1. **Brain-warmer.** Prove the Gordon identities

\[
\bar{u}_2 (q^\nu \sigma_{\mu \nu}) u_1 = i \bar{u}_2 ((p_1 + p_2)_\mu - (m_1 + m_2) \gamma_\mu) u_1
\]

and

\[
\bar{u}_2 ((p_1 + p_2)^\nu \sigma_{\mu \nu}) u_1 = i \bar{u}_2 ((p_2 - p_1)_\mu - (m_2 - m_1) \gamma_\mu) u_1
\]

where \( q \equiv p_2 - p_1 \) and \( \bar{\psi}_1 u_1 = m_1 u_1, \bar{\psi}_2 u_2 = m_2 u_2 \), using the definitions and the Clifford algebra.

2. **Pauli-Villars practice.**

Consider a field theory of two scalar fields with

\[
\mathcal{L} = -\frac{1}{2} \phi \square \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} \Phi \square \Phi - \frac{1}{2} M^2 \Phi^2 - g \phi \Phi^2 + \text{counterterms}
\]

Compute the one-loop contribution to the self-energy of \( \Phi \). Use a Pauli-Villars regulator – introduce a second copy of the \( \phi \) field of mass \( \Lambda \) with the wrong-sign propagator.

\[
\Sigma_\Phi(p) = \int \frac{d^D k}{k^2 - m^2 (k + p)^2} \frac{i}{M^2 (-ig)^2} \tag{1}
\]

\[
= g^2 \int_0^1 dx \int \frac{d^D k}{(1 - x)(k^2 - m^2) + x((k + p)^2 - M^2))^2} \tag{2}
\]

\[
= g^2 \int_0^1 dx \int \frac{dD \ell}{(\ell^2 - \Delta + i\epsilon)^2}, \quad \ell = k - px, \Delta = xM^2 + (1 - x)m^2 - p^2 x(1 - x) \tag{3}
\]

\[
\equiv g^2 \int_0^1 dx J(\Delta(m)) \tag{4}
\]

The Pauli-Villars regulator replaces the \( \phi \) propagator by

\[
\frac{i}{p^2 - m^2} \xrightarrow{\Lambda} \frac{i}{p^2 - m^2} - \frac{i}{p^2 - \Lambda^2}
\]
so that the self energy is replaced by

\[
\Sigma_\Phi(p) = g^2 \int_0^1 dx \left( J(\Delta(m)) - J(\Delta(\Lambda)) \right)
\]

(5)

\[
= \frac{g^2}{8\pi^2} \int_0^1 dx \log \frac{\Delta(\Lambda^2)}{\Delta(m^2)}
\]

(6)

\[
= \frac{g^2}{8\pi^2} \int_0^1 dx \log \frac{\Delta(\Lambda^2)}{\Delta(m^2)}
\]

(7)

\[
\Lambda \gg \text{everyone} \quad \Rightarrow \quad \frac{g^2}{8\pi^2} \int_0^1 dx \log \left( \frac{x\Lambda^2}{xM^2 + (1-x)m^2 - p^2x(1-x)} \right)
\]

(8)

Actually there is also second diagram, where the fermion emits a single scalar which ends at a fermion bubble:

\[
-i\Sigma_{tadpole}^{\text{pole}}(p) = (-ig)^2 \int d^4k \frac{i}{k^2 - M^2 - m^2}.
\]

This is independent of the external momentum, and so only contributes to the mass renormalization. A complication that arises here is that the loop contains only a fermion propagator, so our PV regulator involving only a heavy scalar above will not regularize this divergence. We must also add a heavy fermion ghost field. (Such a step is also required to regulate the corrections to the scalar propagator from a fermion bubble.) I’m going to ignore this diagram below.

Determine the counterterms required to impose that the \( \Phi \) propagator has a pole at \( p^2 = M^2 \) with residue 1.

To do this, expand (8) about \( p^2 = M^2 \):

\[
\Sigma_\Phi(p) = \frac{g^2}{8\pi^2} \int_0^1 dx \log \frac{x\Lambda^2}{M^2(1-x)^2 + m^2x} + (p^2 - M^2) \frac{g^2}{8\pi^2} \int_0^1 dx \frac{x(1-x)}{M^2(1-x)^2 + m^2x}
\]

(9)

\[
\equiv S_1 + (p^2 - M^2)S_2
\]

(10)

and do the \( x \) integrals. The mass correction depends on the cutoff like \( \log \Lambda \), but \( \delta_x \) is independent of the cutoff. The actual expressions (which Mathematica can tell you if you are patient enough) are not very illuminating and I don’t want to type them, but it’s worth noticing that they are singular when \( m = 2M \). Why?

When \( m = 2M \) the intermediate state can be on shell.

The total contribution to \( \Sigma \), including the counterterms \( \mathcal{L}_{ct} = -\delta Z^\frac{1}{2}(\partial\phi)^2 + \)
\[ \delta_{M^2} \frac{1}{2} \Phi^2 \text{ (note the signs) is} \]

\[ 0 + (p^2 - M^2)0 + \mathcal{O}(p^2 - M^2)^2 \]

\[ = \Sigma(p) - \delta_{M^2} - p^2 \delta_Z \]

\[ = S_1 + (p^2 - M^2)S_2 - \delta_{M^2} - p^2 \delta_Z + \mathcal{O}(p^2 - M^2)^2 \]

\[ = S_1 - M^2 S_2 - \delta_{M^2} + p^2 (S_2 - \delta_Z) + \mathcal{O}(p^2 - M^2)^2. \]

We conclude that we need to set

\[ \delta_{M^2}^2 = S_1 - M^2 S_2, \quad \delta_Z = S_2 \]

to satisfy the stated renormalization conditions. Notice that in this process, we not only remove the cutoff dependence, but we also determine the finite parts of the counterterms.

3. **Bosons have worse UV behavior than fermions.**

Consider the Yukawa theory

\[ S[\phi, \psi] = - \int d^Dx \left( \frac{1}{2} \phi (\Box + m_\phi) \phi + \bar{\psi} (-\mathcal{D} + m_\psi) \psi + y \phi \bar{\psi} \psi + \frac{g}{4!} \phi^4 \right) + \text{counterterms.} \]

(a) Show that the superficial degree of divergence for a diagram \( A \) with \( B_E \) external scalars and \( F_E \) external fermions is

\[ D_A = D + (D - 4) \left( V_g + \frac{1}{2} V_y \right) + B_E \left( \frac{2 - D}{2} \right) + F_E \left( \frac{1 - D}{2} \right) \]

where \( V_g \) and \( v_y \) are the number of \( \phi^4 \) and \( \phi \bar{\psi} \psi \) vertices respectively.

All the discussion below is about one loop diagrams.

(b) Draw the diagrams contributing to the self energy of both the scalar and the spinor in the Yukawa theory.

(c) Find the superficial degree of divergence for the scalar self-energy amplitude and the spinor self-energy amplitude.

(d) In the case of \( D = 3 + 1 \) spacetime dimensions, show that (with a cutoff on the Euclidean momenta) the spinor self-energy is actually only logarithmically divergent. (This type of thing is one reason for the adjective ‘superficial’.)

Hint: the amplitude can be parametrized as follows: if the external momentum is \( p^\mu \), it is

\[ \mathcal{M}(p) = A(p^2) \phi + B(p^2). \]

Show that \( B(p^2) \) vanishes when \( m_\psi = 0. \)
See Zee, page 180. In four dimensions, the scalar self-energy has $D = 2$ and indeed depends quadratically on the cutoff. The fermion self-energy has $D = 1$, but the would-be leading divergence of the fermion self-energy is an integral with an odd integrand and therefore vanishes, leaving behind a mere log.

A better argument for this conclusion follows from the chiral transformation $\Psi \rightarrow e^{i\gamma^\alpha}\Psi$, which becomes a symmetry when the fermion mass is zero. This means that the correction to the fermion mass $B(0)$ must be proportional to the mass itself (it must go to zero when the mass goes to zero, and must be analytic in the mass for some reason I am unable to summon at the moment). Combining this statement with the dimensional analysis above, we conclude that there cannot be linear dependence on the cutoff.


(a) In what number of space dimensions does a four-fermion interaction such as $G\bar{\psi}\psi\bar{\psi}\psi$ have a chance to be renormalizable? Assume Lorentz invariance. [optional] Generalize the formula (14) for $D_A$ to include a number $V_G$ of four-fermion vertices.

I find

$$D_A = D + (D - 4) \left( V_g + \frac{1}{2} V_y \right) (D - 2) V_G + B_E \left( \frac{2 - D}{2} \right) + F_E \left( 1 - \frac{D}{2} \right).$$

Therefore the four-fermion interaction is scale invariant in $D = 2$ spacetime dimensions.

(b) If we violate Lorentz invariance the story changes. Consider a non-relativistic theory with kinetic terms of the form $\int dt d^d x \left( \bar{\psi} \left( i\partial_t - D\nabla^2 \right) \psi \right).$ (Here $D$ is a dimensionful constant. In a relativistic theory we relate dimensions of time and space by setting the speed of light to one; here, there is no such thing, and we can choose units to set $D$ to one.) For what number of space dimensions might the four-fermion coupling be renormalizable?

You actually already know the answer to this from our study in the first lecture of the delta function potential. The $\bar{\psi}\psi\bar{\psi}\psi$ is exactly such a contact interaction between two particles. So it is marginal when $d = 2$. Alternatively, you can count inverse-length dimensions of the time-derivative term to learn that $[\psi] = d/2$, and of the $\nabla^2$ term to learn that $[t] = -2$, i.e. we must scale $t$ twice as fast as space to make the free theory scale invariant. Then $0 = \left[ G \int dt d^d x \bar{\psi}\psi\bar{\psi}\psi \right] = \left[ G \right] - 2 - d + 2d$ gives $\left[ G \right] = 2 - d$. If $d > 2$ it is an irrelevant perturbation of the free theory.
(c) In the previous example, the scale transformation preserving the kinetic terms acted by \( t \rightarrow \lambda^2 t, x \rightarrow \lambda x \). More generally, the relative scaling of space and time is called the *dynamical exponent* \( z \) (\( z = 2 \) in the previous example). Suppose that the kinetic terms are first order in time and quadratic in the fields. Ignoring difficulties of writing local quadratic spatial kinetic terms, what is the relationship between \( d \) and \( z \) which gives scale invariant quartic interactions? What if the kinetic terms are second order in time (as for scalar fields)?

To get dynamical exponent \( z \) with first-order-in-time derivatives, we’d need a kinetic term like

\[
S_0 = \int dt dx \bar{\psi} \left( i\partial_t - \nabla^2 \right) \psi.
\]

So \([\psi] = -d/2\) still, but \([t] = -z\). Therefore \(0 = [G \int dt dx \bar{\psi} \psi \psi] = [G] - z - d + 2d\) gives \([G] = z - d\) and it is scale invariant if \( d = z \).

With second-order-in-time derivatives, we have

\[
S_0 = \int dt dx (\partial_t^2 - \nabla^2) \phi
\]

so \(0 = -z - d + 2z + 2[\phi]\) says \([\phi] = (d - z)/2\), and so \(0 = [g \int dt dx \phi^4] = -z - d + 4(d - z) + [g]\) says \([g] = 3z - d\). It is classically scale invariant if \( d = 3z \).

5. **The magnetic moment of a Dirac fermion.** [This problem is optional, but highly recommended.] In this problem we consider the hamiltonian density

\[
h_I = q \bar{\Psi} \gamma^\mu \Psi A_\mu.
\]

As we’ve discussed, this describes a local, Lorentz invariant, and gauge invariant interaction between a Dirac fermion field \( \Psi \) and a vector potential \( A_\mu \). In this problem we will treat the vector potential, representing the electromagnetic field, as a fixed, classical background field.

Define single-particle states of the Dirac field by \( \langle 0 | \Psi(x) | \vec{p}, s \rangle = e^{-ipx} u_s(p) \). We wish to show that these particles have a magnetic dipole moment, in the sense that in their rest frame, their (single-particle) hamiltonian has a term \( h_{NR} \ni \mu B \vec{S} \cdot \vec{B} \) where \( \vec{S} = \frac{1}{2} \vec{\sigma} \) is the particle’s spin operator.

(a) \( q \) is a real number. What is required of \( A_\mu \) for \( H_I = \int d^3 x h_I \) to be hermitian?

\( A = A^\dagger \).  

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(b) How must $A_\mu$ transform under parity $P$ and charge conjugation $C$ in order for $H_I$ to be invariant? How do the electric and magnetic fields transform? Show that this allows for a magnetic dipole moment but not an electric dipole moment.

A look at the transformation of $\bar{\Psi} \gamma^\mu \Psi$ under $P$ and $C$ (see e.g. Peskin page 71 for a reminder) tells us that we require $P : A_\mu \to (-1)^\mu A_\mu, C : A_\mu \to A_\mu$. Therefore $P : E \to -E, C : E \to -E, P : B \to B, C : B \to -B$. Since $P : \vec{S} \to +\vec{S}, C : q\vec{S} \to -q\vec{S}$, the EDM $\vec{E} \cdot \vec{S}$ is odd under $P$, while the magnetic dipole moment is invariant.

$A$ must be parity even and odd under charge conjugation, and hence odd under $CP$. $B$ is even under $CP$ while $E$ is odd. $\vec{S}$ is even under $C$ but odd under $P$. So $CP$ forbids $\vec{E} \cdot \vec{S}$, and nothing forbids $\vec{B} \cdot \vec{S}$.

(c) Show that in the non-relativistic limit

$$\bar{u}(p') \gamma^\mu u'(p) F_{\mu\nu} = a \xi^\dagger \sigma \cdot \vec{B} \xi'$$

for some constant $a$ (find $a$). Recall that $\gamma^\mu \equiv \frac{1}{2} [\gamma^\mu, \gamma^\nu]$. Here $u, u'$ are positive-energy solutions of the Dirac equation with mass $m$ and

$$u = \left( \frac{\sqrt{p \cdot \sigma} \xi}{\sqrt{p \cdot \sigma}} \right), \quad u' = \left( \frac{\sqrt{p' \cdot \sigma} \xi'}{\sqrt{p' \cdot \sigma}} \right).$$

$$u = \left( \frac{\sqrt{p \cdot \sigma} \xi}{\sqrt{p \cdot \sigma}} \right) = \sqrt{m} \left( \frac{1 - \frac{p \cdot \sigma}{2m}}{1 + \frac{p \cdot \sigma}{2m}} \right) + O \left( \frac{|p|}{m} \right)$$

(d) Suppose that $A_\mu$ describes a magnetic field $\vec{B}$ which is uniform in space and time.

Show that in the non-relativistic limit

$$\langle \vec{p'}, s' | H_I | \vec{p}, s \rangle = \delta^3 (\vec{p} - \vec{p'}) h(\xi, \xi', \vec{B})$$

and find the function $h(\xi, \xi', \vec{B})$. You may wish to use the Gordon identity. Rewrite the result in terms of single-particle states with non-relativistic normalization (i.e. $\langle \vec{p} | \vec{p} \rangle_{NR} = \delta^3 (p - p')$). Interpret $h$ as a non-relativistic hamiltonian term saying that the gyromagnetic ratio of the electron is $-g \frac{|q|}{2m}$ with $g = 2$.

The interaction term is $H_I = \int d^3x \bar{\Psi} q \gamma^\mu A_\mu \Psi(x)$. Treating $A(x)$ as a background field, the matrix element is

$$\langle \vec{p'}, s' | H_I | \vec{p}, s \rangle = \int d^3x \bar{u}(p') e^{i\vec{p'} \cdot \vec{x}} q \gamma^\mu u(p) e^{-i\vec{p} \cdot \vec{x}} A_\mu(x)$$
(plus a disconnected term which we can ignore). One way to proceed is to use the Gordon identity here:

\[
\langle \vec{p}', s' | H_I | \vec{p}, s \rangle = \frac{q}{2m} \int d^3x \bar{u}'(p') e^{-i(p-p')x} (-i\sigma^{\mu\nu}(p-p')_\nu + (p+p')^{\mu}) u^s(p) A_\mu(x)
\]

(15)

\[
= \frac{q}{2m} \int d^3x \bar{u}'(p') \left( \partial_\nu e^{-i(p-p')x} \sigma^{\mu\nu} \right) u^s(p) A_\mu(x) + \text{spin-independent}
\]

(16)

\[
\text{IBP} = \frac{q}{2m} \bar{u}'(p') \sigma^{\mu\nu} u^s(p) \int d^3x e^{-i(p-p')x} (-\partial_\nu A_\mu)
\]

(17)

\[
= \frac{q}{2m} \bar{u}'(p') \sigma^{\mu\nu} u^s(p) \frac{1}{2} \tilde{F}_{\mu\nu}(p - p')
\]

(18)

Now if \( F_{\mu\nu} \) represents a constant magnetic field, then \( \sigma^{\mu\nu} \tilde{F}_{\mu\nu}(p - p') = \delta^{(3)}(p - p')i\sigma^i B_i \).

Alternatively, we can choose \( A_0 = 0 \) gauge, and take the non-relativistic limit after the first step

\[
\bar{u}' \gamma^i u \overset{NR}{\approx} -\xi'(\vec{p}' \cdot \sigma \sigma^i + \sigma^i \vec{p} \cdot \vec{\sigma}) \xi = \xi' \sigma^k \xi (p' - p)^j \epsilon_{ijk} + \xi' \xi (p + p')^i.
\]

The second term is again independent of the spin and we can ignore it.

The NR hamiltonian is \( h = \mu_B \vec{\sigma} \cdot \vec{B} \), with \( \mu_B = \frac{2q}{2m} \).

(e) How does the result change if we add the term

\[
\Delta H = \frac{c}{M} \bar{\Psi} F_{\mu\nu} [\gamma^\mu, \gamma^\nu] \Psi
\]

\( \frac{cm}{M} \) gets added to the magnetic dipole moment.

A similar calculation shows that \( \frac{c}{M} \bar{\Psi} \gamma^5 F_{\mu\nu} [\gamma^\mu, \gamma^\nu] \Psi \) produces an electric dipole moment (and as you can see, violates parity).