University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021 Assignment 2

Due 12:30pm Monday, January 18, 2020

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Toric code as $\mathbb{Z}_2$ gauge theory with matter.** Consider a model with qubits on the links of a lattice (with Pauli operators X_ℓ, Z_ℓ, $X_\ell Z_\ell = -Z_\ell X_\ell$) and qubits on the sites of the lattice (with Pauli operators σ^x_j, σ^z_j, $\sigma^x_j \sigma^z_j = -\sigma^z_j \sigma^x_j$).

   (a) Show that the operator
   $$G_j \equiv A_j \sigma^z_j$$
   generates the gauge transformation
   $$\sigma^x_j \to (-1)^{s_j} \sigma^x_j, \quad X_{ij} \to (-1)^{s_i} X_{ij} (-1)^{s_j}, \; \sigma^z_j \to \sigma^z_j, \; Z_{ij} \to Z_{ij} \tag{1}$$
   (where i, j are the sites at the ends of the link labelled ij). By *generates* here I mean that an operator $\mathcal{O}$ transforms as
   $$\mathcal{O} \to \mathcal{G}_s^\dagger \mathcal{O} \mathcal{G}_s, \quad \mathcal{G}_s \equiv \prod_j G_j^{s_j}$$
   with $s_j = 0, 1$.

   (b) Show that the Hamiltonian
   $$\mathbf{H} = -\sum_j G_j - \sum_p B_p - h \sum_{ij} \sigma^x_i X_{ij} \sigma^x_j - g \sum_{\langle ij \rangle} Z_{ij}$$
   is gauge invariant.

   Here we can identify σ^x_j as the operator which creates an e particle at site j. And we can identify $\sigma^z_j = (-1)^{n_j}$ as the parity of the number operator.

   (c) Show that if we set $\sigma^x_j = 1$ and $\sigma^z_j = 1$ for all j we get back the (perturbed) toric code.

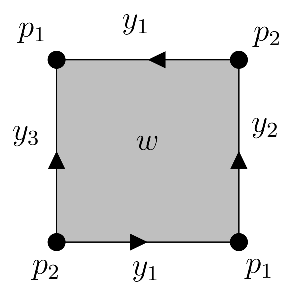
   Bonus problem: interpret this operation as a choice of gauge.

2. **3-ball.** Find two cellulations of the 3-dimensional ball (*e.g.* the region $x^2 + y^2 + z^2 \leq 1$ in $\mathbb{R}^3$) and compute the resulting homology groups.

3. **Who am I?**

(a) Compute the homology of the following cell complex: take a 2-simplex $[v_0, v_1, v_2]$ (recall that the simplex $[v_0, \cdots, v_n]$ is the set of all convex combinations of the points v_i: $[v_0, \cdots, v_n] \equiv \{\lambda_i v_i, \sum_i \lambda_i = 1, \lambda_i \geq 0\}$) and identify the edges $[v_0, v_1]$ and $[v_1, v_2]$ (with the orientation preserving the order of the vertices). Also identify the 0-cells in their boundaries.

(b) Compute the homology of the following cell complex: take a square. Identify one pair of opposite edges with a twist:



The other pair of sides remains distinct. Is this the same space from the previous part? What space is it?

4. Consider a sphere with an extra 1-cell attaching the north pole to the south pole. Compute the homology of this space.