1. **Brain-warmer.** Check that \( (\Delta_T)_{\rho}^{\mu} \equiv \delta_{\rho}^{\mu} - \frac{q_{\mu} q_{\rho}}{q^2} \) is a projector onto momenta transverse to \( q^\rho \).

   This requires showing both that \( \Delta q = 0 \) and that \( \Delta^2 = \Delta \).

2. **Tadpole diagrams.**

   \[ \text{(a) Why don’t we worry about the following diagram as a correction to the electron self-energy in QED?} \]

   It has to vanish by Lorentz symmetry: the object \( \partial \) would be a source \( j^{\mu} \) for the electromagnetic field in the vacuum. At one loop, we can check that \( \int d^4 k \gamma^{\mu} \frac{k + m}{k^2 - m^2} = 0 \) by \( \text{tr} \gamma^{\mu} = 0 \) and Lorentz symmetry, \( \int d^d k k^{\mu} f(k^2) = 0 \).

   The one-point function for the photon also has to vanish by charge-conjugation symmetry (in fact any odd-point function of the photon does for the same reason; this is called Furry’s theorem).

   More generally, a \textit{tadpole diagram} – a diagram with a single field line coming out of it – represents a source for the field. When we developed our Feynman rules, we expanded around a minimum of the potential for the field, and this is why there is no one-point vertex in the Feynman rules. A tadpole diagram is saying that radiative effects are producing a shift in the minimum of the potential. The (quadratic part of the) action wants to change to \( \int ((\partial A)^2 - m^2 A^2 + Aj) \). The equations of motion for the zero-momentum field tell us that the minimum is at \( A = j/m_\gamma \). In the case of a massless field, the shift is arbitrarily large (in this linear approximation). This is the source of the IR divergence in the tadpole diagram as \( m_\gamma \to 0 \). In QED, this is moot because \( j = 0 \).

For the remainder of the problem, we consider \( \phi^3 \) theory with a (small) mass:

\[
S = \int d^D x \left( \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{g}{3!} \phi^3 \right).
\]
(b) Notice that unlike $\phi^4$ theory (or QED), there is no symmetry which forbids a one-point function for the scalar. Why don’t we lose generality by not adding a term linear in $\phi$ to the Lagrangian? We can shift it away by a field redefinition, $\phi \to \phi - a$. It is convenient to choose $a$ to make the linear term vanish, since then the solution to the equations of motion has $\phi_0 = 0$.

(c) Now think about the following contribution to the scalar self-energy: Show that in the limit $m \to 0$ there is an IR divergence. By thinking about the significance for the scalar potential of this part of the diagram explain the meaning of this divergence.

The object is a one-point function for the scalar. As explained in the answer to the previous part of the problem, the presence of such a one-point function ($V_{\text{eff}} \ni v\phi$, with $v \propto g$) means we are doing perturbation theory about a configuration which is not a solution to the equations of motion at order $g$. The correct solution to the equations of motion is $\phi_0$ with $0 = m^2\phi_0 + v$ so $\phi_0 = -v/m^2$, which diverges when $m \to 0$. This is the origin of the IR divergence – the field theory is trying to find its minimum which, when $m \to 0$, is arbitrarily far away in field space.

3. **Symmetry is attractive.** Consider a field theory in $D = 3+1$ with two massless (for simplicity) scalar fields which interact via the interaction

$$V = -\frac{g}{4!} (\phi_1^4 + \phi_2^4) - \frac{2\lambda}{4!} \phi_1^2 \phi_2^2.$$

(a) Show that when $\lambda = g$ the model possesses an $O(2)$ symmetry.

At this special point, the potential is $(\phi_1^2 + \phi_2^2)^2$, which depends only on the distance from the origin of the field space.

(b) Will you need a counterterm of the form $\phi_1 \phi_2$ or $\phi_1 \Box \phi_2$ (for general $g, \lambda$)? If not, why not?

A very important point: such terms can’t be generated because they violate the $\mathbb{Z}_2$ symmetry which takes $(\phi_1, \phi_2) \to (-\phi_1, \phi_2)$. In general, radiative effects (i.e. loops) will not violate symmetries of the bare action. Exceptions to this statement are called *anomalies*; this only happens when no regulator preserves the symmetry in question.
(c) Renormalize the theory to one loop order by regularizing (for example with a
euclidean momentum cutoff or Pauli Villars), adding the necessary counter-
terms, and imposing a renormalization condition on the masses and \(2 \to 2\)
scattering amplitudes at some values of the kinematical variables \(s_0, t_0, u_0\).

I’ll use a hard euclidean momentum cutoff since then we can reuse our results
from \(\phi^4\) theory. To save typing let me define \(L(x) \equiv \frac{1}{32\pi^2} \log x\). Every loop
integral we will encounter is the same as in the pure massless \(\phi^4\) theory that
we did in lecture.

The symmetry which interchanges \(\phi_1 \leftrightarrow \phi_2\) guarantees that their self-
couplings \(g\) (and the masses) stay equal (using the same principle as above).
This means we have only three counterterms to determine altogether: \(\delta m^2\)
and two four-point counterterms \((\delta g, \delta \lambda)\). That is, we have to impose two
renormalization conditions on the four-point functions.

First an annoying point: with the given normalization, the 1122 vertex is
actually \(-i\lambda/3\).

The self-energy for \(\phi_1\) is

\[
-i\Sigma(p^2) = \begin{array}{c}
\text{I}
\end{array} = \begin{array}{c}
\text{I}
\end{array} + \begin{array}{c}
\text{I}
\end{array} + \ldots = -i(g+\lambda/3)c\Lambda^2 + O(g, \lambda)^2
\]

where \(c\) is a numerical constant that I can’t remember right now and which
we don’t need. To put the pole at \(p^2 = m^2_p = 0\), we need the bare mass to be

\[m^2(\Lambda) = -\Sigma(p^2 = 0) = (g + \lambda/4)c\Lambda^2.\]

As in \(\phi^4\) theory, there is no wavefunction renormalization at one loop because
\(\Sigma\) is independent of \(p^2\).

There are three different \(2 \to 2\) scattering processes to consider: 11 \(\to\)
11, 11 \(\to\) 22, 12 \(\to\) 12. (The corrections to 22 \(\to\) 22 are the same as those
for 11 \(\to\) 11, and similarly 22 \(\to\) 11 is the same as 11 \(\to\) 22, by the exchange
symmetry.) Then using the notation \(\langle \phi_1 \phi_2 \rangle = \langle \phi_1 \phi_2 \rangle\)
we have

\[
\mathcal{M}_{11 \leftrightarrow 11} = -g + (g^2 + \left(\frac{\lambda}{3}\right)^2)(L(s/\Lambda^2) + L(t/\Lambda^2) + L(u/\Lambda^2)) + \delta g
\]

(1)

\[
\begin{array}{c}
\text{I}
\end{array} = \begin{array}{c}
\text{I}
\end{array} + \begin{array}{c}
\text{I}
\end{array} + \begin{array}{c}
\text{I}
\end{array}
\]

(2)
The $\lambda^2$ term involves $\phi_2$ running in the loop. (Note that I am writing $iM = -ig + (-ig)^2...$ and dividing the BHS by $i$.) Beware the symmetry factor of $\frac{1}{2}$ in each loop diagram.

$$M_{22-11} = -\frac{\lambda}{3} + \frac{\lambda}{3}g^2L(s/\Lambda^2) + \left(\frac{\lambda}{3}\right)^2(2L(t/\Lambda^2) + 2L(u/\Lambda^2)) + \delta\lambda$$ (3)

where the 2 in the s-channel term is from the fact that either $\phi_1$ or $\phi_2$ can run in the loop. The last two diagrams have a different symmetry factor from the others, since we can’t exchange the two propagators in the loop – so they get an extra factor of 2.

$$M_{12-12} = -\frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2(2L(s/\Lambda^2) + 2L(u/\Lambda^2)) + 2\frac{\lambda}{3}gL(t/\Lambda^2) + \delta\lambda$$ (5)

Using the renormalization conditions $M_{11-11}(s_0 = t_0 = u_0) = -g_P$ and $M_{22-11}(s_0 = t_0 = u_0) = -\frac{\lambda_P}{3}$ we find

$$\lambda(\Lambda) \equiv \lambda + \delta\lambda = \lambda_P + 2\lambda_P g_Pl + \frac{4}{3}L + O(\lambda_P, g_P)^2$$ (7)

$$g(\Lambda) \equiv g + \delta g = g_P + \left(g_P^2 + \left(\frac{\lambda_P}{3}\right)^2\right)3L + O(\lambda_P, g_P)^2$$ (8)

where $L \equiv L(s_0/\Lambda^2)$. We’ve solved for the couplings perturbatively, to second order in both, which means we ignored the difference between e.g. $g$ and $g_P$ in the quadratic term, as we must. From now on I will drop the $P$ subscripts on the physical coupling.

Notice that we would get the same answer if we defined $\lambda_P$ by fixing a value of $M_{12-12}$ instead.

(d) Consider the limit of low energies, i.e. when $s_0, t_0, u_0 \ll \Lambda^2$ where $\Lambda$ is the cutoff scale. Tune the location of the poles in both propagators to $p^2 = 0$. Show that the coupling goes to the $O(2)$-symmetric value if it starts nearby (nearby means $\lambda/g < 3$).

A nice trick for doing this is to compute the beta functions.

$$\beta_g \equiv 32\pi^2\Lambda^2\partial_{\Lambda^2}g(\Lambda) = 3\left(g^2 + \left(\frac{\lambda}{3}\right)^2\right), \quad \beta_\lambda \equiv 32\pi^2\Lambda^2\partial_{\Lambda^2}\lambda(\Lambda) = \left(2\lambda g + \frac{4\lambda^2}{3}\right)$$
where I’ve pulled out a factor of $32\pi^2$ in the definition of $\beta$ for convenience—it only affects how fast the flow happens. A useful check is that if we set $\lambda = 0$, we reproduce the beta function for $\phi^4$ theory, $\beta_g = +3g^2$ (the 3 comes from the 3 different channels).

To look at the relative flow of $g$ and $\lambda$ let’s compute

$$\beta_{\lambda/g} \equiv 8\pi^2 \Lambda^2 \partial_{\Lambda^2} \frac{\lambda}{g} = \frac{1}{g^2} (g\beta_\lambda - \lambda\beta_g) \propto \left( -\frac{\lambda^3}{3} - \frac{5}{3} g\lambda^2 + 2g^2\lambda \right) = \frac{1}{3} \lambda(\lambda-g)(\lambda+6g).$$

This looks like this:

with the convention I’m using, positive $\beta$ means that as we increase $\Lambda$, the coupling decreases. This means that the couplings approach the point $g = \lambda$ as $\Lambda \to \infty$ fixing $g_P, \lambda_P$. This is the case as long as we start with $\lambda/g < 3$.

4. **Bremsstrahlung.** Show that the number of photons per decade of wavenumber produced by the sudden acceleration of a charge is

$$f_{IR}(q^2) = \frac{\alpha}{\pi} \ln \left( \frac{-q^2}{m^2} \right),$$

where $q_\mu = p'_\mu - p_\mu$ is the change of momentum and $m$ is the mass of the charge. This is explained well on pages 177-182 of Peskin. The energy comes out to

$$U = \int d^3k \frac{2\alpha}{\pi} \ln \left( \frac{-q^2}{m^2} \right) = 2 \int d^3k k N_k$$

where $N_k$ is the number density of photons of momentum $k$, and the RHS used the fact that each photon of momentum $k$ carries energy $k$. (The 2 comes from two polarizations for each momentum) Then the number of photons is

$$N = \int \frac{dk}{k} \frac{\alpha}{\pi} \ln \left( \frac{-q^2}{m^2} \right) = \int d\log k \frac{\alpha}{\pi} \ln \left( \frac{-q^2}{m^2} \right)$$

and hence $\frac{\alpha}{\pi} \ln \left( \frac{-q^2}{m^2} \right)$ is the number of photons per decade of wavenumber. (Note that the integral over $k$ here actually diverges; this is an artifact of the approximation that the momentum change is instantaneous.)
5. **Soft gravitons?** [optional] Photons are massless, and this means that the cross sections we measure actually include soft ones that we don’t detect. If we don’t include them we get IR-divergent nonsense.

Gravitons are also massless. Do we need to worry about them in the same way? Here we’ll sketch some hints for how to think about this question.

(a) Consider the action

\[ S_0[h_{\mu\nu}] = \int d^4x \frac{1}{2} h_{\mu\nu} \Box h^{\mu\nu}. \]

This is a kinetic term for (too many polarizations of a) two-index symmetric-tensor field \( h_{\mu\nu} = h_{\nu\mu} \) (which we’ll think of as a small fluctuation of the metric about flat space: \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), and this is where the coupling below comes from). Like with the photon, we’ll rely on the couplings to matter to keep unphysical polarizations from being made. Write the propagator. We still raise and lower indices with \( \eta_{\mu\nu} \). \(^1\)

The propagator is imply the inverse of the kinetic term. After the gauge fixing (implicit in the expression I gave) it is indeed invertible, just like in Maxwell theory. The graviton propagator you’ll find on Wikipedia is the propagator for \( h_{\mu\nu} \), rather than \( \bar{h}_{\mu\nu} \).

(b) Couple the graviton to the electron field via

\[ S_G = \int d^4x \, G h^{\mu\nu} T_{\mu\nu} \]

\[ T_{\mu\nu} \equiv \bar{\psi} (\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu) \psi. \] \hspace{1cm} (9)

What are the engineering dimensions of the coupling constant \( G \)? What is the new Feynman rule?

\( G = \sqrt{G_N} \) has dimensions of one over mass. This factor pops out of \( L \sim \frac{1}{G_N} \sqrt{g} g^{\mu\nu} + T_{\mu\nu} h^{\mu\nu} \) upon rescaling \( h \) to give it canonical kinetic terms.

\(^1\) A warning: I’ve done two misdeeds in the statement of this problem. First, the einstein-hilbert term is \( \int d^4x \frac{1}{8\pi G_N} \sqrt{g} R = \int d^4x \frac{1}{8\pi G_N} (\partial h)^2 + ... \) has a factor of \( G_N \) in front of it. \( R \) has units of \( \frac{1}{\text{length}^2} \), and \( g \) is dimensionless, so \( G_N \) has units of length\(^2 \) – it is \( 8\pi G_N = \frac{1}{M_{Pl}^2} \), where \( M_{Pl} \) is the Planck mass.

I’ve absorbed a factor of \( \sqrt{G_N} \) into \( h \) so that the coefficient of the kinetic term is unity. Second, the \( (\partial h)^2 \) here involves various index contractions, which I haven’t written. Some gauge fixing (de Donder gauge) is required to arrive at the simple expression I wrote above, and one more thing – the \( h_{\mu\nu} \) I’ve written is actually the ‘trace-reversed’ graviton field

\[ \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \]

where \( h \equiv \eta^{\mu\nu} h_{\mu\nu} \) is the trace. (I didn’t write the bar.) For the details of this, see chapter 10 of my GR notes.
(c) Draw a (tree level) Feynman diagram which describes the creation of gravitational radiation from an electron as a result of its acceleration from the absorption of a photon \( (e\gamma \rightarrow eh) \). Evaluate it if you dare. Estimate or calculate the cross section (hint: use dimensional analysis).

The amplitude has a single factor of \( G \), so the probability goes like \( G^2 \) which has dimensions of length-squared, which is already the right dimensions for a cross section. Apparently, for energies large compared to the electron mass, this cross section is constant in energy.

(d) Now the main event: study the one-loop diagram by which the graviton corrects the QED vertex. Is it IR divergent? If not, why not?

There are extra powers of the momentum in the numerator from the derivative coupling. This paper shows that this is not enough to prevent an IR divergence. So indeed, if we wish to include the (very small!) radiative corrections from gravitons, we must study inclusive amplitudes that allow for soft gravitons.

(e) If you get stuck on the previous part, replace the graviton field by a massless scalar \( \pi(x) \). Compare the case of derivative coupling \( \lambda \partial_\mu \pi \bar{\psi} \gamma^\mu \psi \) with the more direct Yukawa coupling \( y \pi \bar{\psi} \psi \).

In this case, the extra powers of the momentum in the numerator from the derivative coupling do prevent an IR divergence.

(f) Quite a bit about the form of the coupling of gravity to matter is determined by the demand of coordinate invariance. This plays a role like gauge invariance in QED. Acting on the small fluctuation, the transformation is

\[
h_{\mu\nu}(x) \rightarrow h_{\mu\nu}(x) + \partial_\mu \lambda_\nu(x) + \partial_\nu \lambda_\mu(x).
\]

What condition does the invariance under this (infinitesimal) transformation impose on the object \( T_{\mu\nu} \) appearing in (9).

The variation of the action is

\[
\delta S = \int d^4x \left( \partial_\mu \lambda_\nu(x) + \partial_\nu \lambda_\mu(x) \right) T^{\mu\nu} \overset{\text{IBP}}{=} -2 \int d^4x \lambda_\mu \partial_\nu T^{\mu\nu}
\]

which vanishes if \( \partial_\nu T^{\mu\nu} = 0 \), i.e. if \( T^{\mu\nu} \) is a conserved stress tensor.

6. **Equivalent photon approximation.** [optional] Consider a process in which very high-energy electrons scatter off a target. At leading order in \( \alpha \), the electron line is connected to the rest of the diagram by a single photon propagator. If the initial and final energies of the electron are \( E \) and \( E' \), the photon will carry momentum \( q \) with \( q^2 = -2EE'(1-\cos \theta) \) (ignoring the electron mass \( m \ll E \)). In
the limit of forward scattering \((\theta \rightarrow 0)\), we have \(q^2 \rightarrow 0\), so the photon approaches its mass shell. In this problem, we ask: To what extent can we treat it as a real photon?

(a) The matrix element for the scattering process can be written as

\[
\mathcal{M} = -ie\bar{u}(p')\gamma^{\mu}u(p)\frac{-i\eta_{\mu\nu}}{q^2}\hat{\mathcal{M}}^{\nu}(q)
\]

where \(\hat{\mathcal{M}}^{\nu}\) represents the coupling of the virtual photon to the target. Let \(q = (q^0, \vec{q})\) and define \(\vec{q} = (q^0, -\vec{q})\). The electron line can be parametrized as

\[
\bar{u}(p')\gamma^{\mu}u(p) = Aq^{\mu} + B\vec{q}^{\mu} + Ce^{\mu}_1 + D\epsilon^{\mu}_2
\]

where \(\epsilon_{\alpha}\) are unit vectors transverse to \(\vec{q}\). Show that \(B\) is at most of order \(\theta^2\) (dot it with \(q\)), so we can ignore it at leading order in an expansion about forward scattering. Why do we not care about the coefficient \(A\)?

I will wait to post the solutions to this problem, since some folks said they would like more time to think about it.

(b) Working in the frame with \(p = (E, 0, 0, E)\), compute

\[
\bar{u}(p')\gamma^{\mu}\epsilon_{\alpha}u(p)
\]

explicitly using massless electrons, where \(\bar{u}\) and \(u\) are spinors of definite helicity, and \(\epsilon_{\alpha=||,\perp}\) are unit vectors parallel and perpendicular to the plane of scattering. Keep only terms through order \(\theta\). Note that for \(\epsilon_{||}\), the (small) \(\hat{3}\) component matters.

(c) Now write the expression for the electron scattering cross section, in terms of \(|\mathcal{M}^{\mu}|^2\) and the integral over phase space of the target. This expression must be integrated over the final electron momentum \(p'\). The integral over \(p'^d\) is an integral over the energy loss of the electron. Show that the integral over \(p'_\perp\) diverges logarithmically as \(p'_\perp\) or \(\theta \rightarrow 0\).

(d) The divergence as \(\theta \rightarrow 0\) is regulated by the electron mass (which we’ve ignored above). Show that reintroducing the electron mass in the expression

\[
q^2 = -2(EE' - pp'\cos \theta) + 2m^2
\]

cuts off the divergence and gives a factor of \(\log (s/m^2)\) in its place.

(e) Assembling all the factors, and assuming that the target cross sections are independent of photon polarization, show that the largest part of the electron-target cross section is given by considering the electron to be the source of
a beam of real photons with energy distribution given by

\[ N_\gamma(x) dx = \frac{dx}{x} \frac{\alpha}{2\pi} (1 + (1 - x)^2) \log \frac{s}{m^2} \]

where \( x \equiv E_\gamma / E \). This is the Weizsäcker-Williams equivalent photon approximation. It is a precursor to the theory of jets and partons in QCD.