University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021 Assignment 3 – Solutions

Due 12:30pm Wednesday January 27, 2021

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Brain-warmer on the definitions.** Show that $H_p(X, A)$ is a group, where the group law is just addition of representatives: if C and C' are cycles, then the sum of their equivalence classes modulo boundaries is $[C] + [C'] = [C + C?]$. Show that this is independent of the choice of representatives.

   If we chose different representatives $C + \partial p$ and $C' + \partial p'$, the representative of the sum would be $C + \partial p + C' + \partial p'$, but $[C + \partial p + C' + \partial p'] = [C + C']$.

2. **Brain-warmer on exact sequences.**

   Consider the following collection of homomorphisms between abelian groups:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "i'"'] & B' \arrow[r, "j'"'] & C' \arrow[r, "k'"'] & D' \arrow[r, "l'"'] & E'
\end{tikzcd}$$

   The rows are exact sequences, and all the maps commute.

   (a) If β and δ are surjective and ε is injective, show that γ is surjective.

   Given $c' \in C'$ we want to find $c \in C$ such that $\gamma c = c'$. If $k'(c') = 0$ then $c' = j'(b')$ by exactness of the bottom row at C'. This implies there exists $b \in B$ with $\beta(b) = b'$ and $j(b) = c$, the desired preimage of c'.

   It is easier to figure this out yourself than to read the following solution, but here it is anyway.

   If on the other hand $k'(c') \neq 0$, then $k'(c') \in \ker \ell'$. That means $\exists d \in D$ such that $\delta(d) = k'c' \in D'$. Then we must have $\ell(d) = 0$ or else $\epsilon\ell(d) = \ell k'(c') \neq 0$, a contradiction. Therefore $d \in \ker \ell$ which means by exactness at D $d \in \mathrm{im}(k)$, that is $d = k(c)$ for some $c \in C$, with $\gamma(c) = c'$.

(b) If β and δ are injective and α is surjective show that γ is injective (that is, $\gamma(c) = 0$ implies $c = 0$).

We want to show that $\gamma(c) = 0$ implies that $c = 0$. $\gamma(c) = 0$ implies that $\delta(k(c)) = k'\gamma(c) = 0$, which says $k(c) = 0$, since δ is onto. Exactness at C then says $c = j(b)$ for some $b \in B$. Now how can we resist taking $\beta(b) = b'$. By commutativity of the diagram, $j'(b') = 0$ and therefore $b' = i'(a') = i'\alpha(a)$ by exactness at B' and commutativity of the leftmost square. Therefore $b = i(a)$. But then $c = j(b) = jia = 0$ since $ji = 0$.

Conclude that if the outer four maps $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is too.

If all of these maps are isomorphisms, then in particular the hypotheses of the previous parts of the problem are satisfied.

This is called the Five-Lemma and is used frequently in algebraic topology. Specifically it is used in proving that various varieties of homology (simplicial, cellular, singular...) produce isomorphic groups.

3. **Coefficients.**

(a) Check that our answers for the homology of the Klein bottle with coefficients $\mathbb{Z}_{2,3,6}$ are consistent with the long exact sequence on homology induced by the short exact sequence of coefficient groups:

$$0 \to \mathbb{Z}_2 \overset{i}{\hookrightarrow} \mathbb{Z}_6 \to (\mathbb{Z}_6/\mathbb{Z}_2 = \mathbb{Z}_3) \to 0. \tag{1}$$

The long exact sequence has the form

$$\begin{aligned} 0 \to & H_2(K, \mathbb{Z}_2) \xrightarrow{i_\star} H_2(K, \mathbb{Z}_6) \xrightarrow{\pi_\star} H_2(K, \mathbb{Z}_3) & (2)\\ \xrightarrow{\partial_\star} & H_1(K, \mathbb{Z}_2) \xrightarrow{i_\star} H_1(K, \mathbb{Z}_6) \xrightarrow{\pi_\star} H_1(K, \mathbb{Z}_3) & (3)\\ \xrightarrow{\partial_\star} & H_0(K, \mathbb{Z}_2) \xrightarrow{i_\star} H_0(K, \mathbb{Z}_6) \xrightarrow{\pi_\star} H_0(K, \mathbb{Z}_3) \to 0 \quad . & (4) \end{aligned}$$

The answers we found were

$$\begin{aligned} 0 \to & \mathbb{Z}_2 \xrightarrow{i_\star} \mathbb{Z}_2 \xrightarrow{\pi_\star} 0 & (5)\\ \xrightarrow{\partial_\star} & \mathbb{Z}_2^2 \xrightarrow{i_\star} \mathbb{Z}_2 \times \mathbb{Z}_6 \xrightarrow{\pi_\star} \mathbb{Z}_3 & (6)\\ \xrightarrow{\partial_\star} & \mathbb{Z}_2 \xrightarrow{i_\star} \mathbb{Z}_6 \xrightarrow{\pi_\star} \mathbb{Z}_3 \to 0 \quad . & (7) \end{aligned}$$

So actually both the Bocksteins are just the zero map. The bottom row is just the original sequence (1).

(b) Construct the 0-form, 1- form and 2-form toric codes with gauge group $A = \mathbb{Z}_6$ on the Klein bottle and find their groundstate subspaces. Use whatever cell decomposition you like, for example the minimal one in the lecture notes. Do the groundstate subspaces agree with the homology groups we found?

Recall that the general p-form $\mathbb{Z}_N$ toric code hamiltonian is

$$H = -\sum_{(p-1)\text{-cells } s} \prod_{\sigma \in v(s)} Z_\sigma - \sum_{(p+1)\text{-cells } w} \prod_{\sigma \in \partial(w)} X_\sigma + h.c..$$

Let's use the minimal cell complex. The 0-form toric code has degrees of freedom on the sites. Here there is only one site. The site appears in the boundary of each link twice with opposite orientations, so the 'plaquette' operators are just constants, $Z_p Z_p^\dagger = 1$. There are no -1-cells so there are no star operators. The hamiltonian is a constant, so there are 6 groundstates.

The 1-form toric code has dofs on the links. The minimal cellulation has two links, y_1, y_2. The star term associated to the single 0-cell p is $A_p = Z_1 Z_2 Z_1^\dagger Z_2^\dagger = 1$, a constant. The plaquette term associated to the 2-cell w is $B_w = X_1 X_2 X_1 X_2^\dagger = X_1^2$. Minimizing $H = -B_w + h.c. + \text{const}$ tells us $X_1^2 = 1$ and X_2 is not constrained. Therefore the groundstates have the same multiplicity as $\mathbb{Z}_2 \times \mathbb{Z}_6$.

The 2-form toric code has dofs on the 2-cells. The minimal cellulation has only one two cell. It appears twice in the vicinity of y_2 with opposite orientation, so the star term associated with y_2 is a constant. The interesting thing is that this cell appears in the vicinity of the link y_1 twice with *the same* orientation. So the star term associated with y_1 is $H = -Z_w^2$. Minimizing this term tells us that $Z_w^2 = 1$. In $\mathbb{Z}_6$ there are two solutions to this condition.