \( \bar{\mu} \) is an arbitrary mass scale which will appear in the regulated answers, which we put here to preserve dim’l analysis – i.e. the couplings in dim reg will have the same engineering dimensions they had in the unregulated theory (dimensionless couplings remain dimensionless). \( \bar{\mu} \) will parametrize our RG, i.e. play the role of the RG scale. (It is often called \( \mu \) at this step and then suddenly replaced by something also called \( \mu \); I will instead call this \( \bar{\mu} \) and relate it to the thing that ends up being called \( \mu \).)

[Zinn-Justin 4th ed page 233] Dimensionally regularized integrals can be defined systematically with a few axioms indicating how the \( D \)-dimensional integrals behave under

1. translations \( \int d^D p f(p + q) = \int d^D p f(p) \)
2. scaling \( \int d^D p f(s p) = |s|^{-D} \int d^D p f(p) \)
3. factorization \( \int d^D p \int d^D q f(p) g(q) = \int d^D p f(p) \int d^D q g(q) \)

The (obvious?) third axiom implies the following formula for the sphere volume as a continuous function of \( D \):

\[ \left( \frac{\pi}{a} \right)^{D/2} = \int d^D x e^{-a x^2} = \Omega_{D-1} \int_0^\infty x^{D-1} dx e^{-a x^2} = \frac{1}{2} a^{-\frac{D}{2}} \Gamma \left( \frac{D}{2} \right) \Omega_{D-1} . \quad (4.44) \]

This defines \( \Omega_{D-1} \) for general \( D \).

In dim reg, the one-loop vacuum polarization correction does satisfy the gauge-invariance Ward identity \( \Pi^{\mu \nu} = \Delta^{\mu \nu} T \delta \Pi \). A peek at the tables of dim reg integrals shows that \( \Pi_2 \) is:

\[ \Pi_2(p^2) \overset{\text{Peskin p. 252}}{=} - \frac{8e^2}{(4\pi)^{D/2}} \int_0^1 dx x(1-x) \frac{\Gamma(2-D/2)\Delta^{2-D/2}}{\Delta^{2-D/2}} \bar{\mu}^\epsilon \]
\[ \overset{D \rightarrow 4}{=} - \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \left( \frac{2}{\epsilon} - \log \left( \frac{\Delta}{\bar{\mu}^2} \right) \right) \]

where we have introduced the heralded \( \mu \):

\[ \mu^2 \equiv 4\pi \bar{\mu}^2 e^{-\gamma_E} \]

\[ \text{Note that this rule fails for the euclidean momentum cutoff. Also note that this is the property we needed to demonstrate the Ward identity for the vertex correction using (4.40).} \]
where $\gamma_E$ is the Euler-Mascheroni constant; we define $\mu$ in this way so that, like Rosen-
crantz and Guildenstern, $\gamma_E$ both appears and disappears from the discussion at this point.

In the second line of (8.9), we expanded the $\Gamma$-function about $D = 4$. Notice that what was a log divergence, becomes a $\frac{1}{\epsilon}$ pole in dim reg. There are other singularities of this function at other integer dimensions. It is an interesting question to ponder why the integrals have such nice behavior as a function of $D$. That is: they only have simple poles. A partial answer is that in order to have worse (e.g. essential) singularities at some $D$, the perturbative field theory would have to somehow fail to make sense at larger $D$.

Now we are in a position to choose a renormalization condition (also known as a renormalization scheme), which will specify how much of the finite bit of $\Pi$ gets sub-
tracted by the counterterm. One possibility is to demand that the photon propagator is not corrected at $q = 0$, i.e. demand $Z_\gamma = 1$. Then the resulting one-loop shift is

$$\delta \Pi_2(q^2) \equiv \Pi_2(q^2) - \Pi_2(0) = \frac{e^2}{2\pi^2} \int_0^1 dx x(1 - x) \log \left( \frac{m^2 - x(1 - x)q^2}{m^2} \right).$$

We’ll use this choice below.

Another popular choice, about which more later, is called the \MS scheme, in which $\Pi$ is defined by the rule that we subtract the $1/\epsilon$ pole. This means that the counterterm is

$$\delta^{(\MS)}_{\epsilon^2} = -\frac{e^2}{2\pi^2} \int_0^1 dx x(1 - x).$$

(Confession: I don’t know how to state this in terms of a simple renormalization condition on $\Pi_2$. Also: the bar in $\MS$ refers to the (not so important) distinction between $\bar{\mu}$ and $\mu$.) The resulting vacuum polarization function is

$$\delta \Pi_2^{(\MS)}(p^2) = \frac{e^2}{2\pi^2} \int_0^1 dx x(1 - x) \log \left( \frac{m^2 - x(1 - x)p^2}{\mu^2} \right).$$

### 4.8.2 Physics from vacuum polarization

One class of physical effects of vacuum polarization arise from attaching the corrected photon propagator to a static delta-function charge source. The resulting effective Coulomb potential is the fourier transform of

$$\tilde{V}(q) = \frac{1}{q^2} \frac{e^2}{1 - \Pi(q^2)} \equiv \frac{e_{\text{eff}}(q)}{q^2}. \quad (4.46)$$
This has consequences in both IR and UV.

**IR:** In the IR \((q^2 \ll m^2)\), it affects the spectra of atoms. The leading correction is
\[
\delta \Pi_2(q) = \int dx x(1-x) \ln \left(1 - \frac{q^2}{m^2} x(1-x)\right) \overset{q^2 \ll m^2}{\approx} \int dx x(1-x) \left(-\frac{q^2}{m^2} x(1-x)\right) = -\frac{q^2}{30m^2}
\]
which means
\[
\tilde{V}(q) \overset{q^2 \ll m^2}{\approx} e^2 \frac{q^2}{q^2} + e^2 \frac{q^2}{q^2} \left(-\frac{q^2}{30m^2}\right) + \cdots
\]
and hence
\[
V(r) = -\frac{e^2}{4\pi r^2} - \frac{e^4}{60\pi^2 m^2} \delta(r) + \cdots \equiv V + \Delta V.
\]
This shifts the energy levels of hydrogen s-orbitals (the ones with support at the origin) by \(\Delta E_s = \langle s|\Delta V|s\rangle\) which contributes to lowering the 2S state relative to the 2P state (the Lamb shift).

This delta function is actually a long-wavelength approximation to what is called the Uehling potential; its actual range is \(1/m_e\), which is the scale on which \(\Pi_2\) varies. The delta function approximation is a good idea for atomic physics, since \(1/m_e \ll a_0 = 1/\alpha m_e\), the Bohr radius. See Schwartz p. 311 for a bit more on this.

**UV:** In the UV limit \((q^2 \gg m^2)\), we can approximate \(\ln \left(1 - \frac{q^2}{m^2} x(1-x)\right) \simeq \ln \left(-\frac{q^2}{m^2} x(1-x)\right)\) to get
\[
\Pi_2(q^2) = \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \left(1 - \frac{q^2}{m^2} x(1-x)\right) \simeq \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \left(-\frac{q^2}{m^2}\right) = \frac{e^2}{12\pi^2} \ln \left(-\frac{q^2}{m^2}\right).
\]
Therefore, the effective charge in (4.46) at high momentum exchange is
\[
e_{\text{eff}}^2(q^2) \overset{q^2 \gg m^2}{\approx} \frac{e^2}{1 - \frac{e^2}{12\pi^2} \ln \left(-\frac{q^2}{m^2}\right)}. \tag{4.47}
\]
(Remember that \(q^2 < 0\) for t-channel exchange, as in the static potential, so the argument of the log is positive and this is real.)

Two things: if we make \(q^2\) big enough, we can make the loop correction as big as the 1. This requires \(|q| \sim 10^{286}\) eV. Good luck with that. This is called a Landau pole. The second thing is: this perspective of a scale-dependent coupling is very valuable, and is a crucial ingredient in the renormalization group.

\[\text{45The last step is safe since the } x(1-x) \text{ suppresses the contributions of the endpoints of the } x \text{ integral, so we can treat } x(1-x) \text{ as finite.}\]