University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021
Assignment 2 – Solutions

Due 12:30pm Monday, January 18, 2020

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Toric code as $\mathbb{Z}_2$ gauge theory with matter.** Consider a model with qubits on the links of a lattice (with Pauli operators X_ℓ, Z_ℓ, $X_\ell Z_\ell = -Z_\ell X_\ell$) and qubits on the sites of the lattice (with Pauli operators σ^x_j, σ^z_j, $\sigma^x_j \sigma^z_j = -\sigma^z_j \sigma^x_j$).

   (a) Show that the operator
   $$G_j \equiv A_j \sigma^z_j$$
   (where A_j is the star operator) generates the gauge transformation
   $$\sigma^x_j \to (-1)^{s_j} \sigma^x_j, \quad X_{ij} \to (-1)^{s_i} X_{ij} (-1)^{s_j}, \ \sigma^z_j \to \sigma^z_j, \ Z_{ij} \to Z_{ij} \qquad (1)$$
   (where i, j are the sites at the ends of the link labelled ij). By *generates* here I mean that an operator $\mathcal{O}$ transforms as
   $$\mathcal{O} \to \mathcal{G}_s^\dagger \mathcal{O} \mathcal{G}_s, \quad \mathcal{G}_s \equiv \prod_j G_j^{s_j}$$
   with $s_j = 0, 1$.

   The relations involving Z and σ^z follow because they commute with G_j. X_ℓ transforms under G_j if the site $j \in \partial \ell$.

   (b) Show that the Hamiltonian
   $$\mathbf{H} = -\sum_j G_j - \sum_p B_p - h \sum_{ij} \sigma^x_i X_{ij} \sigma^x_j - g \sum_{\langle ij \rangle} Z_{ij}$$
   is gauge invariant.

   G_j commutes with $G_{j'}$. B_p commutes since it is a closed loop of Xs. The third term is a kinetic term for the e particles, which is invariant because the transformation of the σ^xs cancels that of the X. The last term is invariant because it is made of Zs.

Here we can identify σ^x_j as the operator which creates an e particle at site j. And we can identify $\sigma^z_j = (-1)^{n_j}$ as the parity of the number operator.

(c) Show that if we set $\sigma^x_j = 1$ and $\sigma^z_j = 1$ for all j we get back the (perturbed) toric code.

Bonus problem: interpret this operation as a choice of gauge in the model where $G_j = 1$ is imposed as a constraint on physical states.

Clearly if we just erase all the σ^x_js and σ^z_js we get back the toric code Hamiltonian.

And clearly we can choose the gauge parameter s_j in (1) to set $\sigma^x = 1$. The tricky part of this is that we can also erase all the appearances of σ^z. This wouldn't make sense on the full Hilbert space $\otimes\mathcal{H}_2$, since σ^z and σ^x do not commute. The claim is that on the space of physical states of the gauge theory, which satisfy $G_j |\text{phys}\rangle = |\text{phys}\rangle$ for all j, we can do this. It's because on such states, the action of σ^z_j can be replaced by A_j. If everything is gauge-invariant, any appearance of G_j can be moved onto the states, and replaced with 1.

2. **3-ball.** Find two cellulations of the 3-dimensional ball (*e.g.* the region $x^2 + y^2 + z^2 \leq 1$ in $\mathbb{R}^3$) and compute the resulting homology groups.

One cellulation is one 3-cell and then the cellulation of the boundary 2-sphere by a single 2-cell and a single 0-cell. The boundary maps are $\partial\sigma_3 = \sigma_2, \partial\sigma_2 = 0$ and we find $H_i(B_3, \mathbb{Z}) = \delta_{i,0}\mathbb{Z}$.

Alternatively, we could take a different cellulation of the boundary 2-sphere, say the iterative scheme described in the lecture notes. The cell complex is then

$$0 \to A \overset{(1,-1)}{\to} A^2 \overset{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}}{\to} A^2 \overset{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{\to} A^2 \to 0.$$

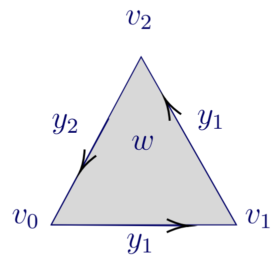
Compared to the cell complex for S^2, the extra step kills the 2d homology, and we again find $H_i(B_3, \mathbb{Z}) = \delta_{i,0}\mathbb{Z}$.

3. **Who am I?**

(a) Compute the homology of the following cell complex: take a 2-simplex $[v_0, v_1, v_2]$ (recall that the simplex $[v_0, \cdots, v_n]$ is the set of all convex combinations of the points v_i: $[v_0, \cdots, v_n] \equiv \{\lambda_i v_i, \sum_i \lambda_i = 1, \lambda_i \geq 0\}$) and identify the edges $[v_0, v_1]$ and $[v_1, v_2]$ (with the orientation preserving the order of the vertices). Also identify the 0-cells in their boundaries.

In order to identify the edges $[v_0, v_1]$ and $[v_1, v_2]$, we have to identify all the vertices. The edge $[v_2, v_0]$ is still distinct, so there are two 2-cells, y_1, y_2.

Here is a picture of the resulting cell complex:

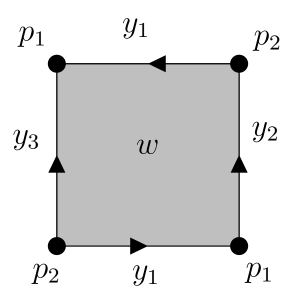


The cell complex is

$$0 \to A \stackrel{(2,1)}{\to} A^2 = \langle y_1 = [v_0, v_1] = [v_1, v_2], [v_2, v_0] \rangle \stackrel{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}{\to} A \to 0.$$

For $A = \mathbb{Z}$, this gives $H_2 = 0, H_1 = \mathbb{Z} \oplus \mathbb{Z}_2, H_0 = \mathbb{Z}$.

(b) Compute the homology of the following cell complex: take a square. Identify one pair of opposite edges with a twist:



The other pair of sides remains distinct. Is this the same space from the previous part? What space is it?

Both of these spaces describe a Mobius strip, an unoriented 2d manifold with a single boundary. This cell complex is

$$0 \to \mathbb{Z} \stackrel{(2,1,1)}{\to} \mathbb{Z}^3 \stackrel{\begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}}{\to} \mathbb{Z}^2 \to 0$$

whose homology is

$$H_2 = 0, H_1 = \langle y_1, y_2 + y_3 | 2y_1 = 0 \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}, H_2 = \mathbb{Z} = \langle p_1 = p_2 \rangle .$$

4. Consider a sphere with an extra 1-cell attaching the north pole to the south pole. Compute the homology of this space.

   Let's decompose the sphere as a single 2-cell w whose boundary is a single line of longitude, e_1, traversed twice: $\partial w = e_1 - e_1 = 0$. Then the extra 1-cell is e_2 with $\partial e_1 = N - S = \partial e_2$. The complex is

$$0 \to A \xrightarrow{0} A^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}} A^2 \to 0.$$

   ∂_2 clearly has rank 1, so we find

$$H_2 = A, H_1 = A = \langle e_1 - e_2 \rangle, H_0 = A = \langle N = S \rangle.$$

   Later we'll see that homology is invariant under homotopy. We can find a homotopy that moves the north pole end of the extra 1-cell down to the south pole. So this space has the same homology as a sphere with a loop attached to it.