University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021
Assignment 3

Due 12:30pm Wednesday January 27, 2021

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Brain-warmer on the definitions.** Show that $H_p(X, A)$ is a group, where the group law is just addition of representatives: if C and C' are cycles, then the sum of their equivalence classes modulo boundaries is $[C] + [C'] = [C + C?]$. Show that this is independent of the choice of representatives.

2. **Brain-warmer on exact sequences.**

   Consider the following collection of homomorphisms between abelian groups:

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ A' & \xrightarrow[i']{} & B' & \xrightarrow[j']{} & C' & \xrightarrow[k']{} & D' & \xrightarrow[l']{} & E' \end{array}$$

   The rows are exact sequences, and all the maps commute.

   (a) If β and δ are surjective and ε is injective, show that γ is surjective.

   (b) If β and δ are injective and α is surjective show that γ is injective (that is, $\gamma(c) = 0$ implies $c = 0$).

   Conclude that if the outer four maps $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is too.

3. **Coefficients.**

   (a) Check that our answers for the homology of the Klein bottle with coefficients $\mathbb{Z}_{2,3,6}$ are consistent with the long exact sequence on homology induced by the short exact sequence of coefficient groups:

$$0 \to \mathbb{Z}_2 \overset{i}{\hookrightarrow} \mathbb{Z}_6 \to (\mathbb{Z}_6/\mathbb{Z}_2 = \mathbb{Z}_3) \to 0. \qquad (1)$$

(b) Construct the 0-form, 1- form and 2-form toric codes with gauge group $A = \mathbb{Z}_6$ on the Klein bottle and find their groundstate subspaces. Use whatever cell decomposition you like, for example the minimal one in the lecture notes. Do the groundstate subspaces agree with the homology groups we found?