University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2020
Assignment 1

Due 12:30pm Monday, January 13, 2020

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1. **Brain-warmer.** Convince yourself that

$$(\partial_g)^n \, e^{-1/g}|_{g=0} = 0 \quad \forall n.$$

This means that a function of the form $e^{-1/g}$ does not have a useful series expansion about $g = 0$.

2. **Scale invariant quantum mechanics.**

Consider the action for one quantum variable r with $r > 0$ and

$$S[r] = \int dt \left(\frac{1}{2} m \dot{r}^2 - V(r) \right), \qquad V(r) = \frac{\lambda}{r^2} \, .$$

(a) Show that the (non-relativistic) mass parameter m can be eliminated by a multiplicative redefinition of the field r or of the time t. As a result, convince yourself that the physics of interest here should only depend on the combination $m\lambda$. Show that the coupling $m\lambda$ is dimensionless: $[m\lambda] = 0$.

(b) Show that this action is *scale invariant*, *i.e.* show that the transformation

$$r(t) \to s^\alpha \cdot r(st) \tag{1}$$

(for some α which you must determine), (with $s \in \mathbb{R}^+$) is a symmetry. Find the associated Noether charge $\mathcal{D}$. For this last step, it will be useful to note that the infinitesimal version of (1) is ($s = e^a, a \ll 1$)

$$\delta r(t) = a \left(\alpha + t \frac{d}{dt} \right) r(t).$$

(c) Find the position-space Hamiltonian $\mathbf{H}$ governing the dynamics of r. Show that the Schrödinger equation is Bessel's equation

$$\left(-\frac{\partial_r^2}{2m} + \frac{\lambda}{r^2} \right) \psi_E(r) = E \psi_E(r).$$

Show that the Noether charge associated $\mathcal{D}$ with scale transformations ($\equiv$ dilatations) satisfies: $[\mathcal{D}, \mathbf{H}] = -\mathbf{i}\mathbf{H}$. This equation says that the Hamiltonian has a definite scaling dimension, *i.e.* that its scale tranformation is $\delta \mathbf{H} = \mathbf{i} a [\mathcal{D}, \mathbf{H}] = +a\mathbf{H}$. Note that you should not need to use arcane facts about Bessel functions, only the asymptotic analysis of the equation, in subsequent parts of the problem.

(d) Describe the behavior of the solutions to this equation as $r \to 0$. [Hint: in this limit you can ignore the RHS. Make a power-law ansatz: $\psi(r) \sim r^\Delta$ and find Δ.]

(e) What happens if $2m\lambda < -\frac{1}{4}$? It looks like there is a continuum of negative-energy solutions (boundstates). This is another example of a *too-attractive* potential.

(f) A hermitian operator has orthogonal eigenvectors. We will show next that to make $\mathbf{H}$ hermitian when $2m\lambda < -\frac{1}{4}$, we must impose a constraint on the wavefunctions:

$$\left(\psi_E^\star \partial_r \psi_E - \psi_E \partial_r \psi_E^\star\right)|_{r=0} = 0 \tag{2}$$

There are two useful perspectives on this condition: one is that the LHS is the probability current passing through the point $r = 0$.

The other perspective is the following. Consider two eigenfunctions:

$$\mathbf{H}\psi_E = E\psi_E, \quad \mathbf{H}\psi_{E'} = E'\psi_{E'}.$$

Multiply the first equation by $\psi_{E'}^\star$ and integrate; multiply the second by $\psi_E^\star$ and integrate; take the difference. Show that the result is a boundary term which must vanish when $E = E'$.

(g) Show that the condition (2) is empty for $2m\lambda > -\frac{1}{4}$. Impose the condition (2) on the eigenfunctions for $2m\lambda < -\frac{1}{4}$. (It will help to impose it at $r = \epsilon$ for some UV cutoff ϵ.) Show that the resulting spectrum of boundstates has a *discrete* scale invariance.

[Cultural remark: For some reason I don't know, restricting the Hilbert space in this way is called a *self-adjoint extension*.]

(h) [Extra credit] Consider instead a particle moving in $\mathbb{R}^d$ with a central $1/r^2$ potential, $r^2 \equiv \vec{x} \cdot \vec{x}$,

$$S[\vec{x}] = \int dt \left(\frac{1}{2} m\dot{\vec{x}} \cdot \dot{\vec{x}} - \frac{\lambda}{r^2}\right).$$

Show that the same analysis applies (*e.g.* to the s-wave states) with minor modifications.

[A useful intermediate result is the following representation of (minus) the laplacian in $\mathbb{R}^d$:

$$\vec{p}^2 = -\frac{1}{r^{d-1}}\partial_r\left(r^{d-1}\partial_r\right) + \frac{\hat{L}^2}{r^2}, \quad \hat{L}^2 \equiv \frac{1}{2}\hat{L}_{ij}\hat{L}_{ij}, \quad L_{ij} = -\mathbf{i}\left(x_i\partial_j - x_j\partial_i\right),$$

where $r^2 \equiv x^i x^i$. By 's-wave states' I mean those annihilated by $\hat{L}^2$.]